Non replication of options

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Abstract

In this paper we study the scarcity of replication of options in the two period model of financial markets with a finite set of states. Especially we study this problem in financial markets without binary vectors and in strongly resolving markets. We start our study by proving that a financial market does not have binary vectors if and only if for any portfolio, at least one non trivial option is replicated. After this characterization we prove that in these markets, for any portfolio, at most \( m - 3 \) options can be replicated where \( m \) is the number of states, therefore for any portfolio, the number of the replicated options is between the natural numbers \( 1 \) and \( m - 3 \). Note that by the existing result of Baptista (2007), the set of non replicated options is of measure zero, and as it is known there are infinite sets with measure zero.

In the sequel we generalize the definition of strongly resolving markets to a more general class of financial markets by considering the payoff matrix of primitive securities, not with respect to the usual basis of \( \mathbb{R}^m \), but with respect to the positive basis of the financial completion of the market. This allows us to generalize the result of Aliprantis-Tourky (2002) about the non-replication of options in a bigger class of financial markets.

In this study, the theory of positive bases developed in [5] and [6] plays a central role. This theory simplifies and unifies the theory of options.

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1 The economic model and some essential notions

In this article we study a two-period security market with a finite number of states \( \Omega = \{1, 2, \ldots, m\} \) during the date 1, a finite number of primitive securities (assets) with payoffs the linearly independent vectors \( x_1, x_2, \ldots, x_n \) of the payoff space \( \mathbb{R}^m \).
A portfolio is a vector \( \theta = (\theta_1, \theta_2, ..., \theta_n) \) of \( \mathbb{R}^n \) where \( \theta_i \) is the number of the units of the \( i \) security. Then \( T(\theta) = \sum_{i=1}^{n} \theta_i x_i \in \mathbb{R}^m \) is the payoff of \( \theta \). Since the operator \( T \) is one-to-one, it identifies portfolios with their payoffs. So the vectors \( x_1, x_2, ..., x_n \) will be mentioned as primitive securities, the subspace

\[ X = [x_1, x_2, ..., x_n], \]

of \( E \), generated by the vectors \( x_i \) as the space of marketed securities or the asset span and the vectors of \( X \) will be also referred as portfolios. A vector \( x \in \mathbb{R}^m \) is marketed or \( x \) is replicated if it is the payoff of some portfolio \( \theta \), or equivalently if \( x \in X \).

Recall that the vector space \( \mathbb{R}^m = \{ x = (x(1), x(2), ..., x(m)) | x(i) \in \mathbb{R} \text{ for each } i \} \) is ordered by the pointwise ordering i.e. for any \( x, y \in \mathbb{R}^m \) we have: \( x \geq y \) if \( x(i) \geq y(i) \) for each \( i \). \( \mathbb{R}^m_+ = \{ x \in \mathbb{R}^m | x(i) \geq 0 \text{ for each } i \} \) is the positive cone of \( \mathbb{R}^m \). For any \( x, y \in \mathbb{R}^m \) \( x + y = (x(1) + y(1), x(2) + y(2), ..., x(m) + y(m)) \) is the supremum and \( x \wedge y = (x(1) \wedge y(1), x(2) \wedge y(2), ..., x(m) \wedge y(m)) \) is the infimum of \( \{ x, y \} \) in \( \mathbb{R}^m \). \( x^+ = x \vee 0 = (x(1) \vee 0, x(2) \vee 0, ..., x(m) \vee 0) \) and \( x^- = (-x) \vee 0 \) are the positive and the negative part of \( x \). Note also that for any two real numbers \( a, b, a \vee b \) is the supremum and \( a \wedge b \) is the infimum of \( \{ a, b \} \). A linear subspace \( Z \) of \( \mathbb{R}^m \) is a sublattice or a Riesz subspace of \( \mathbb{R}^m \) if for any \( x, y \in Z \), \( x \vee y \) and \( x \wedge y \) belong to \( Z \). Also for any \( x = (x(1), x(2), ..., x(m)) \in \mathbb{R}^m \), the set \( \text{supp}(x) = \{ i = 1, 2, ..., m | x(i) \neq 0 \} \) is the support of \( x \).

For any subset \( B \) of \( \mathbb{R}^m \), the sublattice \( S(B) \) of \( \mathbb{R}^m \) generated by \( B \) is the intersection of the sublattices of \( \mathbb{R}^m \) which contain \( B \). The riskless bond \( 1 \) is the vector of \( \mathbb{R}^m \) whose every coordinate is equal to \( 1 \). The call option written on the vector \( x \in \mathbb{R}^m \) with exercise price \( a \) is the vector \( c(x, a) = (x - a1)^+ \) of \( \mathbb{R}^m \) and the put option of \( x \) with exercise price \( a \) is \( p(x, a) = (a1 - x)^+ \). We have \( x - a1 = c(x, a) - p(x, a) \).

If both \( c(x, a) > 0 \) and \( p(x, a) > 0 \), we say that the call option \( c(x, a) \) is non trivial and also we say that the put option \( p(x, a) \) is non trivial. In this case we say that \( a \) is a non trivial exercise price of \( x \). We denote by \( K_a \) the set of non trivial exercise prices of \( x \). If \( c(x, a) \in X \) we say that \( a \) is a call-replicated exercise price of \( x \) and if \( p(x, a) \in X \), we say that \( a \) is a put-replicated exercise price of \( x \). If both \( c(x, a), p(x, a) \) are in \( X \) we say that \( a \) is a replicated exercise price of \( x \). If \( 1 \in X \), we have: \( c(x, a) \in X \) if and only if \( p(x, a) \in X \). If the riskless bond is not contained in \( X \) it is possible only one of the call and put options to be replicated. In this paper we do not suppose always that the riskless bond \( 1 \) belongs to \( X \). Of course \( 1 \) belongs to the completion by options of \( X \).

The completion by options of \( X \) is the subspace of \( \mathbb{R}^m \) which arises inductively by adding in the market the call and put options of the marketed securities and by taking again call and put options which are added again in the market. In [3] a mathematical definition of the completion by options in infinite securities markets is given. Especially in the above article, a more general study of the completion by options of the market is presented by Kountzakis and Polykrakis (2006) were the options are not taken with respect to the riskless bond \( 1 \) but with respect to some risky vectors from a standard subspace \( U \) of \( \mathbb{R}^m \) and the completion by options of \( X \) is denoted by \( \mathcal{F}_U(X) \). This study is very general and also includes the case of exotic options. In the classical case where the options are taken with respect to the riskless bond \( 1 \), the completion
by options of $X$ is denoted in [3] by $F_1(X)$ and we will preserve this notation in the present article. In the above article it is proved that if the payoff space is a general vector lattice $E$ then $F_1(X)$ is the sublattice of $E$ generated by the set $X \cup U$. In our case where the payoff space is the space $\mathbb{R}^m$ and the call and put options are taken with respect to the riskless bond 1, the completion by options $F_1(X)$ of $X$ is the sublattice of $\mathbb{R}^m$ generated by the set $X \cup \{1\}$.

In the above results and also in the results of the present article, the theory of lattice-subspaces and positive bases developed by Polyrakis in the papers of the bibliography is very important. This theory simplifies and unifies the theory of options. Ross (1976) proved in [7] that the existence of an efficient fund in $X$ is equivalent to the fact that the span of call and put options written on the elements of $X$ is equal to the entire space $\mathbb{R}^m$. In Aliprantis and Tourky (2002), a scarcity result for replicated options for strongly resolving markets is proved. Especially it is proved that if $1 \in X$, $n \leq \frac{m+1}{2}$ and the asset span is strongly resolving, i.e. any $n \times n$ submatrix of the payoff matrix of the primitive securities is non singular, then any non-trivial option written on some element of $X$ is not replicated.

In the sequel, Theorem 6, we prove that if $F_1(X)$ is a proper subset of $\mathbb{R}^m$ the market cannot be strongly resolving. The market $X$ of Example 13 is strongly resolving with respect to the basis $\{b_i\}$ but $X$ is not strongly resolving, therefore our definition is a generalization of the existing one. In Theorem 11 we extent the result of [1] for strongly resolving markets with respect to the basis $\{b_i\}$. Our proof is the analogous with the excellent proof of Aliprantis-Tourky.

For a study of the two-periods security markets we refer to the book of LeRoy and Werner (2001), [4].
2 Determination of the completion $F_1(X)$ of $X$

In this section we describe the method of determination of the completion by options of $X$ as it is presented in [3]. According to this method we consider the set

$$A = \{x_1^+, x_1^-, x_2^+, x_2^-, \ldots, x_n^+, x_n^-, 1\}.$$

Any maximal subset $\{y_1, y_2, \ldots, y_r\}$ of linearly independent vectors of $A$ is a basic set of the market, where $x_i^+$, $x_i^-$ are the positive and negative parts of the vectors $x_i$. Note that a basic set is not necessarily unique. In general it is possible to find different basic sets of the market but all these sets have the same cardinal number $r$. Especially $r$ is the dimension of the linear subspace of $R^m$ generated by $A$ and a basic set is a basis of it.

**Theorem 1 ([3], Theorem 11).** $F_1(X)$ is the sublattice of $R^m$ generated by a basic set $\{y_1, y_2, \ldots, y_r\}$ of the market.

After this result we use the theory of lattice-subspaces and positive bases developed by Polyrakis in [5] and [6] for the determination of $F_1(X)$. Since $F_1(X)$ is a sublattice of $R^m$ which contains 1, we have that $F_1(X)$ has a positive basis $\{b_1, b_2, \ldots, b_n\}$ which is a partition of the unit, i.e. the vectors $b_i$ have disjoint supports and $\sum_{i=1}^n b_i = 1$, see in Theorem 16 of the Appendix. This basis is unique. So we have:

**Theorem 2.** $F_1(X)$ has a positive basis $\{b_1, b_2, \ldots, b_n\}$ which is a partition of the unit.

For the determination of the positive basis $\{b_i\}$ of $F_1(X)$ which is a partition of the unit we follow the steps of Polyrakis algorithm, see Theorem 18 in the appendix, where a positive basis of the sublattice of $R^m$ generated by a finite set of positive and linearly independent vectors is determined. We start by the determination of a basic function set of $\{y_1, y_2, \ldots, y_r\}$ of the market. In the sequel we determine the basic function of $y_1, y_2, \ldots, y_r$ which is very important for the theory of lattice-subspaces and positive bases. This function has been defined in [5] and is the following:

$$\beta(i) = \left(\frac{y_1(i)}{y(i)}, \frac{y_2(i)}{y(i)}, \ldots, \frac{y_r(i)}{y(i)}\right), \text{ for each } i = 1, 2, \ldots, m, \text{ with } y(i) > 0,$$

where $y = y_1 + y_2 + \ldots + y_r$. This function takes values in the simplex $\Delta_r = \{\xi \in R^r_+: \sum_{i=1}^r \xi_i = 1\}$ of $R^r_+$.

Denote by $R(\beta)$ is the range (i.e. the set of values) of $\beta$ and by $\text{card}R(\beta)$ the cardinal number of $R(\beta)$ (i.e. the number of the different values of $\beta$).

We continue the algorithm and we obtain a positive basis $\{d_1, d_2, \ldots, d_n\}$ of $F_1(X)$. The elements of this basis have disjoint support and each $d_i$ is constant on its support. So by a normalization of the basis $\{d_i\}$ we obtain the basis positive basis $\{b_i\}$ of $F_1(X)$ which is also a partition of the unit.

Except the determination of the positive basis $\{b_i\}$ of $F_1(X)$, by the basic function we can ask very easy to two important questions of the theory of options. Especially we
can check directly whether any option is replicated and also if the completion by options of $X$ is the whole space $\mathbb{R}^m$. Recall that if $X = F_1(X)$, any option is replicated and we say then that $X$ is **complete by options** (with respect to 1). As it is proved in Polyrakis (1999), Theorem 3.7 (see also in the Appendix) the dimension of the sublattice generated by $\{y_1, y_2, \ldots, y_r\}$, i.e. the dimension of $F_1(X)$, is equal to the cardinal number of $R(\beta)$. So if $R(\beta)$ has $n$ elements we have that $F_1(X) = X$ and if $R(\beta)$ has $m$ elements we have that $F_1(X) = \mathbb{R}^m$, so we have:

**Theorem 3.** The dimension of $F_1(X)$ is equal to the cardinal number of the range $R(\beta)$, therefore we have:

(i) $F_1(X) = X$ if and only if $\text{card}R(\beta) = n$,

(ii) $F_1(X) = \mathbb{R}^m$ if and only if $\text{card}R(\beta) = m$,

(iii) $F_1(X) \subseteq \mathbb{R}^m$ if and only if $\text{card}R(\beta) < m$.

3 Markets without binary vectors

Throughout this paper we will denote by $\{b_1, b_2, \ldots, b_\mu\}$, or for simplicity by $\{b_i\}$, the positive basis of $F_1(X)$ which is a partition of the unit. For any $x = \sum_{i=1}^\mu \lambda_i b_i \in F_1(X)$, $\lambda_1, \lambda_2, \ldots, \lambda_\mu$ are the coefficients of $x$ in the basis $\{b_i\}$. We put

$$a_1 = \min\{\lambda_i | i = 1, 2, \ldots, \mu\} \text{ and } \Phi_1 = \{i | \lambda_i = a_1\},$$

$$a_2 = \min\{\lambda_i | \lambda_i > a_1\} \text{ and } \Phi_2 = \{i | \lambda_i = a_2\},$$

and by continuing this process we take the real numbers $a_1, a_2, \ldots, a_k$ and the subsets $\Phi_1, \Phi_2, \ldots, \Phi_k$ of $\{1, 2, \ldots, \mu\}$. The numbers $a_1, a_2, \ldots, a_k$ will be referred as the **essential coefficients** and the sets $\Phi_1, \Phi_2, \ldots, \Phi_k$ as the **essential sets of states** of $x$, with respect to the basis $\{b_i\}$.

The essential coefficients are in increasing order, i.e. $a_i < a_j$ for any $i < j$. Of course for the number $k$ of the essential coefficients of $x$ we have: $k \leq \mu$.

If for example $\mu = 5$ and $x = 2b_1 - 3b_2 + 2b_3 + b_4 + b_5$, then $a_1 = -3, a_2 = 1, a_3 = 2$ are the essential coefficients and $\Phi_1 = \{2\}, \Phi_2 = \{4, 5\}, \Phi_3 = \{1, 3\}$ the essential sets of states of $x$. We have $x = a_1 \sum_{i \in \Phi_1} b_i + a_2 \sum_{i \in \Phi_2} b_i + a_3 \sum_{i \in \Phi_3} b_i$.

**Proposition 4.** For any $x = \sum_{i=1}^\mu \lambda_i b_i \in F_1(X)$ we have:

(i) $c(x, a) = \sum_{i=1}^\mu (\lambda_i - a)^+ b_i$ and $p(x, a) = \sum_{i=1}^\mu (a - \lambda_i)^+ b_i$,

(ii) if $a_1, a_2, \ldots, a_k$ are the essential coefficients of $x$, the interval $K_x = (a_1, a_k)$ is the set of non-trivial exercise prices of $x$.

**Proof.** (i): The basis $\{b_i\}$ is a partition of the unit therefore the vectors $b_i$ have disjoint supports and $\sum_{i=1}^\mu b_i = 1$. Therefore, the vectors $b_i$ have disjoint supports and $b_i(j) = 1$ for any $j \in \text{supp}(b_i)$. So we have $c(x, a) = (x - a \mathbf{1})^+ = (\sum_{i=1}^\mu \lambda_i b_i - a \sum_{i=1}^\mu b_i)^+ = (\sum_{i=1}^\mu (\lambda_i - a) b_i)^+$. Since the basis $\{b_i\}$ is a partition of the unit, for any $j \in \text{supp}(b_i)$ we have that the $j$-coordinate of $c(x, a)$ in the usual basis $\{e_j\}$ of
\( \mathbf{R}^m \) is \((\lambda_i - a)^+\), therefore it is easy to show that \( c(x, a) = \sum_{i=1}^{m} (\lambda_i - a)^+ b_i \). The proof for the put option is analogous.

(ii): If \( a \leq a_1 \) then \((a - \lambda_i)^+ = 0 \) for any \( i \), therefore \( p(x, a) = 0 \) because \( a_1 \) is the minimum of the coefficients \( \lambda_i \) of \( x \). Therefore \( a \) is a trivial exercise price of \( x \). If \( a \geq a_k \), similarly we have that \((\lambda_i - a)^+ = 0 \) for any \( i \), therefore \( c(x, a) = 0 \) and \( a \) is a trivial exercise price. For any \( a \in (a_1, a_k) \) we have that \((\lambda_i - a)^+ > 0 \) for at least one \( i \) and also \((a - \lambda_j)^+ > 0 \) for at least one \( j \). Since the elements \( b_i \) of the basis are positive we have that \( c(x, a) > 0 \) and \( p(x, a) > 0 \), hence \( a \) is a non-trivial exercise price of \( x \).

We give below a characterization of the markets without binary vectors. We say that a vector \( x \in \mathbf{R}^m \) is a non-constant vector if \( x \) is not a multiple of \( 1 \) i.e. \( x \neq \lambda 1 \) for some \( \lambda \in \mathbf{R} \). According to this definition, \( 0 \) is a constant vector of \( \mathbf{R}^m \).

**Theorem 5.** If \( 1 \in X \), we have: \( X \) does not contain binary vectors if and only if for any non-constant vector \( x \in X \) at least one non-trivial option of \( x \) is non-replicated.

**Proof.** Suppose that \( X \) does not contain binary vectors. Suppose also that there exist \( x \in X \) so that \( c(x, \alpha) \in X \) for each \( \alpha \in K_x \). Also for any \( \alpha \notin K_x \) we have that \( c(x, \alpha) = 0 \) or \( p(x, \alpha) = 0 \), therefore \( c(x, \alpha) \) and \( p(x, \alpha) \) are elements of \( X \) because \( x - \alpha 1 = c(x, \alpha) - p(x, \alpha) \). Therefore if \( L = [x] \) is the one-dimensional subspace generated by \( x \), then by [3], Theorem 21, the completion by options \( F_1(L) \) of \( L \) is the subspace generated by the set of call options written on the elements of the subspace \( Y \) of \( \mathbf{R}^m \) generated by the set \( L \cup \{1\} \), i.e. the completion is taken during the first steep of the process. Each vector \( y \) of \( Y \) is of the form \( y = \lambda x + \xi 1 \) therefore \( c(y, a) = (y - a 1)^+ = (\lambda x - (a - \xi) 1) = c(\lambda x, (a - \xi)) \). So we have that any call option written on an element of \( Y \) is a call option written on an element of \( L \), therefore it belongs to \( X \) as we have proved before. So we have that \( F_1(L) \subset X \). Since \( 1 \in F_1(L) \) we have that \( F_1(L) \) is an at least two-dimensional sublattice, therefore \( F_2(L) \) has a positive basis which is also a partition of the unit. The elements of this basis are binary vectors, and these elements belong to \( X \), contradiction. So for any \( x \in X \) at least one non-trivial option of \( x \) is non-replicated.

For the converse suppose for any \( x \in X \) at least one non-trivial option of \( x \) is non-replicated. If we suppose that \( x \) is a binary vector of \( X \), then it is easy to show that the essential coefficients of \( x \) in the basis \( \{b_i\} \) of \( F_1(X) \) are \( a_1 = 0, a_2 = 1 \), therefore \( K_x = (0, 1) \) and \( x = \sum_{i \in \Phi_2} b_i = b_2 \in X \). For any \( \alpha \in (0, 1) \) we have that \( p(x, \alpha) = (1 - \alpha)b_2 \in X \) which is a contradiction. Therefore \( X \) does not contain binary vectors.

**Theorem 6.** Suppose that the asset span \( X \) does not contain binary vectors and \( x \) is a non-constant vector of \( X \). If \( a_1, a_2, ..., a_k \) are the essential coefficients of \( x \) with respect to the basis \( \{b_i\} \), then:

(i) If \( k = 2 \), each non-trivial call option of \( x \) is non-replicated. If \( k > 2 \), each of the intervals \( (a_1, a_2), [a_2, a_3], ..., [a_{k-2}, a_{k-1}] \) contains at most one call-replicated exercise price, therefore there are at most \( k - 2 \) call-replicated exercise prices of \( x \).
(ii) If $k = 2$, each non-trivial put option of $x$ is non-replicated. If $k > 2$, each of the intervals $(a_2, a_3), \ldots, (a_{k-2}, a_{k-1}), (a_{k-1}, a_k)$ contains at most one put-replicated exercise price, therefore there are at most $k - 2$ put-replicated exercise prices of $x$.

(iii) If we suppose moreover that $1 \in X$, we have: If $k = 3$, each non-trivial option of $x$ is non-replicated. If $k > 3$, each of the intervals $(a_2, a_3), [a_3, a_4), \ldots; (a_{k-2}, a_{k-1})$ contains at most one replicated exercise price, therefore there are at most $k - 3$ replicated exercise prices of $x$.

**Proof.** Suppose that $x \in X$, $x \neq \lambda 1$, $x = \sum_{i=1}^{n} \lambda_i b_i$ is the expansion of $x$ in the basis $\{b_i\}$ and suppose that $a_1, a_2, \ldots, a_k$ are the essential coefficients and $\Phi_1, \Phi_2, \ldots, \Phi_k$ are the essential sets of states of $x$ with respect to the basis $\{b_i\}$. Since we have supposed that $x$ is not a positive multiple of the riskless bond we have that $x$ has at least two essential coefficients, hence $k \geq 2$.

Then

$$x = a_1 \sum_{i \in \Phi_1} b_i + a_2 \sum_{i \in \Phi_2} b_i + \ldots + a_k \sum_{i \in \Phi_k} b_i.$$  

We put $\bar{b}_j = \sum_{i \in \Phi_j} b_i, j = 1, 2, \ldots, k$ and we remark that every such vector is a binary vector. Also, we have

$$x = \sum_{j=1}^{k} a_j \bar{b}_j.$$  

The set of non trivial exercise prices is the interval $K_x = (a_1, a_k)$. For any $a \in (a_1, a_k)$ we have

$$c(x, a) = \sum_{j=r+1}^{k} (a_j - a) \bar{b}_j,$$

where $r = 1$ if $a \in (a_1, a_2)$ and $r = \nu$ if $a \in [a_{\nu}, a_{\nu+1})$ for $\nu = 2, 3, \ldots, k - 1$.

If $a \in [a_{k-1}, a_k)$ then $c(x, a) = (a_k - a) \bar{b}_k$ is a positive multiple of a binary vector, therefore $c(x, a) \not\in X$. So for any $a \in [a_{k-1}, a_k)$, $c(x, a)$ is non-replicated.

This means also that if $k = 2$, i.e. if $a_1, a_2$ are the essential coefficients of $x$, then any call option of $x$ is non-replicated.

Suppose now that $a, a'$ are different exercise prices belonging to the same subinterval of $(a_1, a_k)$, i.e. $a, a' \in (a_1, a_2)$ or $a, a' \in [a_r, a_{r+1})$ for some $r = 2, 3, \ldots, k - 2$.

Then we have

$$c(x, a) - c(x, a') = \sum_{j=r+1}^{k} ((a_j - a) - (a_j - a')) \bar{b}_j = (a' - a) \sum_{j=r+1}^{k} \bar{b}_j.$$

If we suppose that $c(x, a)$ and $c(x, a')$ belong to $X$ we have that

$$(a' - a) \sum_{j=r+1}^{k} \bar{b}_j \in X.$$
which is a contradiction because \( \sum_{j=r+1}^{k} \delta_j \) is a binary vector. This implies that at most one of \( c(x, a) \), \( c(x, a') \) belongs to \( X \).

So any of the subintervals \((a_1, a_2], [a_2, a_3], \ldots, [a_{k-2}, a_{k-1}]\) of \((a_1, a_k)\) contains at most one call-replicated exercise price, therefore there are at most \( k-2 \) call-replicated exercise prices.

Similarly for the case of put options we write \((a_1, a_k) = (a_1, a_2] \cup (a_2, a_3] \cup \ldots \cup (a_{k-1}, a_k)\) and we have: For any \( a \in (a_1, a_2] \) the put option \( p(x, a) \) is a binary vector, therefore the interval \((a_1, a_2]\) does not contain put-replicated exercise prices. For any two \( a, a' \) different exercise prices belonging in the same subinterval of \((a_1, a_k)\), \( p(x, a) - p(x, a') \) is a multiple of a binary vector and as above we have that any of the intervals \((a_2, a_3], \ldots, (a_{k-2}, a_{k-1}], (a_{k-1}, a_k)\) contains at most one put-replicated exercise price, therefore there are at most \( k-2 \) put-replicated exercise prices.

If we suppose that \( 1 \in X \) and at least one of \( c(x, a), p(x, a) \) is replicated, then both of them are replicated because

\[
x - a1 = c(x, a) - p(x, a),
\]

therefore an exercise price \( a \) is call-replicated if and only if \( a \) is put-replicated. So, if \( 1 \in X \), by \((i)\) and \((ii)\) we have that there are at most \( k-3 \) replicated exercise prices because strike prices in the intervals \((a_1, a_2]\) and \([a_{k-1}, a_k)\) are excluded. So in the case where \( 1 \in X \), each interval \((a_2, a_3], [a_3, a_4], \ldots, [a_{k-2}, a_{k-1}]\) has at most one replicated exercise price for \( x \), therefore there are at most \( k-3 \) replicated exercise prices.

If the dimension of \( F_1(X) \) is at most three, then the basis \( \{b_i\} \) of \( X \) has at most three elements and for any \( x \in X \) the essential coefficients of \( x \) are at most three real numbers \( a_1, a_2, a_3 \) and the next corollary is obvious:

**Corollary 7.** Suppose that the asset span \( X \) doesn’t contain binary vectors.

\((i)\) If \( \dim F_1(X) = 2 \), then any non-trivial option written on some element of \( X \) is non-replicated.

\((ii)\) If \( \dim F_1(X) = 3 \) and \( 1 \in X \), any non-trivial option written on some element of \( X \) is non-replicated.

**Example 8.** Suppose that \( x_1 = (1, 1, 2, 2, 0, 0, 0, 0) \), \( x_2 = (0, 0, 0, 0, 3, 3, 4, 4) \), \( x_3 = (1, 1, 1, 1, 1, 1, 1, 1) \) are the primitive securities and \( X = [x_1, x_2, x_3] \) is the marketed space. It is easy to show that \( X \) does not contain binary vectors.

According to the methodology of the determination of \( F_1(X) \) we start by the determination of a basic set and we find that \( \{y_1, y_2, y_3\} = \{x_1, x_2, x_3\} \) is a basic set of the market. In order to determine a positive basis of \( F_1(X) \) we follow the algorithm of Theorem 18. So we determine the basic function \( \beta \) of \( y_1, y_2, y_3 \). \( \beta = \frac{1}{y}(y_1, y_2, y_3) \), where \( y \) is the sum of \( y_i \) and we find that

\[
\beta(1) = \beta(2) = \frac{1}{2}(1, 0, 1) = P_1, \beta(3) = \beta(4) = \frac{1}{3}(2, 0, 1) = P_2
\]
\[ \beta(5) = \beta(6) = \frac{1}{4} (0, 3, 1) = P_3, \beta(7) = \beta(8) = \frac{1}{5} (0, 5, 1) = P_4. \]

So we have that \( \text{card}(R(\beta)) = 4 \) therefore the completion \( F_1(X) \) is a four-dimensional sublattice of \( \mathbb{R}^8 \).

The three first vectors \( P_1, P_2, P_3 \) of \( R(\beta) \) are linearly independent, so we preserve the enumeration of \( R(\beta) \). According to the algorithm, \( I_4 = \beta^{-1}(P_4) = \{ 7, 8 \} \) and we define the new vector \( y_4 = (0, 0, 0, 0, 0, 0, 5, 5) \). We determine the basic function \( \gamma = \frac{1}{y'}(y_1, y_2, y_3, y_4) \) where \( y' \) is the sum of these vectors. We find that

\[
\gamma(1) = \gamma(2) = \frac{1}{2} (1, 0, 1, 0) = P'_1, \quad \gamma(3) = \gamma(4) = \frac{1}{3} (2, 0, 1, 0) = P'_2
\]

\[
\gamma(5) = \gamma(6) = \frac{1}{4} (0, 3, 1, 0) = P'_3, \quad \gamma(7) = \gamma(8) = \frac{1}{10} (0, 4, 1, 5) = P'_4
\]

A positive basis of \( F_1(X) \) is given by the formula \( (d_1, d_2, d_3, d_4)^T = A^{-1}(y_1, y_2, y_3, y_4)^T \) where \( A \) is the matrix whose columns are the vectors \( P'_i, i = 1, ..., 4 \). We find that the vectors

\[
d_1 = (2, 2, 0, 0, 0, 0, 0, 0), \quad d_2 = (0, 0, 3, 3, 0, 0, 0, 0),
\]

\[
d_3 = (0, 0, 0, 0, 4, 0, 0, 0), \quad d_4 = (0, 0, 0, 0, 0, 10, 10),
\]

define a positive basis of \( F_1(X) \). By a normalization of this basis we have that the vectors

\[
b_1 = (1, 1, 0, 0, 0, 0, 0, 0), \quad b_2 = (0, 0, 1, 1, 0, 0, 0, 0),
\]

\[
b_3 = (0, 0, 0, 0, 1, 1, 0, 0), \quad b_4 = (0, 0, 0, 0, 0, 0, 1, 1),
\]

define the positive basis of \( F_1(X) \) which is a partition of the unit.

Consider the portfolio \( x = -x_1 + x_2 = (-1, -1, -2, -2, 3, 4, 4) \). The expansion of \( x \) in the basis \( \{ b_i \} \) is \( x = -b_1 - 2b_2 + 3b_3 + 4b_4 \) and according to the above theorem \( a_1 = -1, a_2 = -2, a_3 = 3, a_4 = 4 \) are the essential coefficients of \( x \). For any \( \alpha \in (-1, 2) \) or \( \alpha \in [3, 4) \), any option is non replicated. By the previous results \( x \) has exactly one replicated exercise \( \alpha \in (2, 3) \). We can determine it as follows:

\[
c(x, \alpha) = (3 - \alpha)b_3 + (4 - \alpha)b_4 \in X,
\]

therefore

\[
c(x, \alpha) = \rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 = \rho_1 (b_1 + 2b_2) + \rho_2 (3b_3 + 4b_4) + \rho_3 (b_1 + b_2 + b_3 + b_4).
\]

We find that \( \rho_1 = \rho_3 = 0, 3\rho_2 = 3 - \alpha, 4\rho_2 = 4 - \alpha \), therefore \( \alpha = 0 \).

Indeed, \( c(x, 0) = 3b_3 + 4b_4 = x_2 \in X \) and \( p(x, 0) = b_1 + 2b_2 = x_1 \in X \).
4 Strongly resolving markets

The replication of options in strongly resolving markets has been studied by Aliprantis-Tourky (2002). If we expand the vectors $x_i$ in the usual basis \{$e_1, ..., e_m$\} of $\mathbb{R}^m$ we have the $m \times n$ matrix

$$
A(x_i, e_i) = \begin{bmatrix}
  x_1(1) & x_2(1) & ... & x_n(1) \\
  x_1(2) & x_2(2) & ... & x_n(2) \\
  . & . & . & . \\
  x_1(m) & x_2(m) & ... & x_n(m)
\end{bmatrix},
$$

which is the payoff matrix of vectors $x_i$. The notion of strongly resolving market is defined in the above article as follows: If any $n \times n$ submatrix of $A(x_i, e_i)$ is non-singular, the asset span $X = [x_1, ..., x_n]$ (or the market) is called strongly resolving.

As we have noted in the introduction, Aliprantis and Tourky prove that if $1 \in X$, $n \leq m+1$ and the asset span is strongly resolving, any non-trivial call and put option written on some element of $X$ is not replicated.

In this paper we extend the definition of strongly resolving markets by taking the payoff matrix of the payoff vectors in the basis \{$b_i$\} of $F_1(X)$. So if \{($b_1, ..., b_\mu$)\} is the positive basis of $F_1(X)$ which is a partition of the unit, we expand each $x_i$ in this basis and suppose that $x_i = \sum_{j=1}^{\mu} x_i^j b_j$. The $\mu \times n$ matrix

$$
A(x_i, b_i) = \begin{bmatrix}
  x_1^1(1) & x_2^1(1) & ... & x_n^1(1) \\
  x_1^1(2) & x_2^1(2) & ... & x_n^1(2) \\
  . & . & . & . \\
  x_1^\mu(\mu) & x_2^\mu(\mu) & ... & x_n^\mu(\mu)
\end{bmatrix},
$$

is the payoff matrix of the basic securities $x_i$ in the basis \{$b_i$\}.

**Definition 9.** If any $n \times n$ submatrix of $A(x_i, b_i)$ is non-singular, the market $X$ is strongly resolving with respect to the basis \{$b_i$\}.

In the next theorem we prove that if $F_1(X) \neq \mathbb{R}^m$, the market cannot be strongly resolving. So if the market is strongly resolving, then $F_1(X) = \mathbb{R}^m$ and the two definitions coincide because the basis \{($b_1, ..., b_\mu$)\} of $F_1(X)$ is the usual basis \{($e_1, ..., e_m$)\} of $\mathbb{R}^m$ and therefore $A(x_i, b_i) = A(x_i, e_i)$. Also the market of Example 13 is strongly resolving with respect to the basis \{($b_1, ..., b_\mu$)\} but not strongly resolving, therefore our definition of strongly resolving markets is a generalization of the existing one.

**Theorem 10.** If $n \geq 2$ and the completion by options $F_1(X)$ of $X$ is a proper subspace of $\mathbb{R}^m$, then the market is not strongly resolving.

**Proof.** The assumption that $F_1(X) \neq \mathbb{R}^m$, implies $\mu < m$, where \{($b_1, b_2, ..., b_\mu$)\} is the positive basis of $F_1(X)$ which is also a partition of the unit. Since $\mu < m$, the support of at least one of the elements of the basis is not a singleton. So we may
Suppose that \( i_1, i_2 \in \text{supp}(b_r) \) for some \( r \). For any \( x_i \) we have \( x_i = \sum_{j=1}^{n} x^i_j b_j \), therefore \( x_i(i_1) = x^i_1(r) b_r(i_1) = x^i_1(r) \) because \( b_r(i_1) = 1 \). Similarly \( x_i(i_2) = x^i_2(r) \), therefore for any vector \( x_i \) we have \( x_i(i_1) = x_i(i_2) \). This implies that the \( i_1 \) and \( i_2 \)-row of the matrix \( A(x_i, e_i) \) coincide and the theorem is true.

For any \( x \in F_1(X) \) we expand \( x \) in the basis \( \{b_i\} \) of \( F_1(X) \) which is a partition of the unit and suppose that \( x = \sum_{i=1}^{n} \lambda_i b_i \). Then \( \text{supp}_b(x) = \{i|\lambda_i \neq 0\} \) is the support of \( x \) and \( \text{zeros}_b(x) = \{i|\lambda_i = 0\} \) is the set of zeros of \( x \) with respect to the basis \( \{b_i\} \). Also \( \#\text{supp}_b(x) \) and \( \#\text{zeros}_b(x) \) is the cardinal number of the sets \( \text{supp}_b(x) \) and \( \text{zeros}_b(x) \).

**Theorem 11.** Suppose that the riskless bond \( I \) is contained in \( X \). If the market is strongly resolving with respect to the basis \( \{b_i\} \) and \( n \leq \frac{n+1}{2} \) then any nontrivial option written on some element of \( X \) is non replicated.

**Proof.** Let \( x = \sum_{i=1}^{n} \lambda_i b_i \in X \) and suppose that \( y = c(x, \alpha) = \sum_{i=1}^{n} (\lambda_i - \alpha)^+ b_i \) is a nontrivial call option. Then \( y > 0 \) and also the corresponding put option \( z = p(x, \alpha) = \sum_{i=1}^{n} (\alpha - \lambda_i)^+ b_i \) is also greater of zero, \( z > 0 \). Let

\[
\#\text{supp}_b(y) = \beta, \#\text{zeros}_b(y) = \gamma, \#\text{supp}_b(z) = \beta', \#\text{zeros}_b(z) = \gamma'.
\]

We shall show that

\[
\max\{\gamma, \gamma'\} \geq \frac{\mu}{2}.
\]

It is clear that \( i \in \text{supp}_b(y) \Rightarrow i \in \text{zeros}_b(z) \) and \( i \in \text{supp}_b(z) \Rightarrow i \in \text{zeros}_b(y) \), therefore \( \gamma' \geq \beta \) and \( \gamma \geq \beta' \).

Also \( \beta + \gamma = \beta' + \gamma' = \mu \). If \( \beta \geq \gamma \) then \( \beta \geq \frac{\mu}{2} \) therefore \( \gamma' \geq \frac{\mu}{2} \). If \( \gamma \geq \beta \) then \( \gamma \geq \frac{\mu}{2} \) and the assertion is true. Since the riskless bond belongs to \( X \) we have that both \( y, z \) are replicated or not. Suppose that \( y, z \) are replicated. Then as we have proved above at least one of them, for example the call option \( y \) has a number of zero coordinates in the basis \( \{b_i\} \) greater or equal to \( \frac{\mu}{2} \), i.e. \( \gamma \geq \frac{\mu}{2} \).

Since \( y \in X \), it can be expanded in the basis \( \{x_1, ..., x_n\} \) of \( X \) and suppose that \( y = \sum_{i=1}^{n} \rho_i x_i \). Then we have

\[
\begin{array}{c}
(\lambda_1 - \alpha)^+ \\
(\lambda_2 - \alpha)^+ \\
\vdots \\
(\lambda_n - \alpha)^+
\end{array}
\begin{bmatrix}
x^1_1(1) & x^1_2(2) & \cdots & x^1_n(1) \\
x^2_1(1) & x^2_2(2) & \cdots & x^2_n(2) \\
\vdots & \vdots & \ddots & \vdots \\
x^n_1(\mu) & x^n_2(\mu) & \cdots & x^n_n(\mu)
\end{bmatrix}
\begin{bmatrix}
\rho_1 \\
\rho_2 \\
\vdots \\
\rho_n
\end{bmatrix}
\end{array}

(2)

By our assumption that \( n \leq \frac{n+1}{2} \) we have that \( n \leq \frac{n}{2} + \frac{1}{2} \leq \gamma + \frac{1}{2} \), therefore we have that \( n \leq \gamma \) because \( n, \gamma \) are natural numbers. Therefore that at least \( n \) coordinates of \( y \) in the basis \( \{b_i\} \) are equal to zero and suppose that \( (\lambda_{i_1} - \alpha)^+ = (\lambda_{i_2} - \alpha)^+ = \ldots = (\lambda_{i_n} - \alpha)^+ = 0 \).
... = (λ_n - α)^+ = 0. Then

\[
\begin{bmatrix}
  x_1^b(i_1) & x_2^b(i_1) & \ldots & x_n^b(i_1) \\
  x_1^b(i_2) & x_2^b(i_2) & \ldots & x_n^b(i_2) \\
  \vdots & \vdots & \ddots & \vdots \\
  x_1^b(i_n) & x_2^b(i_n) & \ldots & x_n^b(i_n)
\end{bmatrix}
\begin{bmatrix}
  \rho_1 \\
  \rho_2 \\
  \vdots \\
  \rho_n
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{bmatrix},
\]

(3)

where (ρ_1, ρ_2, ..., ρ_n) ≠ (0, 0, ..., 0) because ρ_i are the coordinates of y in the basis \{x_i\} and y > 0. This is a contradiction because the matrix of the system is non-singular. So none of y, z belong to X and the theorem is true.

Suppose that x ∈ X and suppose that a_1, a_2, ..., a_k and Φ_1, Φ_2, ..., Φ_k are the essential coefficients and the essential sets of states of x in the basis \{b_i\}. For any r = 1, 2, ..., k we define c_x(r) = card(Φ_1 ∪ ... ∪ Φ_r) and p_x(r) = card(Φ_{r+1} ∪ ... ∪ Φ_k), i.e. c_x(r) and p_x(r) are the cardinal numbers of Φ_1 ∪ ... ∪ Φ_r and Φ_{r+1} ∪ ... ∪ Φ_k.

**Proposition 12.** Suppose that the security market X is strongly resolving with resect to the basis \{b_i\} of F_1(X), x ∈ X and a_1, a_2, ..., a_k are the essential coefficients of x with respect to the basis \{b_i\}.

(i) If c_x(r) ≥ n, the interval [a_r, a_{r+1}) does not contain call-replicated exercise prices of x.

(ii) If p_x(r) ≥ n, the interval (a_r, a_{r+1}] does not contain put-replicated exercise prices of x.

(iii) If I ∈ X and \max(c_x(r), p_x(r)) ≥ n, the interval [a_r, a_{r+1}] does not contain replicated exercise prices of x.

**Proof.** Suppose that c_x(r) ≥ n and that a ∈ [a_r, a_{r+1}) is a call-replicated exercise price. Then

\[
y = c(x, a) = \sum_{j=r+1}^{k} (a_j - a)b_j \in X,
\]

where \(b_j = \sum_{i \in \Phi_j} b_i\) for any \(j = 1, 2, ..., k\).

We expanded y in the basis \{x_1, ..., x_n\} of X and suppose that \(y = \sum_{i=1}^{n} \lambda_i x_i\) and suppose also that \(x_i = \sum_{j=1}^{n} x_i^b(j)b_j\). By our hypothesis we have that zero\(y = c_x(r) ≥ n\), therefore at least \(n\) of the coordinates \(ξ_i\) of y in the basis \{b_i\} are equal to zero and suppose that \(ξ_1, ξ_2, ..., ξ_n\) are \(n\) such coordinates. This leads to the system

\[
\begin{bmatrix}
  x_1^b(i_1) & x_2^b(i_1) & \ldots & x_n^b(i_1) \\
  x_1^b(i_2) & x_2^b(i_2) & \ldots & x_n^b(i_2) \\
  \vdots & \vdots & \ddots & \vdots \\
  x_1^b(i_n) & x_2^b(i_n) & \ldots & x_n^b(i_n)
\end{bmatrix}
\begin{bmatrix}
  \lambda_1 \\
  \lambda_2 \\
  \vdots \\
  \lambda_n
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{bmatrix},
\]
The fact that $X$ is strongly resolving implies that the system has the unique solution
$$\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$$
therefore $y = c(x, a) = 0$, which is a contradiction because $a$ is a non trivial exercise price of $x$. Hence $(a_r, a_{r+1})$ does not contain call-replicated exercise prices and statement (i) is true.

The proof of statement (ii) is analogous and (iii) follows by (i) and (ii) and by the fact that $1 \in X$. ■

**Example 13.** Let $x_1 = (4, 4, 3, 3, 2, 2, 1, 1)$, $x_2 = (1, 1, 2, 2, 3, 3, 4, 4)$, and $X = [x_1, x_2]$. 1 is contained in $X$ because $x_1 + x_2 = 51$. The payoff matrix

$$A(x_i, e_i) = \begin{bmatrix} 4 & 1 \\ 4 & 1 \\ 3 & 2 \\ 3 & 2 \\ 2 & 3 \\ 1 & 4 \\ 1 & 4 \end{bmatrix},$$

has singular $2 \times 2$ submatrices, therefore the market is not strongly resolving.

In order to apply our theorem, we determine the positive basis of $F_1(X)$ which is a partition of the unit and we find that the vectors

$$b_1 = (1, 1, 0, 0, 0, 0, 0, 0), b_2 = (0, 0, 1, 1, 0, 0, 0, 0), b_3 = (0, 0, 0, 0, 1, 1, 0, 0), b_4 = (0, 0, 0, 0, 0, 0, 1, 1),$$

define this basis. We expand the vectors $x_i$ in the basis $\{b_i\}$ and we find that

$$A(x_i, b_i) = \begin{bmatrix} 4 & 1 \\ 3 & 2 \\ 2 & 3 \\ 1 & 4 \end{bmatrix},$$

is the payoff matrix with respect to this basis and we remark that the market is strongly resolving with respect to the basis $\{b_i\}$. Since $n = 2 \leq \frac{\mu + 1}{2} = \frac{5}{2}$ we have that any option written on elements of $X$ is non-replicated.

## 5 Appendix: Lattice-subspaces and positive bases in $C(\Omega)$

In this section we give the basic mathematical notions and results, which are needed for this article. $C(\Omega)$ is the space of real valued functions defined on a compact Hausdorff topological space $\Omega$. $C(\Omega)$ is ordered by the pointwise ordering, i.e for any $x, y \in C(\Omega)$ we have: $x \geq y$ if and only if $x(t) \geq y(t)$ for each $t \in \Omega$. $C_+(\Omega) = \{x \in C(\Omega) \mid x(t) \geq 0\}$ for each $t \in \Omega \}$ is the positive cone of $C(\Omega)$. Recall that if the set $\Omega$ is finite, for example if $\Omega = \{1, 2, ..., m\}$, then $C(\Omega)$ is the vector space $\mathbb{R}^m$, therefore the results presented below hold also for the space $\mathbb{R}^m$ which we use in this paper. But
we present the results in $C(\Omega)$ as they are formulated in [5] and [6]. The results of these articles are presented below.

The space $C(\Omega)$, ordered by the pointwise ordering is a vector lattice i.e. for any $x, y \in C(\Omega)$ the supremum $x \lor y$ and the infimum $x \land y$ of $\{x, y\}$ in $C(\Omega)$ exists. Suppose that $L$ is an ordered subspace of $C(\Omega)$, i.e. $L$ is a linear subspace of $C(\Omega)$ ordered again by the pointwise ordering. Then $L_+ = C_+(\Omega) \cap L$ is the positive cone of $L$. If $L$ is a vector lattice, i.e. if for any $x, y \in L$ the supremum $\sup_L \{x, y\}$ and the infimum $\inf_L \{x, y\}$ of $\{x, y\}$ in $L$ exist, then $L$ is a lattice-subspace of $C(\Omega)$. Then we have

$$\sup_L \{x, y\} \geq x \lor y \geq x \land y \geq \inf_L \{x, y\}.$$

If for any $x, y \in L$, $x \lor y \in L$ and $x \land y \in L$, $L$ is a sublattice of $C(\Omega)$, i.e. $L$ is a lattice-subspace of $C(\Omega)$. It is clear that any sublattice of $C(\Omega)$ is a lattice-subspace but the converse is not true. In general an ordered subspace $L$ of $C(\Omega)$ is not a lattice-subspace and also a lattice-subspace is not always a sublattice.

For any subset $B$ of $C(\Omega)$, the intersection of all sublattices of $C(\Omega)$ which contain $B$ is a sublattice of $C(\Omega)$ and it is the minimum sublattice of $C(\Omega)$ which contains $B$. This subspace is the sublattice of $C(\Omega)$ generated by $B$.

Suppose that $L$ is finite dimensional. A basis $\{b_1, b_2, \ldots, b_r\}$ of $L$ is a positive basis of $L$ if $L_+ = \{x = \sum_{i=1}^r \lambda_i b_i \mid \lambda_i \in \mathbb{R}_+ \text{ for each } i\}$. In other words, the basis $\{b_i\}$ of $L$ is positive if for any $x \in L$ we have: $x(i) \geq 0$ for any $i \in \Omega$ if and only if the coefficients $\lambda_i$ of $x$ in the basis $\{b_i\}$ are positive. Although $L$ has infinitely many bases the existence of a positive basis of $L$ is not always ensured.

Suppose that $\{b_1, b_2, \ldots, b_r\}$ is a positive basis of $L$. Then it is easy to show that for any $x = \sum_{i=1}^r \lambda_i b_i$, $y = \sum_{i=1}^r \mu_i b_i \in L$ we have $x \geq y$ if and only if $\lambda_i \geq \mu_i$ for each $i$. This property implies that $\sup_L \{x, y\} = \sum_{i=1}^r (\lambda_i \lor \mu_i) b_i$ and $\inf_L \{x, y\} = \sum_{i=1}^r (\lambda_i \land \mu_i) b_i$, therefore $L$ is a lattice-subspace. The converse is also true. One can prove it directly or by using the Choquet-Kenthal theorem, see in [6], Proposition 1.1. So we have the following:

**Theorem 14.** A finite-dimensional ordered subspace $L$ of $C(\Omega)$ is a lattice-subspace if and only if $L$ has a positive basis.

Also each vector $b_i$ of the positive basis of $L$ is an extremal point of $L_+$. (A vector $x_0 \in L_+, x_0 \neq 0$ is an extremal point of $L_+$ if for any $x \in L, 0 \leq x \leq x_0$ implies $x = \lambda x_0$ for some real number $\lambda$). This property implies that a positive basis of $L$ is unique in the sense of positive multiples.

**Theorem 15.** [6], Proposition 2.2] A finite-dimensional ordered subspace $L$ of $C(\Omega)$ is a sublattice of $C(\Omega)$ if and only if $L$ has a positive basis $\{b_1, b_2, \ldots, b_r\}$ with the property: $b_i^{-1}(0, +\infty) \cap b_j^{-1}(0, +\infty) = \emptyset$ for any $i \neq j$.

As an application of the above result we have:

**Theorem 16.** Suppose that $L$ is a sublattice of $\mathbb{R}^m$. If the constant vector $1 = (1, 1, \ldots, 1)$ is an element of $L$, then $L$ has a positive basis $\{b_1, b_2, \ldots, b_r\}$ which is a partition of the unit, i.e. the vectors $b_i$ have disjoint supports and $1 = \sum_{i=1}^r b_i$. This basis is unique.
Indeed, by the previous proposition $L$ has a positive basis $\{d_1, d_2, ..., d_r\}$ with disjoint supports. Since $1 \in L$ we have $1 = \sum_{i=1}^r \lambda_i d_i$ and for each $j \in \supp(d_i)$ we have $1 = 1(j) = \lambda_i d_i(j)$, therefore $d_i(j) = \frac{1}{\lambda_i}$ for any $j \in \supp(d_i)$. So each $d_i$ is constant on its support, therefore the basis $\{b_i = \lambda_i d_i\}$ is a positive basis of $L$ which is a partition of the unit.

We suppose now that $z_1, z_2, ..., z_r$ are fixed, linearly independent, positive vectors of $C(\Omega)$ and that

$$L = [z_1, z_2, ..., z_r],$$

is the subspace of $C(\Omega)$ generated by the vectors $z_i$. We study the problem: under what conditions $L$ is a lattice-subspace or a sublattice of $C(\Omega)$? In the case where $L$ fails to be a lattice-subspace we study if $L$ is contained in a finite-dimensional minimal lattice-subspace of $C(\Omega)$ or if the sublattice generated by $L$ is finite-dimensional.

The function

$$\beta(t) = \left(\frac{z_1(t)}{z(t)}, \frac{z_2(t)}{z(t)}, ..., \frac{z_r(t)}{z(t)}\right), \text{ for each } t \in \Omega, \text{ with } z(t) > 0,$$

where $z = z_1 + z_2 + ... + z_r$, is the basic function of $z_1, z_2, ..., z_r$. This function is very important for the study of lattice-subspaces and positive bases and has been defined in [5]. The set $R(\beta) = \{\beta(t) | t \in \Omega \text{ with } z(t) > 0\}$, is the range of $\beta$ and the cardinal number $\text{card}R(\beta)$ of $R(\beta)$ is the number of the (different) elements of $R(\beta)$. Under the above notations we have, see in [6] Theorem 3.6:

**Theorem 17 (Polyrakis).** $L$ is a sublattice of $C(\Omega)$ if and only if $\text{card}R(\beta) = r$.

If $R(\beta) = \{P_1, P_2, ..., P_r\}$, a positive basis $\{b_1, b_2, ..., b_r\}$ of $L$ is given by the formula:

$$(b_1, b_2, ..., b_r)^T = A^{-1}(z_1, z_2, ..., z_r)^T,$$

where $A$ is the $r \times r$ matrix whose the $i^{th}$ column is the vector $P_i$, for each $i = 1, 2, ..., r$, and $(b_1, b_2, ..., b_r)^T, (z_1, z_2, ..., z_r)^T$ are the matrices with rows the vectors $b_1, b_2, ..., b_r, z_1, z_2, ..., z_r$.

### 5.1 The algorithm for the sublattice generated by $L$

The next result gives an algorithm for the construction of the sublattice $Z$ of $C(\Omega)$ generated by a finite set $\{z_1, z_2, ..., z_r\}$ of linearly independent and positive vectors, in the case where $Z$ is finite-dimensional. In this case a positive basis of $Z$ is determined. As in the previous theorem, $\beta$ is the basic function of $z_1, z_2, ..., z_r$. Statement (d) determines the positive basis of $Z$. In fact (d) is an application of the previous theorem for the determination of a positive basis of $Z$. For more details, see in [6], Theorem 3.7.

**Theorem 18 (Polyrakis).** Let $Z$ be the sublattice of $C(\Omega)$ generated by $\{z_1, z_2, ..., z_r\}$ and let $\mu \in \mathbb{N}$. Then the statements (i) and (ii) are equivalent:

(i) $\dim(Z) = \mu$.

(ii) $R(\beta) = \{P_1, P_2, ..., P_\mu\}$.
If statement (ii) is true then $Z$ is constructed as follows:

(a) Enumerate $R(\beta)$ so that its $r$ first vectors to be linearly independent (such an enumeration always exists). Denote again by $P_i$, $i = 1, 2, \ldots, \mu$ the new enumeration and we put $I_{r+k} = \{ t \in \Omega | \beta(t) = P_{r+k} \}$, for each $k = 1, 2, \ldots, \mu - r$.

(b) Define the vectors $z_{r+k}$, $k = 1, 2, \ldots, \mu - r$ as follows:

$$z_{r+k}(i) = z(i) \text{ if } i \in I_{r+k} \text{ and } z_{r+k}(i) = 0 \text{ if } i \notin I_{r+k},$$

where $z = z_1 + z_2 + \ldots + z_r$ is the sum of the vectors $z_i$.

(c) $Z = [z_1, z_2, \ldots, z_r, z_{r+1}, \ldots, z_{\mu}]$.

(d) A positive basis $\{b_1, b_2, \ldots, b_{\mu}\}$ of $Z$ is constructed as follows:

Consider the basic function $\gamma$ of $z_1, z_2, \ldots, z_r, z_{r+1}, \ldots, z_{\mu}$ and suppose that $\{P'_1, P'_2, \ldots, P'_\mu\}$ is the range of $\gamma$ (the range of $\gamma$ has exactly $\mu$ points). Then

$$(b_1, b_2, \ldots, b_{\mu})^T = D^{-1}(z_1, z_2, z_3)^T,$$

where $D$ is the $\mu \times \mu$ matrix with columns the vectors $P'_1, P'_2, \ldots, P'_\mu$.

References


