Trade and the Value of Information under Unawareness

Spyros Galanis*
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Abstract

The value of information and the possibility of speculation are examined in an environment with unawareness. Although agents have “correct” prior beliefs about events they are aware of and have a clear understanding of their available actions and payoffs, their unawareness may lead them to commit information processing errors and to behave suboptimally. As a result, more information is not always valuable and agents can speculate with each other. Two specific information processing errors are responsible. First, the level of the agent’s awareness may be too low. Second, his varying awareness provides a signal that the agent comprehends only partially and may misuse. Trade is analysed in three different settings. Assume common priors. First, there cannot be common knowledge trade. Hence, being unaware of an event is behaviourally distinct from assigning probability zero to it. Second, if all agents do not misuse their awareness signal, then an always mutually beneficial trade does not exist. If, in addition, agents do not make wrong inferences about others due to their low awareness, there cannot be trade in equilibrium.

1 Introduction

Consider an agent who is contemplating investing in the stock market today. His payoff is determined by the prices of shares tomorrow, and his particular buy and sell orders. The agent is aware of all his possible actions (investments) and all the possible prices of the shares.

Suppose now that there are other contingencies, expressed by questions, which indirectly influence the prices of shares tomorrow, and therefore the agent’s payoff. Examples of such questions are whether there will be a merger, what are the characteristics of a new CEO or whether an innovation will be announced. The agent may be aware of some of these questions, and unaware of others. Being unaware of a question means that he does not know its answer and he does not know that he does not know. In other words, he misses some information and at the same time fails to recognise it.

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*Economics Division, School of Social Sciences, University of Southampton, Highfield, Southampton, SO17 1BJ, UK, s.galanis@soton.ac.uk.
Although the agent has an an incomplete understanding of the world and there are questions that he has never thought about, he nevertheless has correct prior beliefs about events he is aware of and, within the bounds of his reasoning, he is perfectly rational. He is aware of all of his possible actions and he does not err when contemplating their deterministic payoff. In such an environment is information valuable? That is, will the more informed be better off ex ante when compared with a less informed but otherwise identical agent? In an interactive setting, will unaware agents who share a common prior refrain from speculating?

As subsequent examples show, the answer to both questions is “not always”. Since unawareness is a mistake of reasoning, this is not surprising. Even in the context of the standard model of knowledge both of these phenomena can occur when agents have a non partitional information structure. However, the standard model only distinguishes between a fully rational agent who makes no mistakes, and an irrational agent who makes some (unspecified) mistakes. Modelling unawareness explicitly allows us to analyse in detail the nature of these mistakes and distinguish between those that agents are most likely to commit systematically (because of being unaware), from those that are random or of no economic interest (like mistakes in computing one’s payoff).

It turns out that two specific information processing errors are responsible for both the value of information being negative and for agents engaging in speculative trade.

The first mistake arises when the level of one’s awareness is too low. For the first problem, a low level of awareness means that even though the agent is more informed about events he is aware of, there are many events that he is unaware of. As he cannot condition his actions on events he is unaware of, he is more inflexible and this comes at a cost. For the second problem, a low level of awareness implies that the agent may make wrong inferences about other agents’ information and actions. The reason is that the agent cannot reason beyond his awareness and as a result he may miss some connections (theorems) between events he is aware of and events he unaware of. Unavoidably, he will also be unaware that others actually know these theorems and therefore underestimate their knowledge.

In order to understand the second mistake in reasoning, we need to interpret awareness as a signal. When the agent receives his awareness and updates his beliefs, he excludes all states which specified that his awareness would be lower. However, he cannot exclude states describing higher awareness, since he cannot reason beyond his current level. In other words, although he is aware of event $E$, he cannot distinguish the finer details (or dimensions) that $E$ describes. In the case where these higher dimensions describe that the agent’s awareness would most likely be higher if $E$ occurs, the agent effectively overestimates (puts more probability on) the occurrence of $E$. Accordingly, he underestimates the occurrence of events describing that his awareness would most likely not increase. In both cases, the agent uses the signal provided by his awareness in a biased way.

Two different properties address this information processing error. The first, nested awareness, requires that awareness is ordered, so that an agent can be aware of an issue or a question only if he is also aware of all the other questions that precede it, according to this order. The second, conditional independence, requires that awareness, as a signal, has no value to the decision maker. Note that if one’s awareness is always the same, then awareness as a signal is uninformative and both these conditions are satisfied.
Conditional independence can be phrased in terms of an example. Suppose that an agent who considers investing in a firm were to acquire some private information, that enabled him to consider only two mutually exclusive scenarios as possible. Either there will be a lawsuit against the firm, or the firm will announce a technological breakthrough. Would the lawsuit increase the likelihood of the agent acquiring more awareness? In that case, the agent overestimates the plausibility of a lawsuit. On the contrary, if the announcement of the breakthrough would most likely make him more aware, then the agent underestimates the occurrence of a lawsuit. Conditional independence specifies that neither of the two is true.

1.1 Related Literature

The value of information has been studied in a variety of settings. Blackwell (1951) shows that an experiment is more informative if and only if it is more valuable to the decision maker. Blackwell’s theorem fails when the agent is not an expected utility maximiser, as shown by Safra and Sulganik (1995), Schlee (1991) and Wakker (1988).

The setting that is closest to that of the present paper represents information as a partition of the state space. If the agent’s prior is correct, he updates using Bayes’ rule and chooses an action which maximises his expected utility, then more information, measured by a finer partition, makes him better off ex ante. However, a partition is consistent only with an unboundedly rational agent who makes no information processing errors.

Sher (2005) shows that if the agent has wrong priors and updates nonmonotonically, then a little more information can be bad, but a lot of information is always good. Geanakoplos (1989) shows that when the agent has correct priors but a non-partitional information structure, then it is not always good to know more, unless this structure satisfies non-delusion, positive introspection, and nested.

Geanakoplos provides several reasons why agents in his model make information processing errors. For example, agents may forget, ignore unpleasant information or take no notice when something does not occur. However, the standard model does not provide a rich enough framework that facilitates identifying each of these problems with a specific violation of the partitional structure. As a result, it is not clear which mistake is being captured by which condition. Moreover, as demonstrated by Dekel et al. (1998), his setting has some limitations if the intention is to model agents who make information processing errors because of their unawareness.\footnote{Dekel et al. (1998) show that if unawareness satisfies three properties they propose, then the standard state space model can only accommodate trivial unawareness.} \footnote{Another limitation is that Geanakoplos only compares an agent who makes some information processing errors with another agent who is less informed but makes no errors - so his information is represented by a partition. In this paper we take a more general approach and allow for any agent, not just the more informed, to be unaware and to make information processing errors.} By applying the unawareness model provided by Galanis (2007), we are able to specify exactly what are the mistakes in reasoning that can result in speculative trade and in negative value of information and address the criticism of Dekel et al. (1998).

The literature on no trade theorems stems from Aumann (1976). Agents trade either because they have different priors or because they make information processing errors. In the context of the standard model where agents make no mistakes, Morris
(1994), Feinberg (1995, 1996), Samet (1998), Bonanno and Nehring (1996) and Ng (2003) show that “a necessary and sufficient condition for the existence of a common prior is that there is no bet for which it is always common knowledge that all agents expect a positive gain”. Moreover, Milgrom and Stokey (1982) and Sebenius and Geanakoplos (1983) show that common priors imply that there cannot be common knowledge of speculation. Geanakoplos (1989) shows that if we allow for mistakes, agents can speculate, even with common priors. However, speculation in equilibrium cannot occur, as long as non-delusion, nested and positive introspection are satisfied.

Models of unawareness are either syntactic or semantic. Beginning with Fagin and Halpern (1988), syntactic models have been provided by Halpern (2001), Modica and Rustichini (1994, 1999), Halpern and Rêgo (2005), Heifetz et al. (2007b) and Board and Chung (2007). Geanakoplos (1989) provides one of the first set-theoretic models of boundedly rational agents, using the standard framework. Dekel et al. (1998) argues against the use of a standard state space by proposing three properties for an unawareness operator. Two approaches overcome this impossibility result. First, by arguing against one of the properties (Ely (1998)), or by relaxing them (Xiong (2007)). Second, by introducing multiple state spaces, as in Li (2006), Heifetz et al. (2006) and Galanis (2007).


The main difference between the model of Galanis (2007) and those of Heifetz et al. (2006) and Heifetz et al. (2007b) is that the former allows for knowledge and posteriors to vary across states that differ only in terms of awareness. Since in this paper we make the assumption that agents make no mistakes about their actions and payoffs, the only source of errors in reasoning comes from how knowledge and posteriors change when awareness varies. Hence, in the paper we employ the model of Galanis (2007).

1.2 Overview of the results

The value of information is analysed by comparing a more informed agent 2 with a less informed agent 1. Both have the same preferences, payoffs, prior, and a correct understanding of their payoffs and actions, but may differ in their awareness and information. In the context of the standard model, agent 2 is more informed than agent 1 if whenever 1 knows an event, 2 knows it as well. In this model we apply the same definition, but only for events that both are aware of.

Information is valuable if both mistakes in reasoning are addressed. The low level of awareness is addressed by requiring that whenever agent 1 knows an event $E$ that 2 is unaware of, agent 2 knows another event that logically implies $E$. For the second mistake in reasoning we need either of the following two conditions to be true. First, the more informed agent 2 satisfies conditional independence, so that he does not misuse his awareness signal. Second, agent 1 satisfies conditional independence and agent 2’s awareness is nested and more informative.

Trade is analysed in three different settings. Suppose that all agents share a common

\footnote{For details on this difference see Galanis (2007).}
prior. Then, there cannot be common knowledge trade. Hence, although agents may be boundedly rational due to their unawareness, common knowledge of trade is sufficiently strong to rule out any such possibility, just like in the standard model with partitional structures. This result is in contrast with models of bounded reasoning (Geanakoplos (1989)) and unawareness (Xiong (2007)) that use the standard framework, as common knowledge trade is feasible there. Another implication is that assigning zero probability to an event is behaviourally distinct from being unaware of it. The former agents would engage in common knowledge (or common belief) trade, while the latter would not. Interestingly, we also show that common knowledge of no trade does not imply that there is no trade, the reason being that agents may make wrong inferences about others, due to their low level of awareness. Hence, although agents cannot agree that there are unexploited trading opportunities, such opportunities may exist, as long as they are beyond the awareness of some.

Second, conditional independence implies that there does not exist a trade that provides positive expected gains for everyone, always. That is, as long as agents do not overestimate or underestimate events due to their varying awareness, a mutually beneficial trade does not exist. Moreover, the reverse does not hold, so that even with conditional independence, different priors do not imply the existence of a mutually beneficial trade. The problem of low awareness is not relevant in this setting because agents do not need to reason about the actions of others.

Finally, trade in equilibrium cannot occur if each agent satisfies either conditional independence or nested awareness, and his payoff is not influenced by the level of his awareness. The second condition addresses the issue that, in equilibrium, agents have to reason about the actions of others. Low level of awareness means that they may reason incorrectly, so their actual payoffs may be different from their perceived ones, if this condition is not satisfied.

The paper proceeds as follows. Section 2 provides an example showing that information is not always valuable in the presence of unawareness. Section 3 presents an overview of the model of unawareness of Galanis (2007) and formalises the conditions mentioned above. The value of information problem is analysed in section 4, whereas the no trade theorems are presented in section 5. Proofs are contained in the appendix.

2 Knowing less can be better

In the following example we show that the value of information can be negative in an environment with unawareness. In particular, we compare two agents who share the same awareness, payoffs, action set and prior, but one has more information and he is strictly worse off. The reason is that due to his unawareness and private information, he overestimates the occurrence of some events.

Suppose that the agents contemplate investing in a particular share. We model this by making them aware of question $p$, “What is the price of the share?”. This question has three possible answers: low, medium and high. There are two available actions, buy (B) and not buy (NB). The payoff (in utils) is zero if the action is NB, irrespective of the price. If the action is B then the payoff is one if the price is high and $-1$ otherwise. Therefore, in terms of payoffs there is no difference between a low and a medium price. One can think of a low and a medium price as two distinct bad
scenarios.

Although the payoff depends only on the price, there are two factors that influence the price of the share. We model these factors by introducing two more questions. The first is question \( q \), “Is there going to be an acquisition?”, and the second is question \( r \), “Is there an innovation going to be adopted?”. Both questions have two possible answers, “yes” and “no”.

For each of the three “basic” questions \( p, q \) and \( r \), we add a question \( ap \) (respectively, \( aq, ar \)): “Is the agent aware of question \( p \)?”. These questions express the agents’ awareness and have two answers, “yes” and “no”. For example, suppose that the answer to \( aq \) is “no”. Then the agents are unaware of both \( q \) and \( aq \) - these questions never enter their reasoning. In other words, they do not know whether there is an acquisition, and fail to realise that they are missing this information. Suppose that the answer to \( aq \) is “yes”. Then both agents are aware of both \( q \) and \( aq \) and may or may not know whether there is an acquisition.

The three basic questions \( p, q \) and \( r \), together with their awareness counterparts \( ap, aq \) and \( ar \), constitute the set of all possible questions. A full state \( \omega^* \) specifies an answer to all six questions and thus provides a complete description of the world. Let \( \Omega^* \) denote the full state space, the collection of full states and let \( \pi \) be a prior on \( \Omega^* \).

In this example the full state space contains only three full states:

\[
\omega_1^* = (p_l, q_y, r_n, ap_y, aq_y, ar_n), \quad \pi(\omega_1^*) = 0.3,
\]
\[
\omega_2^* = (p_h, q_y, r_y, ap_y, aq_y, ar_y), \quad \pi(\omega_2^*) = 0.4,
\]
\[
\omega_3^* = (p_m, q_n, r_y, ap_y, aq_y, ar_n), \quad \pi(\omega_3^*) = 0.3.
\]

This state space describes that the agent is always aware of questions \( p, ap \), \( q \) and \( aq \). It also describes that both agents are aware of questions \( r \) and \( ar \) if and only if the price is high. One can think of the two previous sentences as two different “theorems”, expressed in the full state space.

Suppose that \( \omega_1^* \) occurs. It specifies that the agents are aware of questions \( p, q, ap \) and \( aq \) but not \( r \) and \( ar \). Each agent cannot even describe the full state space - he is unaware of it. However, he is aware of his subjective state space, which contains states that specify an answer only for questions \( p, q, ap \) and \( aq \). His subjective state space \( \Omega(\omega_1^*) \) at \( \omega_1^* \) contains the following three elements:

\[
\omega_1 = (p_l, q_y, ap_y, aq_y),
\]
\[
\omega_2 = (p_h, q_y, ap_y, aq_y),
\]
\[
\omega_3 = (p_m, q_n, ap_y, aq_y).
\]

The obvious difference between \( \Omega(\omega_1^*) \) and \( \Omega^* \) is that the former specifies no information about questions \( r \) and \( ar \). Moreover, \( \Omega(\omega_1^*) \), just like \( \Omega^* \), describes that the agent is always aware of questions \( p, q, ap \) and \( aq \). But \( \Omega(\omega_1^*) \), unlike \( \Omega^* \), cannot express the theorem that a high price occurs if and only if the agent is aware of \( r \) and \( ar \). In other words, the agent at \( \omega_1^* \) not only misses information about the question \( r \) that he is unaware of but, more importantly, misses a theorem that involves question \( p \) that he is aware of and questions \( r \) and \( ar \) that he is unaware of. When updating his beliefs, this will mean that he will overestimate the occurrence of a high price \( p_h \).
Although both agents are identical in terms of their awareness, they differ in terms of their information. Suppose that agent 2 is more informed than agent 1. “More informed” means, as in Geanakoplos (1989), that 2 always considers fewer states to be possible. We will represent this difference by assuming that whenever agent 1 learns the answer to a question (that he is aware of) then so does agent 2. Formally, let $X^1$ and $X^2$ be the following sets of questions:

$$X_1 = \{aq, ap, ar\},$$

$$X_2 = \{aq, ap, ar, q, r\}.$$

Whenever a state specifies that agent $i$ is aware of question $q$ and $q \in X^i$, we will assume that $i$ learns its answer. He can therefore exclude any other state which specifies a different answer.

To give an example, suppose that $\omega^*_1$ occurs. It specifies that both agents are aware of $p$, $q$, $ap$, and $aq$ and their subjective state space is $\{\omega_1, \omega_2, \omega_3\}$. Moreover, $\omega^*_1$ specifies “yes” for $q$. Since $q \in X^2$, agent 2 can exclude $\omega_3$ which specifies “no” for $q$. On the other hand, since $q \notin X^1$ agent 1 cannot exclude any state. When $\omega^*_2$ occurs, both agents are aware of all six questions. Since $\omega^*_2$ is the only full state specifying “yes” for question $ar$, which belongs to both $X^1$ and $X^2$, both agents can exclude states $\omega^*_1$ and $\omega^*_3$. The following table summarises the information for each agent, at each full state.

<table>
<thead>
<tr>
<th>Full state</th>
<th>Agent 1</th>
<th>Agent 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega^*_1$</td>
<td>${\omega_1, \omega_2, \omega_3}$</td>
<td>${\omega_1, \omega_2}$</td>
</tr>
<tr>
<td>$\omega^*_2$</td>
<td>${\omega^*_2}$</td>
<td>${\omega^*_2}$</td>
</tr>
<tr>
<td>$\omega^*_3$</td>
<td>${\omega_1, \omega_2, \omega_3}$</td>
<td>${\omega_3}$</td>
</tr>
</tbody>
</table>

Let $\pi$ be a prior on the full state space $\Omega^*$. We assume that whenever the agent has a lower dimensional subjective state space, his prior is the marginal of $\pi$ on that state space.\(^4\)

At each full state $\omega^*$ each agent $i$ receives his awareness, he constructs his subjective state space and receives his information. Having a prior on his subjective state space he updates using Bayes’ rule and chooses an action that maximises his expected utility, given his information. The following table summarises each agent’s posteriors and best action at each full state. For instance, at $\omega^*_1$ agent 2 assigns posterior belief of $3/7$ to the price being low, $4/7$ to the price being high and his best action is to buy.

<table>
<thead>
<tr>
<th>Full state</th>
<th>Agent 1</th>
<th>Action</th>
<th>Agent 2</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega^*_1$</td>
<td>(0.3, 0.4, 0.3)</td>
<td>NB</td>
<td>(3/7, 4/7, 0)</td>
<td>B</td>
</tr>
<tr>
<td>$\omega^*_2$</td>
<td>(0, 1, 0)</td>
<td>B</td>
<td>(0, 1, 0)</td>
<td>B</td>
</tr>
<tr>
<td>$\omega^*_3$</td>
<td>(0.3, 0.4, 0.3)</td>
<td>NB</td>
<td>(0, 0, 1)</td>
<td>NB</td>
</tr>
</tbody>
</table>

\(^4\)Every subjective state $\omega$ gives an answer to some questions. We assume that the probability assigned to $\omega$ is the probability that $\pi$ assigns to the set of full states which give the same answers as $\omega$. For example, the probability assigned to $\omega_1 = (p_1, q_2, ap_2, aq_2)$ is $\pi(\omega_1) = 0.3$, since $\omega^*_1 = (p_1, q_2, r_n, ap_2, aq_2, ar_n)$ is the only full state which gives the same answers as $\omega_1$ and $\pi(\omega^*_1) = 0.3$. Generally, many full states project to each state $\omega$.  

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At $\omega_2^*$ and at $\omega_3^*$ both agents choose the same action, so the interesting state is $\omega_1^*$ where the price is low and the more informed agent mistakenly chooses to buy. At that state there is an acquisition ($q_3$) but only agent 2 receives this information, since $q \not\in X^2$ but $q \not\in X^1$. Hence, he can exclude that a medium price has occurred. But agent 2 is unaware of $r$, which implies that he is unaware of the theorem that a high price occurs if and only he is aware of both $r$ and $ar$. As a result, he overestimates the plausibility of a high price and his optimal action is to buy. On the other hand, the less informed agent 1 cannot exclude any states and his optimal action is Not Buy. His unawareness of the theorem does not hurt him because he does not overestimate the plausibility of a high price, due to his low information.

Summarising, agent 1’s unawareness of the theorem, coupled with his extra information about $q$ leads him to behave suboptimally. When $\omega_2^*$ occurs and the price is high, both agents become aware of the theorem and exclude the two bad scenarios of a low or a medium price. They correctly choose to buy.

The more informed type 2 always considers fewer states to be possible and never excludes the true state. Nevertheless, the less informed agent always makes the correct decision, while the informed agent makes the wrong decision at $\omega_1^*$ and he is strictly worse off.

### 3 The Model

#### 3.1 Overview of Galanis (2007)

This section presents an overview of the model developed in Galanis (2007).

Consider a complete lattice of disjoint state spaces $S = \{S_a\}_{a \in A}$ and denote by $\Sigma = \bigcup_{a \in A} S_a$ the union of these state spaces. Throughout, we assume that $\Sigma$ is finite. A state $\omega$ is an element of some state space $S$. Let $S^*$ be the most complete state space, the join of all state spaces in $S$. We call $S^*$ the full state space. An element $\omega^* \in S^*$ is called a full state.

Let $\preceq$ be a partial order on $S$. For any $S, S' \in S$, $S \preceq S'$ means that $S'$ is more expressive than $S$. Moreover, there is a surjective projection $r_S^S : S' \to S$. Projections are required to commute. If $S \preceq S' \preceq S''$ then $r_S^{S''} = r_S^{S'} \circ r_S^{S''}$. If $\omega \in S'$, denote $\omega_S = r_S^S(\omega)$ and $\omega_{S'} = (r_S^{S'})^{-1}(\omega)$. If $B \subseteq S'$, denote $B_S = \{\omega_S : \omega \in B\}$ and $B_{S'} = \{\omega_{S'} : \omega \in B\}$. Let $g(S) = \{S' : S \preceq S'\}$ be the collection of state spaces that are at least as expressive as $S$. For a set $B \subseteq S$, denote by $B^S = \bigcup_{S' \in g(S)} (r_S^{S'})^{-1}(B)$ the enlargements of $B$ to all state spaces which are at least as expressive as $S$.

Consider a possibility correspondence $P : \Sigma \to 2^\Sigma \setminus \emptyset$ with the following properties:

1. **Confinedness**: If $\omega \in S$ then $P(\omega) \subseteq S'$ for some $S' \preceq S$.
2. **Generalized Reflexivity**: $\omega \in (P(\omega))^\dagger$ for every $\omega \in \Sigma$.
3. **Stationarity**: $\omega' \in P(\omega)$ implies $P(\omega') = P(\omega)$.
4. **Projections Preserve Ignorance**: If $\omega \in S'$ and $S \preceq S'$ then $(P(\omega))^\dagger \subseteq (P(\omega_S))^\dagger$.
5. **Projections Preserve Awareness**: If $\omega \in S'$, $\omega \in P(\omega)$ and $S \preceq S'$ then $\omega_S \in P(\omega_S)$. 

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3.1.1 Events, awareness and knowledge

An event $E$ is a subset of some (necessarily unique) state space $S \in \mathcal{S}$. The negation of $E$, denoted by $\neg E$, is the complement of $E$ with respect to $S$. Denote the complement of $S$ by $\emptyset_S$. Let $\mathcal{E} = \{E \subseteq S : S \in \mathcal{S}\}$ be the union of all events. For each event $E$, let $S(E)$ be the state space of which it is a subset. An event $E$ “inherits” the expressiveness of the state space of which it is a subset. Hence, we can extend $\leq$ to a partial order $\preceq_0$ on $\mathcal{E}$ in the following way: $E \preceq_0 E'$ if and only if $S(E) \preceq S(E')$. Abusing notation, we write $\preceq$ instead of $\preceq_0$.

Before defining knowledge, we need to define awareness. For any event $E$, for any state space $S$ such that $E \preceq S$, define

$$A_S(E) = \{\omega \in S : E \preceq S(\omega)\}$$

to be the event which describes, with the vocabulary of $S$, that the agent is aware of event $E$. The condition $E \preceq S$ imposes that only a state space rich enough to describe $E$, can also describe the agent’s awareness of $E$. The agent is aware of an event if his possibility resides in a state space that is rich enough to express event $E$. Unawareness is defined as the negation of awareness. More formally, the event $U_S(E)$ describes, with the vocabulary of $S$, that the agent is unaware of $E$:

$$U_S(E) = \neg A_S(E).$$

Let $\Omega : \Sigma \rightarrow \mathcal{S}$ be such that for any $\omega \in \Sigma$, $\Omega(\omega) = S$ if and only if $P(\omega) \subseteq S$. $\Omega(\omega)$ denotes the agent’s state space at $\omega$. An agent knows an event $E$ if he is aware of it and in all the states he considers possible, $E$ is true. Formally, for any event $E$ and for any state space $S$ such that $E \preceq S$, define

$$K_S(E) = \{\omega \in A_S(E) : P(\omega) \subseteq E(\Omega(\omega))\}.$$

3.2 Information

In the standard model, more information is represented by a finer partition. Formally, agent 2 is more informed than agent 1 if $P^2(\omega) \subseteq P^1(\omega)$ for all $\omega \in \Omega$. This is equivalent to requiring that whenever agent 1 knows an event, agent 2 knows it as well, so that $K^1(E) \subseteq K^2(E)$ for all events $E \subseteq \Omega$.

In an environment with unawareness these two definitions are not equivalent. In particular, $K^1_{\emptyset_S}(E) \leq K^2_{\emptyset_S}(E)$ for all events $E$ is equivalent to the property that $P^2(\omega^*)_{\emptyset_S} \subseteq P^1(\omega^*)_{\emptyset_S}$ and $\Omega^1(\omega^*) \preceq \Omega^2(\omega^*)$ for all $\omega^* \in S^*$. That is, “knowing more” according to the first property is equivalent to “considering fewer states to be possible and being more aware”, as described by the second property.

In order to disentangle information from awareness we will consider two weaker properties. The first compares the agents’ partitions in the “highest” state space that both are aware of.

**Definition 1.** $P^2$ is weakly more informed than $P^1$ if $P^2(\omega^*)_{\emptyset_S} \subseteq P^1(\omega^*)_{\emptyset_S}$ for all $\omega^* \in S^*$, where $S_{\omega^*} = \Omega^1(\omega^*) \cap \Omega^2(\omega^*)$.

Recall that $P^2(\omega^*)_{\emptyset_S}$ is the projection of 2’s information at $\omega^*$ to state space $S_{\omega^*}$, which is the meet of the agents’ state spaces at $\omega^*$. As the following Lemma shows,
this property is equivalent to requiring that whenever agent 1 knows an event that 2 is aware of, then 2 knows it as well. Hence, this property compares the agents’ knowledge only about events that both are aware of.

The second property compares the agents’ partitions in the full (most complete) state space \( S^* \).

**Definition 2.** \( P^2 \) is strongly more informed than \( P^1 \) if \( P^2(\omega^*)_{S^*} \subseteq P^1(\omega^*)_{S^*} \) for all \( \omega^* \in S^* \).

Recall that agents may be unaware of \( S^* \) and that \( P^2(\omega^*)_{S^*} \) is the enlargement (opposite of restriction) of 2’s information at \( \omega^* \) to the full state space. As the following Lemma shows, this property is equivalent to requiring that 2 is weakly more informed than 1 and whenever agent 1 knows an event that 2 is unaware of, then 2 knows another event that logically implies it. Formally, event \( E' \) logically implies \( E \) if \( E'_{S^*} \subseteq E_{S^*} \). Hence, this property compares the agents’ knowledge on events that both are aware of or that only one is aware of.

Below we enumerate the aforementioned properties. The following Lemma summarises the relations between them.

1. For all events \( E \), \( K^1_{S^*}(E) \subseteq K^2_{S^*}(E) \).
2. For all events \( E \), \( K^1_{S^*}(E) \cap A^2_{S^*}(E) \subseteq K^2_{S^*}(E) \).
3. For any event \( E \), there is event \( E' \) such that \( K^1_{S^*}(E) \cap U^2_{S^*}(E) \subseteq K^2_{S^*}(E') \) and \( E_{S^*} \subseteq E_{S^*} \).
4. \( P^2(\omega^*)_{S_{\omega^*}} \subseteq P^1(\omega^*)_{S_{\omega^*}} \) for all \( \omega^* \in S^* \), where \( S_{\omega^*} = \Omega^1(\omega^*) \land \Omega^2(\omega^*) \).
5. \( P^2(\omega^*)_{S^*} \subseteq P^1(\omega^*)_{S^*} \) for all \( \omega^* \in S^* \).
6. \( \Omega^1(\omega^*) \preceq \Omega^2(\omega^*) \) for all \( \omega^* \in S^* \).

**Lemma 1.**

- (1) \( \iff \) (5) and (6),
- (1) \( \iff \) (2) and (6),
- (4) \( \iff \) (2),
- (5) \( \iff \) (2) and (3).

Properties 2, 3, 4 and 5 compare the agents’ information without specifying anything about their awareness. On the other hand, property 6 compares the agents’ awareness without implying anything about their information. Finally, property 1 compares both information and awareness.

### 3.3 Variable awareness

The second information processing error arises when the agent misuses the signal provided by his varying awareness. The following conditions place restrictions on how awareness varies.

The first property, nested awareness, requires that awareness is ordered. Formally, an agent’s awareness at a full state is either more or less expressive than his awareness at any other full state.
Definition 3. [Nested awareness] Awareness for $P$ is nested if for all $\omega^*, \omega^*_1 \in S^*$, either $\Omega(\omega^*) \preceq \Omega(\omega^*_1)$ or $\Omega(\omega^*_1) \preceq \Omega(\omega^*)$.

This is equivalent to having a partial order on the collection of all state spaces, so that the agent is aware of a state space $S$ only if he is also aware of all states spaces $S'$ that precede it, according to this order. Nested awareness is closely related to the property nested, discussed in Geanakoplos (1989).

For the next two properties we need the following definition. Let $E(\omega^*) = \{ \omega^*_1 \in S^*: \Omega(\omega^*) = \Omega(\omega^*_1) \}$ be the event of the full state space describing that the agent has the same awareness as $\omega^*$ describes. Function $E$ partitions the full state space $S^*$ and provides a signal that the agent can only partially comprehend. If one’s awareness varies a lot across states, then his signal is more informative. This notion is formalised by the following definition.

Definition 4. Awareness for $P^1$ is more informative than for $P^2$ if for all $\omega^* \in S^*$, $E^1(\omega^*) \subseteq E^2(\omega^*)$.

The following property requires that conditional on receiving one’s private information, $P(\omega^*)$, the event describing the agent’s awareness, $E(\omega^*)$, is independent of any other event he is aware of.

Definition 5. [Conditional independence] $(P, \pi)$ satisfies conditional independence if for any $\omega^* \in S^*$ with $\pi(\omega^*) > 0$, for any $E \subseteq \Omega(\omega^*)$,

$$ \pi(E(\omega^*) \cap P(\omega^*)_{S^*}) = \pi(E(\omega^*)|P(\omega^*)_{S^*})\pi(E_{S^*}|P(\omega^*)_{S^*}). $$

Conditional independence specifies that awareness, as a signal, has no informational value. Equivalently, we can write the condition as

$$ \pi(E_{S^*}|P(\omega^*)_{S^*}) = \pi(E_{S^*}|E(\omega^*) \cap P(\omega^*)_{S^*}). $$

In other words, the beliefs of a fully aware (hence fully rational) agent with the same information structure $P$ would not change if he used the signal provided by $E$. As a result, although the agent cannot reason beyond his awareness, this inability does not lead to updating beliefs in a biased way. Note that nested awareness and conditional independence use the full state space $S^*$. Hence, the agent cannot, in principle, check whether he satisfies these conditions. Moreover, if the agent’s awareness is constant across states, then both conditional independence and nested awareness are satisfied.

4 The value of information

Let $C$ be a set of possible actions and define $u: C \times \Sigma \rightarrow \mathbb{R}$ to be the agent’s utility function. When the agent is aware of a state $\omega$, his perception of his payoff at $\omega$ if he chooses action $c$ is $u(c, \omega)$. But $\omega$ may be a coarse description of the world. We will assume that the agent does not err when contemplating the consequences of his actions. For this, we need the following assumption. Let $S_0 = \bigwedge S$ be the meet of all the agent’s subjective states.

---

The set $E_{S^*}$ contains all full states that project to $E$, while $P(\omega^*)_{S^*}$ contains all full states that project to $P(\omega^*)$. Hence, $E_{S^*}$ is the event ‘$E$ occurs’, while $P(\omega^*)_{S^*}$ is the event ‘$P(\omega^*)$ occurs’. 

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**Assumption 1.** For all \( \omega \in \Sigma \), \( u(c, \omega_{S_0}) = u(c, \omega) \) for all \( c \in C \).

The assumption states that in terms of payoffs, only \( S_0 \) matters. Hence, if two states \( \omega, \omega' \) project to the same state \( \omega'' \in S_0 \), (i.e. \( \omega_{S_0} = \omega'_{S_0} = \omega'' \)) then they assign the same payoff for every action. Since the agent is always aware of \( S_0 \), he has a correct understanding of his payoffs for each action. This assumption is consistent with the story, outlined in the example, of an agent who invests in the stock market and has a clear understanding of the payoffs of his buy and sell orders, as they only depend on prices. Although there are many factors that could influence his payoffs (a merger, an innovation, a lawsuit), this can only happen through prices. This assumption is similar to that made in Morris (1994), where signals are not payoff relevant.

A decision problem is a tuple \((C, \Sigma, P, u, \pi)\) where \( C \) is the action set, \( \Sigma \) is the union of all state spaces, \( P \) is the agent’s possibility correspondence, \( u \) is his utility function and \( \pi \) is a prior on \( S^* \).

Suppose that the agent is partially aware and his subjective state space is \( \Omega \). We will assume that although the agent is unaware of \( S^* \) and of \( \pi \), he nevertheless has correct beliefs about events that he is aware of. If he had “wrong” beliefs, then more information could harm him even in the absence of unawareness. Hence, we require that his prior on \( \Omega \) is the marginal of \( \pi \) on \( \Omega \), denoted by \( \pi|_{\Omega} \). For every \( \omega \in \Omega \), \( \pi|_{\Omega}(\omega) = \sum_{\omega^* \in \omega_{S^*}} \pi(\omega^*) \), where \( \omega_{S^*} \) is the set of full states that project to \( \omega \).

A decision function \( f \) maps each full state \( \omega^* \in S^* \) to a specific action \( c \in C \). The agent chooses his best action by maximising his expected value, given his information.

**Definition 6.** A decision function \( f : S^* \rightarrow C \) is optimal for the decision problem \((C, \Sigma, P, u, \pi)\) if and only if

1. For all \( \omega^*, \omega^*_1 \in S^* \), \( P(\omega^*_1) = P(\omega^*_1) \Rightarrow f(\omega^*_1) = f(\omega^*_1) \).
2. For all \( \omega^* \in S^* \) and \( c \in C \),

\[
\sum_{\omega \in P(\omega^*)} u(f(\omega^*), \omega) \pi|_{\Omega(\omega^*)}(\omega) \geq \sum_{\omega \in P(\omega^*)} u(c, \omega) \pi|_{\Omega(\omega^*)}(\omega).
\]

The first condition states that if two full states describe the same awareness and knowledge, then the agent chooses the same action. The second condition says that once the agent receives his awareness and his information, he updates his beliefs using Bayes’ law and chooses the action that maximises his expected utility given his information. A decision problem is more valuable than another if each decision function guarantees a higher ex ante expected utility.

**Definition 7.** Decision problem \( A = (C, \Sigma, P, u, \pi) \) is more valuable than decision problem \( B = (C', \Sigma, Q, u', \pi') \) if whenever \( g \) is optimal for \( A \) and \( f \) is optimal for \( B \), we have

\[
\sum_{\omega^* \in S^*} u(g(\omega^*), \omega^*) \pi(\omega^*) \geq \sum_{\omega^* \in S^*} u'(f(\omega^*), \omega^*) \pi'(\omega^*).
\]

### 4.1 Results

In the standard framework with partitional information structures, a partition \( P^2 \) is finer than \( P^1 \) if and only if it is more valuable for all \( \pi, u \) and \( C \) (Laffont (1989)). The
following Theorem summarises the sufficient conditions for information to be valuable in an environment with unawareness.

**Theorem 1.** Suppose that $P^2$ is strongly more informed than $P^1$. Decision problem $(C, \Sigma, P^2, u, \pi)$ is more valuable than $(C, \Sigma, P^1, u, \pi)$ if one of the following is true:

1. $(P^2, \pi)$ satisfies conditional independence, or,
2. $(P^1, \pi)$ satisfies conditional independence and awareness for $P^2$ is nested and more informative.

Moreover, whenever one of the properties is violated there exist $u, C, \pi$ where all other properties are satisfied and $(C, \Sigma, P^1, u, \pi)$ is strictly more valuable than $(C, \Sigma, P^2, u, \pi)$.

Recall that agent 2 is strongly more informed if whenever agent 1 knows an event $E$, then if agent 2 is aware of $E$ he knows it as well, or, if he is unaware of it, he knows another event $E'$ that logically implies $E$. Hence, a low level of awareness can be compensated by a high level of knowledge. The first part of the Theorem specifies that if the more informed agent 2 does not misuse his awareness signal, information is valuable. If, on the other hand, the less informed agent 1 does not misuse his signal, then information is still valuable provided 2’s awareness is nested and more informative.

One reason why the converse can fail to be true is that the payoff relevant state space $S_0$ may contain too few states. For example, it may be that the only two states that describe different information and awareness for the two agents project to the same payoff relevant state. In that case, differences in information do not imply differences in payoffs. We say that $S_0$ is non-degenerate for information structure $P$ if whenever two states are distinguishable according to $P$, they can be distinguished also in terms of payoffs.

**Definition 8.** The payoff relevant state space $S_0$ is non-degenerate for $P$ if for all $\omega^* \in S^*$, $\omega, \omega' \in P(\omega^*)$ and $\omega \neq \omega'$ imply $\omega S_0 \neq \omega' S_0$.

Under the assumption of non-degeneracy, the following Proposition shows that a necessary condition for an information structure to be more valuable is to be weakly (but not strongly) more informed.

**Proposition 1.** Suppose that for all $u, \pi, C$, decision problem $(C, \Sigma, P^2, u, \pi)$ is more valuable than $(C, \Sigma, P^1, u, \pi)$. If $S_0$ is non-degenerate for $P^2$ then $P^2$ is weakly more informed.

## 5 Speculation

Speculation is examined in three different settings. Both information processing errors outlined above are relevant here but for different reasons. A low level of awareness is responsible for leading agents to make wrong inferences about others’ information. Hence, existing trading opportunities, if they are beyond one’s awareness, can be left unexplored. Below we provide such an example where although it is common knowledge that there should be no trade, such a trade exists. However, under common priors, common knowledge of trade cannot occur. That is, unaware agents are rational enough to exhaust all trading possibilities that can be described within their common awareness.

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Consider the following example of two agents with constant but different awareness. The agent with the low level awareness makes false inferences about the other agent’s information and actions. There are two questions, concerning the possible prices and interest rates, each with two possible answers, “high” and “low”. State space $S_0 = \{\omega_5, \omega_6\}$ specifies whether prices are high or low, whereas state space $S^* = \{\omega_1, \omega_2, \omega_4\}$ specifies whether prices and interest rates are high or low. Agent 1 is always aware of both dimensions, and he always gets information about interest rates. Moreover, he knows a theorem that states “low interest rates imply high prices”. Hence, when interest rates are low, he knows that prices are high. Agent 1 is always aware of both dimensions and his information is as follows.

$$
\begin{align*}
P_1(\omega_1) &= P_1(\omega_2) = \{\omega_1, \omega_2\}, \\
P_1(\omega_4) &= \{\omega_4\}, \\
P_1(\omega_5) &= P_1(\omega_6) = \{\omega_5, \omega_6\}.
\end{align*}
$$

On the other hand, agent 2 is only aware of prices. Since he is unaware of interest rates and of the theorem, he never knows whether prices are high or low. His state space is $S_0$ and his information is represented by the coarsest partition of $S_0$; for all $\omega \in \Sigma = S^* \cup S_0$, $P_1(\omega) = S_0$.

The fact that agent 2’s awareness is constant implies that he does not underestimate or overestimate any event. However, the low level of his awareness leads to wrong reasoning about agent 1’s information. Since he is unaware of the theorem, he falsely concludes that also agent 1 is always unable to know when prices are high or low. In other words, agent 2 thinks that 1’s information is given by $P_1(\omega_5) = P_1(\omega_6) = S_0$.

The information structure of both agents is depicted in Figure 1.

Agent $i$’s posterior at $\omega \in \Sigma$ about event $E \subseteq \Omega(\omega)$ is given by $t_i(\omega, E)$. A probability distribution on the full state space $S^*$ is a prior for $i$ if it generates his posteriors.

**Definition 9.** A prior for agent $i$ is a probability distribution $\mu \in \Delta S^*$ such that for each $\omega \in \Sigma$, if $\mu(P_i(\omega)_{S^*}) > 0$, then $t_i(\omega, E) = \mu|_{\Omega(\omega)}(E|P_i(\omega))$ for each $E \subseteq \Omega(\omega)$.

Both agents share a common prior $\pi$ on $S^*$, where $\pi(\omega_1) = 1/2, \pi(\omega_2) = 1/4$, and $\pi(\omega_4) = 1/2, \pi(\omega_5) = 1/4, \text{ and } \pi(\omega_6) = 1/4$. Hence, the state $\omega_3$, “low interest rates, low prices”, is impossible.

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$^6$Hence, the state $\omega_3$, “low interest rates, low prices”, is impossible.
\( \pi(\omega_4) = 1/4. \) Since agent 2 is always unaware of \( S^* \), his posteriors are derived from the marginal of \( \pi \) on \( S_0 \).

A bet between the two agents generates, at each state \( \omega \in S_0 \), gains and losses that add up to zero.

**Definition 10.** A bet \( b \) is a collection of functions \( b^i : S_0 \rightarrow \mathbb{R} \), one for each agent \( i \), such that for each \( \omega \in S_0 \), \( \sum_{i \in I} b^i(\omega) = 0 \). Agent \( i \) expects positive gain from bet \( b \) at \( \omega^* \in S^* \) if \( \sum_{\omega \in \Omega^i(\omega^*)} t^i(\omega^*, \omega) b^i(\omega_{S_0}) > 0 \).

Recall that the payoff relevant state space \( S_0 \) is the meet of all state spaces: \( S_0 = \bigwedge S \). In this interactive setting, this implies that it is always common knowledge that everyone is aware of \( S_0 \) and of any bet defined on \( S_0 \).

Note that at \( \omega_5, \omega_6 \) both agents have identical posteriors. Hence, at both \( \omega_5 \) and \( \omega_6 \) there is no bet from which both agents expect positive gains. Moreover, at \( \omega_4 \), the event \( S_0 = \{ \omega_5, \omega_6 \} \) is common knowledge at \( \omega_4 \), which means that it is common knowledge that there is no trade. However, this is not true. The bet \( b^1(\omega_5) = -1.1, b^1(\omega_6) = 0.9, b^2 = -b^1 \), is a bet from which both agents expect positive gains at \( \omega_4 \). But this reasoning is above agent 2’s awareness.

Summarising, the example shows that although there is common knowledge that there is no mutually beneficial bet, such a bet exists. This happens because of agent 2’s faulty reasoning about 1’s information, which stems from agent 2 being less aware than agent 1.

### 5.1 Results

The example above shows that although agents can exhaust trading opportunities that can be described within their common awareness, trading opportunities can still exist. The following result shows that with common priors, there cannot be common knowledge trade. Hence, unaware agents are rational enough to understand that common knowledge of trade exhausts all trading opportunities within their common awareness, just like in the context of the standard model of knowledge with partitional informational structures.

**Theorem 2.** If at \( \omega^* \in S^* \) it is common knowledge that there is a bet \( b \) from which all agents expect positive gains, then there is no common prior.

This is in contrast with models of unawareness (Xiong (2007)) and bounded reasoning (Geanakoplos (1989)) that employ the standard framework, where common knowledge trade is possible. Moreover, if we were to model unawareness of an event by assigning zero probability to that event, then “unaware” agents would engage in common knowledge (belief) trade. Hence, this result provides a behavioural implication which distinguishes the present model from standard state space models of bounded

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\(^7\)Since \( \omega_3 \) is impossible we do not include it in the state space \( S^* \).

\(^8\)Formally, for all \( \omega^* \in S^* \), it is commonly known that everyone is aware of \( S_0 \). For definitions of common knowledge and common knowledge of awareness, see Galanis (2007).

\(^9\)An example of common knowledge trade with common priors but non partitional information structures is provided in the appendix.
reasoning. This result has also been shown by Galanis (2007) (in the form of the agreeing to disagree result) and by Heifetz et al. (2007a).

The next question is whether we can have a trade that is always mutually beneficial for everyone. In the context of the standard model where agents make no mistakes, non existence of such a trade is characterised by the existence of a common prior. Since anything that is always true is always common knowledge, common priors are equivalent to not having a bet for which it is always common knowledge that it is mutually beneficial for everyone.

If we allow for unawareness, then this result is false. First, what is always true is not always common knowledge. The reason is that if one is not fully aware, he may fail to realise that something is always true. Second, as in the value of information problem, agents may overestimate or underestimate events depending on how their awareness varies. The following result shows that if the priors of all agents satisfy conditional independence, then they can trade only if they have different priors. Third, the converse does not hold. Even in the presence of conditional independence, different priors do not imply trade.

**Theorem 3.** Suppose that the priors of all agents satisfy conditional independence and there is a bet \( b \) for which all agents always expect a positive gain. Then, there is no common prior. However, no common prior and conditional independence do not imply the existence of such a bet \( b \).

The third setting is trade in equilibrium. For that, we need to define the notion of a Bayesian Nash equilibrium in an environment with unawareness. Let \( I \) be a finite set of agents, each having a finite set of actions \( C^i \) and let \( C = \times_{i \in I} C^i \) denote the set of all action profiles. In the single agent case we defined a decision function to be a mapping from the full state space \( S^* \) to the set of actions \( C^i \). For the multi agent case we generalise by defining the domain of the decision function to be the union of all state spaces \( \Omega \).

**Definition 11.** A Bayesian game with unawareness is a tuple \((I, C, \Sigma, (P^i)_{i \in I}, (t^i)_{i \in I}, (u^i)_{i \in I})\), where \( C \) is the set of all action profiles, \( \Sigma \) is the union of all state spaces, \( P^i \) denotes agent \( i \)'s possibility correspondence, \( u^i : C \times \Sigma \to \mathbb{R} \) denotes his utility function and \( t^i \) his type mapping.

**Definition 12.** A strategy for player \( i \in I \) is a function \( f^i : \Sigma \to C^i \) such that for all \( \omega, \omega' \in \Sigma \), \( t^i(\omega, \cdot) = t^i(\omega', \cdot) \) implies \( f^i(\omega) = f^i(\omega') \).

**Definition 13.** Strategies \((f^i)_{i \in I}\) constitute a Bayesian Nash equilibrium if for all \( \omega \in \Sigma \), all \( i \in I \) and all \( c^i \in C^i \),

\[
\sum_{\omega' \in \Omega^i(\omega)} u^i(f^i(\omega'), f^{-i}(\omega'), \omega')t^i(\omega, \omega') \geq \sum_{\omega' \in \Omega^i(\omega)} u^i(c^i, f^{-i}(\omega'), \omega')t^i(\omega, \omega').
\]

In a Bayesian game with unawareness agents have to reason about the actions of others, before choosing their own best response. If the level of their awareness is too low, then they may reason incorrectly about the information and actions of others. As a result, they will choose suboptimally and agree to trade in equilibrium. The following condition specifies that an agent’s level of awareness does not influence his perception of his payoffs, and as a result does not lead to suboptimal play in equilibrium.
Definition 14 (Projections Preserve Own Payoffs). A strategy profile \( \{f^i\}_{i \in I} \) satisfies \( \text{PPOP} \) if for all \( i \in I \), for all \( \omega \in \Sigma \), \( u^i(c,f^{-i}(\omega),\omega) = u^i(c,f^{-i}(\omega_{S_0}),\omega_{S_0}) \), for all \( c \in C \).

When all equilibria in a game \( G \) satisfy \( \text{PPOP} \) we say that \( G \) satisfies \( \text{PPOP} \). Agent \( i \)'s ex ante expected utility from strategy profile \( \{f^j\}_{j \in I} \) is

\[
U^i(\{f^j\}_{j \in I}) = \sum_{\omega^* \in S^*} \pi^i(\omega^*) u^i(f(\omega^*),\omega^*).
\]

Suppose that the agents have arrived at an ex ante Pareto optimal allocation. This means that each agent can stick to his allocation and can guarantee for himself an ex ante payoff of \( \tilde{u}^i \), irrespective of what everyone else is doing. Moreover, there is no strategy profile that ex ante can make everyone weakly better off and at least one strictly better off. Suppose that the agents receive differential information and awareness. Will they be willing to trade? As the following theorem shows, the answer is negative, as long as low level of awareness and misuse of the awareness signal does not lead to information processing errors.

Theorem 4. Let \( G \) be a Bayesian game with unawareness that satisfies \( \text{PPOP} \) and suppose that each agent satisfies either nested awareness or conditional independence. Suppose that each agent \( i \) has an action \( z^i \in C^i \) such that for all \( \{f^j\}_{j \in I} \), \( U^i([z^i],\{f^j\}_{j \in I-1}) = \tilde{u}^i \). Moreover, suppose that for all \( \{f^j\}_{j \in I} \), if \( U^i(\{f^j\}_{j \in I}) \geq \tilde{u}^i \) for all \( i \in I \), then \( f^i(\omega^*) = [z^i] \) for all \( \omega^* \in S^* \), for all \( j \in I \). Then, \( G \) has a unique equilibrium such that \( f^i(\omega^*) = [z^i] \) for all \( \omega^* \in S^* \).

A Appendix

Proof of Lemma 1. (1) \( \implies \) (5) and (6). By definition, for any \( \omega^* \in S^* \), \( \omega^* \in K^{\omega^*}_S(P^1(\omega^*)) \). From (1) we have \( \omega^* \in K^{\omega^*}_S(P^1(\omega^*)) \), which implies \( \Omega(\omega^*) \subseteq \Omega^1(\omega^*) \) and \( (P^2(\omega^*))_S \subseteq (P^1(\omega^*))_S \). For the reverse direction, suppose that \( \omega^* \in K^{\omega^*}_S(E) \). Then, \( (P^1(\omega^*))_S \subseteq E_{S^*} \) and \( E \leq \Omega(\omega^*) \). From (5) and (6) we immediately have that \( \omega^* \in K^{\omega^*}_S(E) \).

That (1) implies (2) and (6) is immediate. For the reverse direction, (6) implies that for all events \( E, K^S_{\omega^*}(E) \subseteq A^S_{\omega^*}(E) \), so (1) follows from (2).

(4) \( \implies \) (2). Suppose \( \omega^* \in A^S_{\omega^*}(E) \cap K^S_{\omega^*}(E) \). Then, \( P^1(\omega^*) \subseteq E_{\Omega(\omega^*)} \), which implies \( P^1(\omega^*)_{\Omega(E)} \subseteq E \). Since \( E \leq S_{\omega^*} \leq \Omega^1(\omega^*) \), we have \( P^1(\omega^*)_{S_{\omega^*}} \subseteq (P^1(\omega^*))_{S_{\omega^*}} \subseteq E_{S_{\omega^*}} \). This implies \( P^2(\omega^*)_{S_{\omega^*}} \subseteq E_{S_{\omega^*}} \). Since \( S_{\omega^*} \leq \Omega^1(\omega^*) \), we have \( P^2(\omega^*) \subseteq (P^2(\omega^*))_{S_{\omega^*}} \subseteq E_{S_{\omega^*}} \). Hence, \( \omega^* \in K^S_{\omega^*}(E) \).

(2) \( \implies \) (4). Suppose not. Then, there exists \( \omega^* \) such that \( P^2(\omega^*)_{S_{\omega^*}} \not\subseteq P^1(\omega^*)_{S_{\omega^*}} \). Since both agents are aware of \( S_{\omega^*} \), we have that \( P^1(\omega^*) \subseteq (P^1(\omega^*))_{S_{\omega^*}} \) and \( P^2(\omega^*) \subseteq (P^1(\omega^*))_{S_{\omega^*}} \). Hence, \( \omega^* \in A^S_{\omega^*}(P^1(\omega^*))_{S_{\omega^*}} \cap K^S_{\omega^*}(P^1(\omega^*))_{S_{\omega^*}} \), a contradiction.

(5) \( \implies \) (2) and (3). Suppose \( \omega^* \in K^S_{\omega^*}(E) \cap A^S_{\omega^*}(E) \). Then, we have that \( P^1(\omega^*)_{S_{\omega^*}} \subseteq E_{S_{\omega^*}}, E \leq \Omega(\omega^*) \) and \( E \leq \Omega^1(\omega^*) \). From (5) we also have \( P^2(\omega^*)_{S_{\omega^*}} \subseteq E_{S_{\omega^*}} \), which together with \( E \leq \Omega^1(\omega^*) \) imply \( P^2(\omega^*) \subseteq E_{\Omega^1(\omega^*)} \) and therefore \( \omega^* \in K^S_{\omega^*}(E) \).

Suppose now that \( \omega^* \in K^S_{\omega^*}(E) \cap A^S_{\omega^*}(E) \), which implies that \( P^1(\omega^*)_{S_{\omega^*}} \subseteq E_{S_{\omega^*}} \). Let \( E' = P^2(\omega^*) \). Then, from (5) we have \( E_{S_{\omega^*}} \subseteq E_{S_{\omega^*}} \) and \( \omega^* \in K^S_{\omega^*}(E') \).
For the reverse direction, fix \( \omega^* \in S^* \). If \( \Omega^1(\omega^*) \leq \Omega^2(\omega^*) \) then we have \( \omega^* \in K_{12}^* \cap A_{2s}^* \). From (2) we have \( \omega^* \in K_{12}^* \cap A_{2s}^* \), which implies \( P^2(\omega^*) \subseteq P^1(\omega^*) \). If \( \Omega^1(\omega^*) \not\preceq \Omega^2(\omega^*) \) then \( \omega^* \not\in K_{12}^* \cap A_{2s}^* \). Then, there exists \( E' \) such that \( \omega^* \in K_{12}^* \cap E' \) and \( E' \subseteq P^1(\omega^*) \). But \( \omega^* \in K_{12}^* \cap E' \) implies \( P^2(\omega^*) \subseteq P^1(\omega^*) \). So we have \( P^2(\omega^*) \subseteq P^1(\omega^*) \).

\[ \square \]

**Proof of first part of Theorem 1.** Suppose \( f \) is optimal for \( (C, \Sigma, P^1, u, \pi) \) and \( g \) is optimal for \( (C, \Sigma, P^2, u, \pi) \). Fix \( \omega^* \in S^* \) with \( \pi(\omega^*) > 0 \). For all \( c \in C \), we have

\[
\sum_{\omega \in P^2(\omega^*)} u(g(\omega^*), \omega) \pi(\omega) \geq \sum_{\omega \in P^2(\omega^*)} u(c, \omega) \pi(\omega).
\]

Conditional independence and generalized reflexivity imply that for \( \omega, \omega' \in P^2(\omega^*) \), we have

\[
\frac{\pi(\omega^* \cap \omega_s^* \cap \omega_{s^*}^*)}{\pi(\omega^*)} = \frac{\pi(\omega^* \cap \omega_{s^*}^*)}{\pi(\omega^*)} > 0.
\]

Multiplying by that number we have that for all \( c \in C \),

\[
\sum_{\omega \in P^2(\omega^*)} u(g(\omega^*), \omega) \pi(\omega^* \cap \omega_s^* \cap \omega_{s^*}^*) \geq \sum_{\omega \in P^2(\omega^*)} u(c, \omega) \pi(\omega^* \cap \omega_s^* \cap \omega_{s^*}^*) \quad \Rightarrow \quad \sum_{\omega \in P^2(\omega^*)} u(g(\omega^*), \omega) \sum_{\omega \in P^2(\omega^*)} \pi(\omega^* \cap \omega_s^* \cap \omega_{s^*}^*) \geq \sum_{\omega \in P^2(\omega^*)} u(c, \omega) \sum_{\omega \in P^2(\omega^*)} \pi(\omega^* \cap \omega_s^* \cap \omega_{s^*}^*).
\]

Since \( \{\omega_1^*\}_s = \omega S_0 \) for all \( \omega_1^* \in \omega S_0 \), we have

\[
\sum_{\omega \in P^2(\omega^*)} u(g(\omega^*), \omega) \sum_{\omega \in P^2(\omega^*)} \pi(\omega^* \cap \omega_s^* \cap \omega_{s^*}^*) \geq \sum_{\omega \in P^2(\omega^*)} u(c, \omega) \sum_{\omega \in P^2(\omega^*)} \pi(\omega^* \cap \omega_s^* \cap \omega_{s^*}^*).
\]

Hence, conditional independence implies that for any \( \omega^* \in S^* \), the agent’s best action at \( P^2(\omega^*) \) is also the best action at \( (P^2(\omega^*)) \cap E^2(\omega^*) \). Next, we show that the full state space \( S^* \) is partitioned by

\[
\{P^2(\omega^*) \cap E^2(\omega^*)\}_{\omega^* \in S^*}.
\]

Suppose \( \omega_1^* \in (P^2(\omega^*)) \) \( \cap \) \( E^2(\omega^*) \). Then, \( \Omega^2(\omega^*) = \Omega^2(\omega_1^*) \) and \( \{\omega_1^*\} \cap (\omega^*) \) is \( P^2(\omega^*) \). Generalized Reflexivity implies \( \{\omega_1^*\} \cap (\omega^*) \) is \( P^2(\omega_1^*) \) and Stationarity implies \( P^2(\omega_1^*) = P^2(\omega_1^*) \cap (\omega^*) \). Hence, \( (P^2(\omega_1^*)) \cap (\omega^*) \cap (\omega^*) \). Next, we show that the full state space \( S^* \) is partitioned by

\[
\{P^2(\omega^*) \cap E^2(\omega^*)\}_{\omega^* \in S^*}.
\]

Suppose \( \omega_1^* \in (P^2(\omega^*)) \) \( \cap \) \( E^2(\omega^*) \). Then, \( \Omega^2(\omega^*) = \Omega^2(\omega_1^*) \) and \( \{\omega_1^*\} \cap (\omega^*) \) is \( P^2(\omega_1^*) \). Generalized Reflexivity and Stationarity imply \( P^2(\omega_1^*) = P^2(\omega_1^*) \cap (\omega^*) \). Similarly for \( \omega_2^* \), we have \( P^2(\omega_2^*) = P^2(\omega_2^*) \). Hence, \( (\omega_1^*) \cap (\omega^*) \cap (\omega^*) \). The same argument shows that \( \omega_1^*, \omega_2^* \) \( \in \) \( (\omega^*) \) \( \cap \) \( (\omega^*) \) \( \cap \) \( (\omega^*) \). We showed that for each element of the partition \( \{P^2(\omega^*) \cap E^2(\omega^*)\}_{\omega^* \in S^*} \), agent 2 picks an action that maximises his expected utility. Moreover, agent 1 picks
one action for each element of the partition, \(\{(P^1(\omega^*))_{S^*} \cap E^1(\omega^*)\}_{\omega^* \in S^*}\). This action may not necessarily be optimal.

Finally, we need to show that the former partition is finer than the latter if and only if agent 2 is strongly more informed. One direction is obvious, so for the other direction suppose \(\omega_1^* \in (P^2(\omega^*))_{S^*} \cap E^2(\omega^*)\). Then, \(\omega_1^* \in (P^2(\omega^*))_{S^*}\) and Stationarity, together with PPI, implies \(\Omega^1(\omega_1^*) \supseteq \Omega^1(\omega^*)\). Suppose \(\Omega^1(\omega_1^*) \supsetneq \Omega^1(\omega^*)\). Then, \(\omega^* \notin (P^2(\omega_1^*))_{S^*}\), which implies \(\omega^* \notin (P^2(\omega_1^*))_{S^*}\). But this contradicts the fact that \(\{(P^2(\omega^*))_{S^*} \cap E^2(\omega^*)\}_{\omega^* \in S^*}\) is a partition. Hence, \(\Omega^1(\omega_1^*) = \Omega^1(\omega^*)\) and \(\omega_1^* \in (P^1(\omega^*))_{S^*} \cap E^1(\omega^*)\). Note that if the agent is always more aware and weakly more informed, then he is strongly more informed.

The main example shows that if conditional independence is violated but the agent is strongly more informed, then he may be worse off. In Example 1, the agent satisfies conditional independence, he is weakly (but not strongly) more informed and he is strictly worse off.

The proof of the second part of Theorem 1 is given in three steps. First, using \(P^2\) we define a possibility correspondence \(P : S^* \rightarrow 2^{S^*}\) and show that it satisfies non-delusion, KTYK and nested, three properties which are discussed in Geanakoplos (1989). Then, we show that if \(g\) is optimal for \(P^2\) then it is also optimal for \(P\) in a suitably defined problem in the Geanakoplos’ setting. Finally, we apply Theorem 1 in Geanakoplos (1989).

Define the mapping \(P : S^* \rightarrow 2^{S^*}\) such that for all \(\omega^* \in S^*, P(\omega^*) = (P^2(\omega^*))_{S^*}\). We show that \(P\) also satisfies the following three properties, discussed in Geanakoplos (1989):

**Definition 15.**

- **Non-Delusion** For all \(\omega^* \in S^*, \omega^* \in P(\omega^*)\).
- **KTYK** \(\omega_1^* \in P(\omega^*) \implies P(\omega_1^*) \subseteq P(\omega^*)\).
- **Nested** For any \(\omega^*, \omega_1^* \in S^*, \text{ either } P(\omega^*) \cap P(\omega_1^*) = \emptyset, \text{ or } P(\omega^*) \subseteq P(\omega_1^*) \text{ or } P(\omega_1^*) \subseteq P(\omega^*)\).

**Lemma 2.** Suppose that \(P^2\) satisfies nested awareness. Then, the possibility correspondence \(P\) satisfies non-delusion, KTYK and nested.

**Proof.** \(P\) satisfies non-delusion because \(P^2\) satisfies Generalized Reflexivity. For KTYK, suppose \(\omega_1^* \in P(\omega^*) = (P^2(\omega^*))_{S^*}\). Then, \(\{\omega_1^*\}_{\Omega^2(\omega^*)} \in P^2(\omega^*)\). Stationarity implies \(P^2(\{\omega_1^*\}_{\Omega^2(\omega^*)}) = P^2(\omega^*)\). PPI implies \((P^2(\omega_1^*))_{S^*} \subseteq (P^2(\{\omega_1^*\}_{\Omega^2(\omega^*)}))_{S^*} = (P^2(\omega^*))_{S^*}\). Hence \(P(\omega_1^*) \subseteq P(\omega^*)\).

For nested, suppose \(P(\omega^*) \cap P(\omega_1^*) \neq \emptyset\). Take \(\omega_2^* \in P(\omega^*) \cap P(\omega_1^*)\). Then \(\omega_2^* \in (P^2(\omega^*))_{S^*} \cap (P^2(\omega_1^*))_{S^*}\). As in the previous paragraph, this implies that \(P^2(\{\omega_2^*\}_{\Omega^2(\omega^*)}) = P^2(\omega_1^*) \text{ and } P^2(\{\omega_2^*\}_{\Omega^2(\omega^*)}) = P^2(\omega^*)\). Without loss of generality, suppose that \(\Omega^2(\omega^*) \supsetneq \Omega^2(\omega_1^*)\). PPI implies that \((P^2(\{\omega_2^*\}_{\Omega^2(\omega^*)}))_{S^*} \subseteq (P^2(\{\omega_2^*\}_{\Omega^2(\omega^*)}))_{S^*}\) and hence, \(P(\omega_1^*) \subseteq P(\omega^*)\).

The setting in Geanakoplos (1989) specifies a finite state space \(S^*\), a possibility correspondence \(P : S^* \rightarrow 2^{S^*}\) and an action set \(C\). Let \(u : C \times S^* \rightarrow \mathbb{R}\) and let \(\pi\) be a measure on \(S^*\).
Definition 16. In the Geanakoplos’ setting, a decision function \( f : S^* \to C \) is optimal for the decision problem \( (C, S^*, P, u, \pi) \) iff

- \( P(\omega^*) = P(\omega_1^*) \implies f(\omega^*) = f(\omega_1^*) \)
- For all \( \omega^* \in S^* \) and \( c \in C \),
  \[
  \sum_{\omega_1^* \in P(\omega^*)} u(f(\omega^*), \omega_1^*)\pi(\omega_1^*) \geq \sum_{\omega_1^* \in P(\omega^*)} u(c, \omega_1^*)\pi(\omega_1^*)
  \]

Given a decision problem \( (C, \Sigma, P^2, u, \pi) \) we define the decision problem \( (C, S^*, P, u', \pi) \) in the setting of Geanakoplos (1989). The possibility correspondence \( R \) be a partition of \( \omega \) such that for all \( \omega^* \in S^* \), \( P(\omega^*) = (P^2(\omega^*))_{S^*} \). The utility function \( u' : C \times S^* \to \mathbb{R} \) is such that \( u'(c, \omega^*) = u(c, \omega^*_{S_0}) \) for all \( \omega^* \in S^* \), for all \( c \in C \).

Lemma 3. Decision function \( g \) is optimal for \( (C, \Sigma, P^2, u, \pi) \) if and only if \( g \) is optimal for \( (C, S^*, P, u', \pi) \).

Proof. First we show that for any \( \omega^*, \omega_1^* \in S^*, P^2(\omega^*) = P^2(\omega_1^*) \iff (P^2(\omega^*))_{S^*} = (P^2(\omega_1^*))_{S^*} \). One direction is obvious so for the other suppose that \( (P^2(\omega^*))_{S^*} = (P^2(\omega_1^*))_{S^*} \). Since \( \omega^* \in (P^2(\omega_1^*))_{S^*} \) we have \( \omega^*_{\Omega_2(\omega_1^*)} \in P^2(\omega_1^*) \), which implies \( P^2(\omega^*_{\Omega_2(\omega_1^*)}) = P^2(\omega_1^*) \). PPI implies \( P^2(\omega^*) \subseteq P^2(\omega^*_{\Omega_2(\omega_1^*)}) = P^2(\omega_1^*) \). Similarly, since \( \omega_1^* \in (P^2(\omega^*))_{S^*} \), we have \( P^2(\omega_1^*) \subseteq P^2(\omega^*) \), which implies \( P^2(\omega_1^*) = P^2(\omega^*) \).

If \( g \) is optimal for the first problem, then for any \( \omega^* \in S^* \) and any \( c \in C \),
  \[
  \sum_{\omega \in P^2(\omega^*)} u(g(\omega^*), \omega_1^*)\pi(\omega_1^*) \geq \sum_{\omega \in P^2(\omega^*)} u(c, \omega_1^*)\pi(\omega_1^*)
  \]

Fix \( \omega^* \in S^* \) and take \( \omega \in P^2(\omega^*) \). For any \( \omega_1^* \in \omega_{S^*} \), we have that \( u'(g(\omega^*), \omega_1^*) = u(g(\omega^*), \omega_1^*) = u(g(\omega^*), \omega_{S_0}) \). We also have that \( \pi(\omega_{S^*}) = \sum_{\omega_1^* \in \omega_{S^*}} \pi(\omega_1^*) \). Combining these two we have
  \[
  \sum_{\omega \in P^2(\omega^*)} \sum_{\omega_1^* \in \omega_{S^*}} u'(g(\omega^*), \omega_1^*)\pi(\omega_1^*) \geq \sum_{\omega \in P^2(\omega^*)} \sum_{\omega_1^* \in \omega_{S^*}} u'(c, \omega_1^*)\pi(\omega_1^*)
  \]

which is equivalent to
  \[
  \sum_{\omega_1^* \in (P^2(\omega^*))_{S^*}} u'(g(\omega^*), \omega_1^*)\pi(\omega_1^*) \geq \sum_{\omega_1^* \in (P^2(\omega^*))_{S^*}} u'(c, \omega_1^*)\pi(\omega_1^*)
  \]

Since \( P(\omega^*) = (P^2(\omega^*))_{S^*} \), \( g \) is optimal for the second problem in the Geanakoplos’ setting. The other direction is similar.

The following theorem is proved in Geanakoplos (1989).

Theorem 5 (Geanakoplos (1989)). Let \( P \) satisfy non-delusion, nested and KTYK. Let \( R \) be a partition of \( S^* \) that is a coarsening of \( P \). Let \( g, f \) be optimal for \( (C, S^*, P, u', \pi) \) and \( (C, S^*, R, u', \pi) \) respectively. Then,
  \[
  \sum_{\omega^* \in S^*} u'(g(\omega^*), \omega^*)\pi(\omega^*) \geq \sum_{\omega^* \in S^*} u'(f(\omega^*), \omega^*)\pi(\omega^*)
  \]
Proof of the second part of Theorem 1. Suppose \( f \) is optimal for \((C, \Sigma, P^1, u, \pi)\) and \( g \) is optimal for \((C, \Sigma, P^2, u, \pi)\). Define \( Q : S^* \to 2^{S^*} \) such that \( Q(\omega^*) = P^1(\omega^*)_S \cap \mathcal{E}^1(\omega^*) \). Since \( P^1 \) satisfies conditional independence, we know from the proof of the first part of Theorem 1 that \( Q \) partitions the full state space and that \( f \) is optimal for \((C, S^*, Q, u', \pi)\). Define \( P : S^* \to 2^{S^*} \) such that for all \( \omega^* \in S^* \), \( P(\omega^*) = (P^2(\omega^*))_S \). By Lemma 2, \( P \) satisfies non-delusion, nested and KTYK. Since \( \mathcal{E}^2(\omega^*) \subseteq \mathcal{E}^1(\omega^*) \) for all \( \omega^* \in S^* \) and \( P^2 \) is strongly more informed than \( P^1 \), we have that \( P \) is finer than \( Q \). By applying Lemma 3, \( g \) is optimal for \((C, S^*, P, u', \pi)\). By Theorem 5, \( g \) attains a higher ex ante expected utility than what \( f \) attains.

The main example shows that if the less informed agent does not satisfy conditional independence while the awareness of the strongly more informed agent is nested and higher ex ante expected utility than what \( \omega \) attains.

Proof of Proposition 1. Suppose that \( P^2 \) is not weakly more informed than \( P^1 \). Then, there exist \( \omega^*, \omega_1 \) such that \( \omega_1 \in P^2(\omega^*)_S, \omega_1 \notin P^1(\omega^*)_S \), where \( S = \Omega^1(\omega^*) \cap \Omega^2(\omega^*) \). Let \( \omega_2 \in P^2(\omega^*) \) be such that \( \omega_2 := \omega_1 \). By Generalized Reflexivity and since \( \omega_1 \notin P^1(\omega^*)_S \), we have that \( \omega_1 \notin \omega_2^* \), which implies that \( \omega_2 \notin \omega_2^* \). By non-degeneracy of \( P^2 \), \( \omega_2 \notin \omega_2^* \). By Generalized Reflexivity, \( \omega_2^* \in P^2(\omega^*)_S, P^1(\omega^*)_S \). Let \( C = \{c_1, c_2\} \) and consider the following payoffs: \( u(c_1, \omega_2^*) = 1, u(c_1, \omega_2^*) = 1.1, u(c_2, \omega_2^*) = 8 \), \( u(c_1, \omega_2^*) = 8 \).

Let \( \pi(\omega^*) = 1/2 \) and \( \pi(\omega_2^*) = 1/2 \), where \( \omega_2^* \cap \omega^* \). At \( \omega^* \), 2’s optimal action is \( c_1 \), while 1’s optimal action is \( c_2 \). At \( \omega_2^* \), from Generalized Reflexivity, both agents assign probability at least \( 1/2 \) to state \( \omega_2^* \) and their optimal action is \( c_1 \). Hence, the decision problem \((C, \Sigma, P^1, u, \pi)\) is more valuable than \((C, \Sigma, P^2, u, \pi)\), a contradiction.

Lemma 4. Suppose that the agent’s posteriors are generated by a prior which satisfies conditional independence. Then, the set of the agent’s priors which satisfy conditional independence is a convex set.

Proof. Let \( \mu \) and \( \mu' \) be priors which satisfy conditional independence. For \( \alpha \in (0, 1) \), let \( \mu'' = \alpha \mu' + (1 - \alpha) \mu \) and take \( \omega^* \in S^* \) such that \( \mu''(P(\omega^*)_S) > 0 \). It is straightforward to show that \( \mu'' \) is also a prior for the agent, so we only need to show that it satisfies conditional independence. Suppose that either \( \mu(P(\omega^*)_S) = 0 \) or \( \mu'(P(\omega^*)_S) = 0 \) and let, without loss of generality, \( \mu'(P(\omega^*)_S) = 0 \). Since \( \mu \) satisfies conditional independence we have that for each \( \omega \in P(\omega^*) \),

\[
\frac{\mu(\mathcal{E}(\omega^*) \cap \omega_S)}{\mu(\mathcal{E}(\omega^*) \cap P(\omega^*)_S)} = \frac{\mu(\omega_S)}{\mu(P(\omega^*)_S)}.
\]

Multiplying by \( \alpha \), this implies that,

\[
\mu''(\mathcal{E}(\omega^*) \cap \omega_S | P(\omega^*)_S) = \mu''(\mathcal{E}(\omega^*) | P(\omega^*)_S) \mu''(\omega_S | P(\omega^*)_S).
\]

Suppose now that \( \mu(P(\omega^*)_S), \mu'(P(\omega^*)_S) > 0 \). Then, we have that,

\[
\alpha \mu(P(\omega^*)_S) t(\omega^*, \omega) = \alpha \mu(\omega_S),
\]

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\[ \alpha \mu(\mathcal{E}(\omega^*) \cap \omega_{S^*}) t(\omega^*, \omega) = \alpha \mu(\mathcal{E}(\omega^*) \cap P(\omega^*)_{S^*}), \]
\[ (1 - \alpha) \mu'(P(\omega^*)_{S^*}) t(\omega^*, \omega) = (1 - \alpha) \mu'(\omega_{S^*}), \]
\[ (1 - \alpha) \mu'(\mathcal{E}(\omega^*) \cap \omega_{S^*}) t(\omega^*, \omega) = (1 - \alpha) \mu'(\mathcal{E}(\omega^*) \cap P(\omega^*)_{S^*}). \]
Adding, we have the result.

**Proof of Theorem 2.** Suppose there is a common prior \( \pi \), there are bets \( b^i : S_0 \to \mathbb{R}, \ i \in I \), and an event \( E^* \) such that \( S_0 \subseteq E^* \), and for each \( \omega \in E^* \), all agents expect positive gains from their respective bets. Suppose that at \( \omega^* \in S^* \), event \( E^* \) is common knowledge and \( E^* \subseteq \Omega^\wedge(\omega^*) \), which is the common state space at \( \omega^* \). Since we have assumed that \( S^* \) is finite, Theorem 3 in Galanis (2007) states that there is a public event \( E \subseteq \Omega^\wedge(\omega^*) \) such that \( \omega_{\Omega^\wedge(\omega^*)}^* \subseteq E \subseteq E^* \). The proof of Theorem 4 in Galanis (2007) shows that \( E \) is partitioned by \( P^i \), for each \( i \). By adding up we have that for each \( i \), \( \sum_{\omega \in E} \pi|_{\Omega(\omega)} b^i(\omega_{S_0}) > 0 \). By adding over all agents we have \( \sum_{\omega \in E} \sum_{i \in I} \pi|_{\Omega(\omega)} b^i(\omega_{S_0}) > 0 \). Since \( \sum_{i \in I} b^i(\omega_{S_0}) = 0 \) for all \( \omega \in E \), we have a contradiction.

**Proof of Theorem 3.** Suppose there is a common prior \( \pi \). Fix agent \( i \) and a state \( \omega^* \in S^* \) such that \( \pi(\omega^*) > 0 \). Then, \( \pi(P^i(\omega^*)) > 0 \) and \( \sum_{\omega \in P^i(\omega^*)} b^i(\omega_{S_0}) \pi|_{\Omega(\omega)}(\omega) > 0 \).

Conditional independence and generalized reflexivity imply that for \( \omega, \omega' \in P^i(\omega^*) \) such that \( \pi|_{\Omega(\omega)}(\omega), \pi|_{\Omega(\omega')}(\omega') > 0 \), we have
\[
\frac{\pi(\mathcal{E}^i(\omega^*) \cap \omega_{S^*})}{\pi|_{\Omega(\omega')}(\omega')} = \frac{\pi(\mathcal{E}^i(\omega^*) \cap \omega_{S^*})}{\pi|_{\Omega(\omega)}(\omega)} > 0.
\]

Multiplying by that number we have that,
\[
\sum_{\omega \in P^i(\omega^*)} b^i(\omega_{S_0}) \pi(\mathcal{E}^i(\omega^*) \cap \omega_{S^*}) > 0 \iff \\
\sum_{\omega \in P^i(\omega^*)} b^i(\omega_{S_0}) \sum_{\omega^*_1 \in \mathcal{E}^i(\omega^*) \cap \omega_{S^*}} \pi(\omega^*_1) > 0.
\]

Since \( \{\omega^*_1\}_{S_0} = \omega_{S_0} \) for all \( \omega^*_1 \in \omega_{S^*} \), we have
\[
\sum_{\omega \in P^i(\omega^*)} \sum_{\omega^*_1 \in \mathcal{E}^i(\omega^*) \cap \omega_{S^*}} b^i(\{\omega^*_1\}_{S_0}) \pi(\omega^*_1) > 0 \iff \\
\sum_{\omega^*_1 \in \mathcal{E}^i(\omega^*) \cap \omega_{S^*}} b^i(\{\omega^*_1\}_{S_0}) \pi(\omega^*_1) > 0.
\]

From the proof of the first part of Theorem 1 we know that \( \{(P^i(\omega^*))_{S^*} \cap \mathcal{E}^i(\omega^*)\}_{\omega^* \in S^*} \) generates a partition of \( S^* \). By adding all elements of the partition we have that
\[
\sum_{\omega^* \in S^*} b^i(\{\omega^*\}_{S_0}) \pi(\omega^*) > 0.
\]

By adding all agents and since \( \sum_{i \in I} b^i(\omega) = 0 \) for each \( \omega \in S_0 \), we derive the contradiction.

\[\text{\footnotesize ¹For definition of the common state space } \Omega^\wedge(\omega^*), \text{ look at Galanis (2007)}\]
Proof of Theorem 4. The proof is similar to the proof of Theorem 3 in Geanakoplos (1989). Let \( \{f^i\}_{j \in I} \) be an equilibrium and look at \( i \)'s one agent decision problem that is induced when the strategy of each \( j \neq i \) is fixed. Because of PPOP, assumption 1 is satisfied. If agent \( i \) was fully aware but had no information at all, his optimal action would be \( z^i \) and his ex ante payoff would be \( \bar{v}^i \). Since agent \( i \) is strongly more informed, his awareness is more informative, and satisfies either nested awareness or conditional independence, by Theorem 1, his ex ante payoff is weakly higher than \( \bar{v}^i \). Since this is true for all agents, by hypothesis, \( f^i(\omega^*) = z^i \), for all \( \omega^* \in S^* \), for all \( i \in I \).

\[ \square \]

B Counter examples

Example 1. Agent 1 satisfies conditional independence, he is weakly (but not strongly) more informed and he is strictly worse off. There are two basic questions, \( q \) and \( r \). Agent 1 is only aware of the first question, while agent 2 is always fully aware. Since both agents have constant awareness, they satisfy conditional independence and nested awareness. Agent 2 always learns the answer to the question \( r \), while both never learn the answer to question \( q \). There are two state spaces, \( S^* = \{\omega_1^*, \omega_2^*, \omega_3^*, \omega_4^*\} \),

\[
\begin{align*}
\omega_1^* &= (q_y, r_n), \quad \pi(\omega_1^*) = 3/8, \\
\omega_2^* &= (q_n, r_y), \quad \pi(\omega_2^*) = 3/8, \\
\omega_3^* &= (q_y, r_y), \quad \pi(\omega_3^*) = 1/8, \\
\omega_4^* &= (q_n, r_n), \quad \pi(\omega_4^*) = 1/8,
\end{align*}
\]

and \( S_0 = \{\omega_1, \omega_2\} \), where \( \omega_1 = (q_y) \) and \( \omega_2 = (q_n) \). There are two actions, B and NB. We have \( u(\omega_1, NB) = 1, u(\omega_1, B) = 1, u(\omega_2, NB) = 1, u(\omega_2, B) = -1 \). Agent 1 has the trivial partition, so for all \( \omega \in S^* \cup S_0 \), \( P^1(\omega) = \{\omega_1, \omega_2\} \). Agent 2’s possibility correspondence is as follows:

\[
\begin{align*}
P^2(\omega_1^*) &= P^2(\omega_4^*) = \{\omega_1^*, \omega_4^*\}, \\
P^2(\omega_2^*) &= P^2(\omega_3^*) = \{\omega_2^*, \omega_3^*\}, \\
P^2(\omega_1) &= P^2(\omega_2) = \{\omega_1, \omega_2\}.
\end{align*}
\]

Agent 1 is indifferent between both actions and his expected utility is 0. Agent 2 chooses action B at \( \omega_1^*, \omega_4^* \) and action NB at \( \omega_2^*, \omega_3^* \). His expected utility is 1/2.

Example 2. We present an example with two agents whose priors satisfy conditional independence, they have no common prior and there is no trade that ensures positive expected gains at each full state, for both. There are two state spaces, \( S^* = \{\omega_1^*, \omega_2^*, \omega_3^*, \omega_4^*\} \) and \( S_0 = \{\omega_1, \omega_2\} \), such that \( S_0 \leq S^*, \omega_1^*_{S_0} = \omega_2^*_{S_0} = \omega_1 \) and \( \omega_3^*_{S_0} = \omega_4^*_{S_0} = \omega_2 \). Agent 1 is always fully aware and \( P^1(\omega^*) = S^* \) for all \( \omega^* \in S^* \). His prior \( \pi^1 \) on \( S^* \) is \( 1/8, 1/2, 2/8, 1/8 \). In fact, this is the only prior that can generate his posteriors. Agent 2’s possibility correspondence is as follows: \( P^2(\omega_1^*) = P^2(\omega_4^*) = S_0, \)

\[
\begin{align*}
P^2(\omega_2^*) &= \{\omega_1^*, \omega_3^*\}, \\
P^2(\omega_3^*) &= \{\omega_2^*, \omega_4^*\}.
\end{align*}
\]

His prior assigns 1/4 to each \( \omega^* \in S^* \). Since \( \pi^1 \) cannot generate 2’s posteriors, the agents have no common priors. Moreover, the agents’ priors satisfy conditional independence. Suppose there is a trade \( b^i : S_0 \to \mathbb{R}, i = 1, 2, \)}
such that \( \sum_{\omega \in \Omega^1(\omega^*)} t^i(\omega^*, \omega)b^1(\omega_{S_0}) > 0 \), for each \( \omega^* \in S^* \), for each \( i \). For agent 2 this means that \( b^2(\omega_1), b^2(\omega_2) > 0 \). But since \( \sum_{\omega \in \Omega} b^i(\omega) = 0 \) for each \( \omega \in S_0 \), we have \( b^1(\omega_1), b^1(\omega_2) < 0 \), which implies \( \sum_{\omega \in \Omega^1(\omega_1^i)} t^i(\omega^*, \omega)b^1(\omega_{S_0}) < 0 \).

**Example 3.** This is an example of common knowledge trade and common priors, within the standard model and with non-partitional information structures. There are three states \{\( \omega_1, \omega_2, \omega_3 \)\} and the prior is such that \( \pi(\omega_1), \pi(\omega_3) = 1/4 \). \( \pi(\omega_2) = 1/2 \). There are two agents. Agent 1 has the trivial partition \( P^1(\omega) = \Omega \) for all \( \omega \in \Omega \). For agent 2 we have \( P^2(\omega_1) = P^2(\omega_2) = \{\omega_1, \omega_2\} \), \( P^2(\omega_3) = \{\omega_2, \omega_3\} \). Consider the trade \( b^1(\omega_1) = b^1(\omega_3) = 1/4, b^1(\omega_2) = -3/16 \), \( b^2 = -b^1 \). At each state \( \omega \in \Omega \), both agents expect positive gains. Hence, this is always common knowledge.

**Example 4.** This is an example of an information structure \( P^2 \) that is not strongly more informed than \( P^1 \), \( S_0 \) is non-degenerate for both \( P^1 \) and \( P^2 \), but for any \( u, \pi, C \), decision problem \((C, \Sigma, P^2, u, \pi)\) is more valuable than \((C, \Sigma, P^1, u, \pi)\). There are four state spaces, \( S_0 = \{\omega\}, S_1 = \{\omega_1, \omega_2\}, S_2 = \{\omega_3, \omega_4\} \) and \( S^* = \{\omega_1^*, \omega_2^*, \omega_3^*, \omega_4^*\} \). All states project to \( \omega \). Moreover, \( P^1(\omega_1^*) = P^1(\omega_2^*) = \{\omega_1\}, P^1(\omega_3^*) = P^1(\omega_4^*) = \{\omega_2\}, P^2(\omega_1^*) = P^1(\omega_3^*) = \{\omega_3\}, P^2(\omega_2^*) = P^1(\omega_4^*) = \{\omega_4\} \). Neither \( P^1 \) or \( P^2 \) is strongly more informed. Moreover, \( S_0 \) is degenerate for both \( P^1 \) and \( P^2 \) because each agent always considers only one state to be possible. Yet, for any any \( u, \pi, C \), decision problem \((C, \Sigma, P^2, u, \pi)\) is more valuable as \((C, \Sigma, P_1, u, \pi)\).

**References**


Jing Li. Dynamic games of complete information with unawareness. *mimeo, University of Pennsylvania*, 2006b.


