Dynamic Consistency and Subjective Beliefs

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Abstract

Using the notion of subjective beliefs from Rigotti et al. (2008), we provide, for a wide variety of dynamic models with ambiguity aversion, a connection between Dynamic Consistency, prior by prior Bayesian updating of beliefs revealed by trading behavior and positive value of information. We apply these characterizations in a multi agent setting. First, we show that a weakening of DC, consistent with Ellsberg type behavior, precludes speculative trade, generalizing the result of Milgrom and Stokey (1982). Second, we show that, in a risk sharing environment with no aggregate risk, if the value of information is positive for all agents, then the value of public information is negative, generalizing the result of Schlee (2001).

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1 Introduction

In dynamic-choice problems under uncertainty, the decision maker updates his preferences and his beliefs as new information arrives and takes optimal actions in multiple periods. A natural constraint on how these preferences are updated is placed by Dynamic Consistency (DC). It requires that an action is optimal when evaluated with the updated preferences of a later period if and only if it is also optimal when evaluated from the perspective and preferences of an earlier period. DC ensures that an ex ante optimal plan of action will remain optimal at every period and irrespective of how information is updated.

DC can be motivated on the grounds that, without it, the value of information can be negative (Wakker (1988), Epstein and Le Breton (1993)), whereas under subjective expected utility it is implied by Bayesian updating of beliefs. Moreover, under some conditions, DC is equivalent with Bayesian updating.1 On the other hand, Hanany

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1See Ghirardato (2002) and Hanany and Klibanoff (2007) for formal results.
and Klibanoff (2007) argue that DC is inconsistent with Ellsberg type behavior and propose weakening it.

In this paper, we characterize these three notions. In particular, we show that a weakening of the “if” direction of DC is equivalent to prior by prior Bayesian updating, whereas the “only if” direction is equivalent to the value of information being positive and implies Bayesian updating. Moreover, the weakening of DC that we propose is consistent with Ellsberg type behavior.

The novelty of our approach is that we characterize the Bayesian updating of the beliefs revealed by potential trading behavior, instead of the beliefs which are part of the utility representation of preferences, which is the usual route. We call these the subjective beliefs, after Rigotti et al. (2008), who formalized this notion in a static setting, based on an idea of Yaari (1969).

Rigotti et al. (2008) identified the subjective beliefs for a wide variety of static models with ambiguity averse preferences. This enables us to formulate a theory of updating under ambiguity which applies to all these models and therefore is fully general. But more importantly, Bayesian updating of subjective beliefs is, in many cases, enough to analyse behavior in dynamic settings, without the need to get into the details of a specific dynamic model with ambiguity aversion, which could be complicated.

We provide two applications to support this claim. The first is the familiar result of Milgrom and Stoney (1982), that starting from an ex ante efficient allocation, it cannot be common knowledge in the interim stage that there is another allocation which Pareto dominates it. This result, known as the absence of speculative trade, was shown in the subjective expected utility (SEU) model, where both DC and Bayesian updating hold. We weaken both directions of DC, thus retaining only the Bayesian updating of subjective beliefs. This implies that behavior consistent with Ellsberg will still preclude speculative trade.

The second application examines the value of information in a competitive risk sharing environment without aggregate uncertainty, where agents trade state-contingent claims. In the SEU model with risk aversion, it is shown with an example by Hirshleifer (1971) and more generally by Schlee (2001) that public information makes everyone weakly worse off. Again, the full force of DC and Bayesian updating of the common prior is assumed. We show that if information is positively valued at the individual level, and therefore only one direction of DC is assumed, then subjective beliefs are updated using Bayes’ rule and public information makes everyone weakly worse off.

1.1 Related literature

Epstein and Schneider (2003) provide a recursive version of the multiple priors model of Gilboa and Schmeidler (1989) and show that DC is equivalent to prior by prior Bayesian updating of a rectangular set of probability measures used in representing the unconditional preferences of the agent. However, this characterization is applied only on a restricted set of events. Hanany and Klibanoff (2007) relax Consequentialism, allowing the preferences conditional on an event to also depend on the feasible set of acts and on the act that was chosen ex ante. They show that a weakening of DC
is equivalent to Bayesian updating of a subset (not necessarily rectangular) of these measures. Hanany and Klibanoff (2009) generalize this approach to many models of ambiguity aversion, showing that weak DC is equivalent to having at least one measure which supports both the conditional indifference curve at the chosen act and its conditional optimality. However, it is not shown that the there is a set of measures that is updated using Bayes’ rule.

Maccheroni et al. (2006b) and Klibanoff et al. (2009) provide the recursive versions of the static models of Maccheroni et al. (2006a) and Klibanoff et al. (2005), respectively, assuming DC. Siniscalchi (2011) studies ambiguity averse preferences over decision trees and relaxes DC, using the Sophistication Axiom.

Rigotti et al. (2008) have identified the subjective beliefs generated by a large number of models of ambiguity aversion, making our approach very general. These models are the convex Choquet model of Schmeidler (1989), the multiple priors model of Gilboa and Schmeidler (1989), the variational preferences model of Maccheroni et al. (2006a), the multiplier model of Hansen and Sargent (2001), the smooth second-order prior models of Klibanoff et al. (2005) and Nau (2006), the confidence preferences model of Chateauneuf and Faro (2009) and the second-order expected utility model of Ergin and Gul (2009).

Wakker (1988) shows that if Independence is violated, the value of information is not always positive. Independence is related to DC and, under some conditions (e.g., English auctions), equivalent (Karni and Safra (1986)). Grant et al. (2000) provide necessary and sufficient conditions for a weakly dynamically consistent agent to always prefer more information. Our approach differs from theirs in two respects. First, they adopt the definition of “more information” suggested by Blackwell (1951), whereas we adopt the definition of a finer partition. Second, they adopt a different weakening of DC, due to Machina (1989), which requires that an agent conforms to what he would have chosen ex ante only if he were able to commit. Snow (2010) examines the value of information in the special case where it either reduces or eliminates ambiguity, using the model of Klibanoff et al. (2005). Li (2011) studies the link between ambiguity attitudes and aversion to receiving information when one is completely uninformed.

Milgrom and Stokey (1982) show in the SEU model with a common prior and differential information that, starting in the ex ante stage from a Pareto efficient allocation, it cannot be common knowledge in the interim stage that there is an allocation which Pareto dominates it. Geanakoplos (1992) interprets this as the absence of speculative trade. The result relies on Dynamic Consistency and Consequentialism. Halevy (2004) shows that Consequentialism can be weakened, using Resolute Conditional preferences and Conditional Decomposition, but retains the full force of DC. In this paper, we prove the same result by weakening DC and assuming Consequentialism. In the Appendix, we also weaken Consequentialism by assuming Resolute Conditional preferences, but not Conditional Decomposition.

The paper proceeds as follows. Section 2 presents the model, whereas in Section 3 we provide the characterizations between DC, valuable information and Bayesian updating of subjective beliefs. The two applications for the multi-agent case are presented in Section 4. In the Appendix, Section A.1, we show that the characterization of Bayesian

2The two definitions are closely related, as shown by Green and Stokey (1978).
updating and the absence of speculative trade do not require Consequentialism. In Section A.2 we show that under Consequentialism, behavior consistent with Ellsberg can lead to a negative value of information.

2 Model

2.1 Preliminaries

There is a single consumption good. We fix a finite set of payoff relevant states \( S \), with typical element \( s \). The set of consequences is \( \mathbb{R}_+ \), interpreted as monetary payoffs. The set of acts is \( \mathcal{F} = \mathbb{R}^S \) with the natural topology. Acts are denoted by \( f, g, h \), while \( f(s) \) denotes the monetary payoff from act \( f \) when state \( s \) obtains. For any \( x \in \mathbb{R}_+ \) we abuse notation by writing \( x \in \mathcal{F} \), which stands for the constant act with payoff \( x \) in each state of the world. An act \( f \) is strictly positive if \( f(s) > 0 \) for all \( s \in S \). For any two acts \( f, g \in \mathcal{F} \) and event \( E \subseteq S \), we denote by \( f \uparrow E g \) the act \( h \) such that \( h(s) = f(s) \) if \( s \in E \) and \( h(s) = g(s) \) if \( s \notin E \). Define \( f \geq_E g \) if \( f(s) \geq g(s) \) for all \( s \in E \), with strict inequality for some \( s \in E \). Define \( f =_E g \) if \( f(s) = g(s) \) for all \( s \in E \). Let \( E^c \) be the complement of \( E \) with respect to \( S \).

Let \( \mathcal{E} \) be a collection of nonempty events \( E \subseteq S \) which contains \( S \). One can think of \( \mathcal{E} \) as the collection of events generated by a filtration, as in Epstein and Schneider (2003) and Maccheroni et al. (2006a). The decision maker is endowed with a collection of conditional preference relations, \( \{ \succsim_E \}_{E \in \mathcal{E}} \), one for each event \( E \in \mathcal{E} \). We associate \( \succsim_S \) with the ex ante preference relation \( \succsim \). We say that event \( F \subseteq E \) is \( \succsim_E \)-null if, for any acts \( h, h' \), \( h =_F h' \) implies \( h \sim_E h' \). It is weakly \( \succsim_E \)-null if there exists act \( h \) such that, for any act \( h' \), \( h =_F h' \) implies \( h \succsim_E h' \).

2.2 Axioms

We say that preferences \( \{ \succsim_E \}_{E \in \mathcal{E}} \) are convex if they satisfy the following 4 Axioms, for each event \( E \in \mathcal{E} \).

**Axiom 1.** (Preference). \( \succsim_E \) is complete and transitive.

**Axiom 2.** (Continuity). For all \( f \in \mathcal{F} \), the sets \( \{ g \in \mathcal{F} : g \succsim_E f \} \) and \( \{ g \in \mathcal{F} : f \succsim_E g \} \) are closed.

**Axiom 3.** (Monotonicity). For all \( f, g \in \mathcal{F} \), if \( f(s) > g(s) \) for all \( s \in E \), then \( f \succ_E g \).

**Axiom 4.** (Convexity). For all \( f \in \mathcal{F} \), the set \( \{ g \in \mathcal{F} : g \succsim_E f \} \) is convex.

The next three Axioms are only used in the competitive risk sharing model of Section 4.2. The first two are the strong versions of Monotonicity and Convexity. The third is proposed and discussed by Rigotti et al. (2008), who show that it is satisfied by most classes of ambiguity averse preferences.

**Axiom 5.** (Strong Monotonicity). For each event \( E \in \mathcal{E} \), for all \( f \neq_E g \), if \( f \geq_E g \), then \( f \succ_E g \).
Axiom 6. (Strict Convexity). For each event $E \in \mathcal{E}$, for all $f \not\equiv_E g$ and $\alpha \in (0, 1)$, if $f \succeq_E g$, then $\alpha f + (1 - \alpha)g \succ_E g$.

Axiom 7. (Translation Invariance at Certainty). For each event $E \in \mathcal{E}$, for all $g \in \mathbb{R}^S$ and all constant bundles $x, x' > 0$, if $x + \lambda g \succeq_E x$ for some $\lambda > 0$, then there exists $\lambda' > 0$ such that $x' + \lambda' g \succeq_E x$.

The following Axiom, which is stronger than the usual Full Support Axiom but weaker than Strong Monotonicity, ensures that each subjective belief (defined in Section 2.3) assigns positive probability to each event $F \in \mathcal{E}$.

Axiom 8. (Strong Full Support). For all events $E, F \in \mathcal{E}$, where $F \subseteq E$, if $f \succeq_{F} g$ and $f \equiv_{F} g$ then $f \succeq_{E} g$.

The last three Axioms provide restrictions on how preferences are updated when new information arrives. Consequentialism requires that if the agent knows that event $E$ has occurred, his preferences depend only on what acts specify inside $E$. In the Appendix, we relax Consequentialism for some of our results.

Axiom 9. (Consequentialism) For all $f, g$ and each event $E \in \mathcal{E}$, if $f =_{E} g$, then $f \sim_{E} g$.

Finally, DC provides restrictions on how two acts, which are identical outside of the conditioning event $F$, should be compared before and after $F$ is known to have occurred. We break it into two Axioms and adopt the names proposed by Ghirardato (2002).

Axiom 10. (Information is Valuable) For all acts $f, g \in \mathcal{F}$ and events $E, F \in \mathcal{E}$, $F \subseteq E$, if $f \succeq_{F} g$ and $f =_{F} g$ then $f \succeq_{E} g$.

Axiom 11. (Consistency of Implementation) For all acts $f, g \in \mathcal{F}$ and events $E, F \in \mathcal{E}$, $F \subseteq E$, if $f \succeq_{E} g$ and $f =_{F} g$ then $f \succeq_{F} g$.

2.3 Subjective beliefs

Given a measure $p \in \Delta E$, where $E \subseteq S$ and $p(F) > 0$, denote by $p_{F} \in \Delta F$ the measure obtained through Bayesian conditioning of $p$ on $F$. We write $\mathbb{E}_p f := \sum_{s \in E} p(s) f(s)$ for the expectation of $f$ given $p$.

Rigotti et al. (2008) define the subjective beliefs at an act $f$ to be $\pi(f) = \{p \in \Delta S : \mathbb{E}_p g \geq \mathbb{E}_p f \text{ for all } g \succeq f\}$, the set of all (normalized) supporting hyperplanes of $f$. They provide two alternative definitions for subjective beliefs and show that, for strictly positive acts and convex preferences, the three definitions are equivalent. The first set contains the subjective beliefs revealed by unwillingness to trade at $f$, $\pi^u(f) = \{p \in \Delta S : f \succeq g \text{ for all } g \text{ such that } \mathbb{E}_p g = \mathbb{E}_p f\}$. For the second, let $\mathcal{P}(f)$ denote the collection of all compact, convex sets $P \subseteq \Delta S$ such that if $\mathbb{E}_p g \geq \mathbb{E}_p f$ for all $p \in P$, then $\mathbb{E}_p [g + (1 - \epsilon) f] \succ f$ for sufficiently small $\epsilon$. Then, the subjective beliefs revealed by willingness to trade at $f$ are denoted by $\pi^w(f) = \bigcap \mathcal{P}(f)$. We extend these definitions to all events $E \in \mathcal{E}$ and conditional preferences $\succeq_E$. 
2.4 Mutual absolute continuity

A convenient and often assumed property of dynamic models with ambiguity is mutual absolute continuity of the priors used to represent preferences. It says that all priors put positive probability on the same events. This facilitates the Bayesian updating of all the priors when new information arrives, without worrying what to do with priors that assign zero probability on the event. Epstein and Marinacci (2007) characterize mutual absolute continuity in the multiple priors model, using a condition introduced by Kreps (1979). In the following Lemma we characterize mutual absolute continuity of the subjective beliefs, using the notion of a weakly null event.

**Lemma 1.** For all events $E, F \in \mathcal{E}$, $F \subseteq E$, if $F$ is not weakly $\succsim_E$-null then $p \in \pi^u_E(f)$ implies $p(F) > 0$, for all acts $f$. Conversely, if, for all acts $f$, $p \in \pi^u_E(f)$ implies $p(F) > 0$, then $F$ is not weakly $\succsim_E$-null.

**Proof.** Suppose $p \in \pi^u_E(f)$ and $p(F) = 0$. Take any $h$ such that $h =_F f$. Because $\mathbb{E}_p f = \mathbb{E}_p h$, we have $f \succsim_E h$, implying that $F$ is weakly $\succsim_E$-null. Conversely, suppose that for all acts $f$, $p \in \pi^u_E(f)$ implies $p(F) > 0$. Suppose there exists act $f$ such that for all $h$ with $f =_F h$, $f \succsim_E h$. Let $k > 0$ and define act $h$ such that $h(s) = f(s) + k$ if $s \in F$ and $h(s) = f(s)$ otherwise. Then, for all $p \in \pi^u_E(f)$, $\mathbb{E}_p h > \mathbb{E}_p f$. From the definition of $\pi^u_E(f)$, there exists small enough $\epsilon > 0$, such that $h' = \epsilon h + (1 - \epsilon)f$ and $h' \succsim_F f$. Because $f =_F h'$, we have a contradiction. \( \square \)

Using this Lemma, Axiom 8 implies that all subjective beliefs $p \in \pi^u_E(f)$ put positive probability at each event $F \in \mathcal{E}$, where $F \subseteq E$. We next define prior by prior Bayesian updating of subjective beliefs.

**Definition 1.** The subjective beliefs of $\{\succsim_E\}_{E \in \mathcal{E}}$ are updated using Bayes’ rule if, for all $E, F \in \mathcal{E}$, $F \subseteq E$, and all $f \in \mathcal{F}$, if $p \in \pi_E(f)$ and $p(F) > 0$ then $p_F \in \pi_F(f)$. Bayesian updating of subjective beliefs revealed by unwillingness to trade, $\pi^u_E(f)$, is similarly defined.

2.5 Weak Dynamic Consistency

Using the notion of subjective beliefs, we are able to weaken DC. Recall that $\pi^u_E(f)$ are the beliefs revealed by unwillingness to trade at $f$, given preferences $\succsim_E$, where $E \in \mathcal{E}$. We can interpret $\pi^u_E(f)$ as the set of (normalized) Arrow-Debreu prices for which the agent, endowed with $f$, would have zero net demand.

The “dual” of $\pi^u_E(f)$ is the set of acts for which $f$ is revealed preferred to them. In particular, if $\mathbb{E}_p f \geq \mathbb{E}_p g$ then act $g$ is affordable given normalized price $p$ and endowment $f$. If $p \in \pi^u_E(f)$, then $f \succsim_E g$, which means that $f$ is revealed preferred to $g$.

We weaken Axiom 12 by requiring that if $f$ is revealed preferred to $g$ and the two acts are identical outside of $F \subseteq E$, then $f$ is weakly preferred to $g$, conditional on $F$. Formally, for act $f$ and event $E \in \mathcal{E}$, let $\mathcal{R}_E(f)$ be the set of acts such that $f$ is revealed preferred to them given preferences $\succsim_E$,

$$\mathcal{R}_E(f) = \{ g \in \mathcal{F} : \mathbb{E}_p f \geq \mathbb{E}_p g \text{ for some } p \in \pi^u_E(f) \}.$$
Axiom 12. (Weak Consistency of Implementation) For all acts \( f, g \in \mathcal{F} \) and events \( E, F \in \mathcal{E}, F \subseteq E \), if \( g \in \mathcal{R}_E(f) \) and \( f =_{F^c} g \) then \( f \gtrsim F g \).

Axiom 12 is weaker than Axiom 11 for convex preferences. It is also related to the Dynamic Consistency Axiom proposed by Hanany and Klibanoff (2007) in the dynamic version of the multiple priors model. They define a quadruple \( (\succsim_E, F, f, B) \), which consists of the initial preference relation \( \succsim_E \), the conditioning event \( F \subseteq E \) and the act \( f \) that was chosen from feasible set \( B \), before the realization of \( F \). Their Dynamic Consistency Axiom requires that if another act \( g \) is feasible (and therefore \( f \) is revealed preferred to \( g \) according to our terminology) and \( f =_{F^c} g \), then \( f \gtrsim_{F,B} g \).

The other direction of DC, Axiom 10, is weakened in a similar manner. Let \( S_E(f) = \{ g \in \mathcal{F} : \mathbb{E}_p f > \mathbb{E}_p g \ \text{for some} \ p \in \pi_E^u(f) \} \) be the set of acts such that \( f \) is strictly revealed preferred to them given preferences \( \succsim_E \). The following Axiom is weaker than Axiom 10 for convex preferences and strictly positive acts.

Axiom 13. (Weak Value of Information) For all acts \( f, g \in \mathcal{F} \) and events \( E, F \in \mathcal{E}, F \subseteq E \), if \( g \succsim_F f \) and \( f =_{F^c} g \) then \( g \not\in S_E(f) \).

3. DC, Bayesian updating and the value of information

3.1 Characterization of Bayesian updating

In this Section we show that Weak Consistency of Implementation, which is weaker than the second part of DC, is equivalent to Bayesian updating of the subjective beliefs revealed by unwillingness to trade.

Proposition 1. Suppose that convex preferences \( \{\succsim_E\}_{E \in \mathcal{E}} \) satisfy Axioms 8 and 9. Then, subjective beliefs revealed by unwillingness to trade are updated using Bayes’ rule if and only if Axiom 12 is satisfied.

Proof. Fix events \( E, F \in \mathcal{E}, F \subseteq E \). Suppose \( g \in \mathcal{R}_E(f) \) and \( f =_{F^c} g \). Then, \( \mathbb{E}_p f \geq \mathbb{E}_p g \) for some \( p \in \pi_E^u(f) \). Axiom 8 and Lemma 1 imply that \( p(F) > 0 \). Because subjective beliefs are updated using Bayes’ rule and \( f =_{F^c} g \), we have \( \mathbb{E}_{p_F} f \geq \mathbb{E}_{p_F} g \) and \( p_F \in \pi_F^u(f) \). Axiom 3 implies \( f \gtrsim_F g \).

Conversely, suppose that there exists \( p \in \pi_E^u(f) \) with \( p(F) > 0 \) and \( p_F \notin \pi_F^u(f) \). Then, there exists act \( g \) such that \( g \succ F f \) and \( \mathbb{E}_{p_F} g = \mathbb{E}_{p_F} f \). From Axiom 9, we have that \( gF f \sim_F g \succ_F f \). Because \( \mathbb{E}_{p_F}(gF f) = \mathbb{E}_{p_F} g \) and \( gF f =_{F^c} f \), we have that \( \mathbb{E}_p(gF f) = \mathbb{E}_p f \), hence \( gF f \in \mathcal{R}_E(f) \). Axiom 12 implies that \( f \gtrsim_F gF f \), a contradiction.

Note that we have assumed Consequentialism (Axiom 9). In the context of the multiple priors model, Hanany and Klibanoff (2007) show that their Dynamic Consistency Axiom is equivalent to Bayesian updating of a subset of the measures used in
representing the unconditional preferences of the agent. They relax Consequentialism, as they condition the preference relation not only on event $F$ but also on the chosen act $f$ and the feasible set $B$. In the Appendix, we show that Proposition 1 can be proven by weakening Consequentialism, using the notion of Resolute Conditional preferences, suggested by Halevy (2004).

3.2 Characterization of valuable information

In this Section we show that the first part of DC (Axiom 10) is equivalent to information being valuable. Moreover, under Axiom 6 it implies the Bayesian updating of the subjective beliefs revealed by unwillingness to trade.

In order to examine the value of information, we adopt the framework of Geanakoplos (1989). An agent faces some uncertainty ex ante, represented by a finite state space $S$. He has a feasible set of acts. His information structure is represented by a partition of $S$. More information is represented by a finer partition. In the interim stage, the agent is informed that a particular cell of his partition has occurred and chooses an optimal act, out of the feasible set, using his conditional preferences. This is repeated for each cell of the partition, generating an act that the agent can evaluate ex ante. By comparing these generated acts ex ante, the agent can evaluate different partitions. We say that information is valuable if a finer partition always generates an act which is weakly preferred ex ante.

More formally, a decision problem $D = \{E, \Pi, A, \{\succsim_E\}_{E \in E}\}$ consists of an event $E \in \mathcal{E}$, a partition $\Pi \subseteq \mathcal{E}$ of $E$, a set of available acts $A \subseteq \mathcal{F}$ and a collection of conditional preference relations $\{\succsim_E\}_{E \in \mathcal{E}}$. When the agent is informed that $F \in \Pi$ has occurred, he chooses among the available acts in $A$, using his conditional preference relation $\succsim_F$.

The choices for each $F \in \Pi$ generate an act, which is measurable with respect to $\Pi$. Formally, an act $f \in \mathcal{F}$ is measurable for decision problem $D$ if for each $F \in \Pi$ there exists $h \in A$ such that $f =_F h$. Let $\mathcal{F}_D$ be the set of acts which are measurable with respect to decision problem $D$. Note that $A \subseteq \mathcal{F}_D$, whereas if $\Pi = \{E\}$ is the trivial partition, then $A = \mathcal{F}_D$. An act is optimal if, conditional on each element of the partition, it is weakly preferred against any other available act.

**Definition 2.** Act $f \in \mathcal{F}_D$ is optimal for decision problem $D = \{E, \Pi, A, \{\succsim_E\}_{E \in \mathcal{E}}\}$ if for all $F \in \Pi$, $f \succsim_F g$ for all $g \in A$.

We compare two decision problems that differ only in terms of how they partition $E$, by comparing the optimal acts they generate, according to preference relation $\succsim_E$.

**Definition 3.** Decision problem $D_1 = \{E, \Pi_1, A, \{\succsim_E\}_{E \in \mathcal{E}}\}$ is more valuable than decision problem $D_2 = \{E, \Pi_2, A, \{\succsim_E\}_{E \in \mathcal{E}}\}$ if, whenever act $f$ is optimal for $D_1$ and act $g$ is optimal for $D_2$, we have $f \succsim_E g$.

We say that partition $\Pi_1$ is finer than partition $\Pi_2$ if every element of $\Pi_2$ is the union of some elements of $\Pi_1$. Information is valuable for $\{\succsim_E\}_{E \in \mathcal{E}}$ if a decision problem generated by a finer partition is always more valuable.
**Definition 4.** Information is valuable for \( \{\succeq_E\}_{E \in \mathcal{E}} \) if, for all \( E \in \mathcal{E} \), whenever partition \( \Pi_1 \) of \( E \) is finer than partition \( \Pi_2 \) of \( E \), decision problem \( D_1 = (E, \Pi_1, A, \{\succeq_E\}_{E \in \mathcal{E}}) \) is more valuable than decision problem \( D_2 = (E, \Pi_2, A, \{\succeq_E\}_{E \in \mathcal{E}}) \).

We now show that, under Consequentialism, valuable information is equivalent to Axiom 10.

**Proposition 2.** Suppose convex preferences \( \{\succeq_E\}_{E \in \mathcal{E}} \) satisfy Axiom 9. Then, information is valuable for \( \{\succeq_E\}_{E \in \mathcal{E}} \) if and only if Axiom 10 is satisfied.

**Proof.** Suppose that for any event \( F \in \mathcal{E} \) such that \( F \subseteq E \) and all acts \( f, g \), \( f \succeq_F g \) implies \( fFg \succeq_E g \). Consider decision problems \( D_1 = (E, \Pi_1, A, \{\succeq_E\}_{E \in \mathcal{E}}) \) and \( D_2 = (E, \Pi_2, A, \{\succeq_E\}_{E \in \mathcal{E}}) \), where \( \Pi_1, \Pi_2 \subseteq \mathcal{E} \) are partitions of \( E \) and \( \Pi_1 \) is finer than \( \Pi_2 \). Let act \( f \in \mathcal{F}_{D_1} \) be optimal for \( D_1 \) and act \( g \in \mathcal{F}_{D_2} \) be optimal for \( D_2 \). Since \( \Pi_2 \) is coarser than \( \Pi_1 \), \( \mathcal{F}_{D_2} \subseteq \mathcal{F}_{D_1} \). This means that, for all \( F \in \Pi_1 \), \( f \succeq_F g \).

Enumerate the partition cells of \( \Pi_1 = \{F_1, \ldots, F_n\} \). If \( n = 1 \) then \( \Pi_1 = \{E\} \) is the uninformative partition and the result is immediate, so suppose that \( n \geq 2 \). For cell \( 1 \leq k \leq n \) define act \( h_k \) as follows. Let \( h_k(s) = f(s) \) if \( s \in F_j \), where \( 1 \leq j \leq k \), and \( h_k(s) = g(s) \) otherwise. Note that \( h_n = f \) and let \( h_0 = g \). From Axiom 9 we have that, for each \( 1 \leq k \leq n \), \( f \succeq_{F_k} g \) implies \( h_k \succeq_{F_k} h_{k-1} \). Applying Axiom 9 we have \( h_k \succeq_E h_{k-1} \), for each \( 1 \leq k \leq n \). Axiom 1 implies that \( f \succeq_E g \).

Conversely, suppose that information is valuable for \( \{\succeq_E\}_{E \in \mathcal{E}} \) at \( E \in \mathcal{E} \). Suppose that for some event \( F \in \mathcal{E} \), where \( F \subseteq E \) and acts \( f, g \), we have \( f \succeq_F g \) but \( g \succ_E fFg \). Consider partitions \( \Pi_1 = \{F, E \setminus F\} \) and \( \Pi_2 = \{E\} \). Let \( A = \{g, h\} \), where \( h = fFg \). Then, \( g \) is optimal for decision problem \( D_2 = (E, \Pi_2, A, \{\succeq_E\}_{E \in \mathcal{E}}) \). Because \( f \succeq_F g \), Axiom 9 implies that \( fFg \succeq_F g \) and \( fFg \succeq_{E \setminus F} g \). Therefore, \( h = fFg \) is optimal for decision problem \( D_1 = (E, \Pi_1, A, \{\succeq_E\}_{E \in \mathcal{E}}) \). Because information is valuable for \( \{\succeq_E\}_{E \in \mathcal{E}} \) at \( E \) and \( \Pi_1 \) is finer than \( \Pi_2 \), we have that \( D_1 \) is more valuable than \( D_2 \), which implies \( fFg \succeq_E g \), a contradiction. Therefore, \( fFg \succeq_E g \).

Finally, the following Lemma relates valuable information with the Bayesian updating of the subjective beliefs revealed by unwillingness to trade.

**Lemma 2.** Suppose convex preferences \( \{\succeq_E\}_{E \in \mathcal{E}} \) satisfy Axioms 6 and 9. If information is valuable then subjective beliefs revealed by unwillingness to trade are updated using Bayes’ rule.

**Proof.** Fix \( E \in \mathcal{E} \). Let \( p \in \pi_E(f) \) with \( p(F) > 0 \) and suppose that \( p_F \notin \pi_F(f) \). Then, there exists act \( g \) such that \( E_{pF}f = E_{pF}g \) and \( g \succ_F f \). Proposition 2 implies that \( gFf \succeq_E f \). We also have that \( E_pgFf = E_pf \), which implies that \( f \succeq_E gFf \). From Axiom 6, for all \( a \in (0, 1) \), \( af + (1 - a)gFf \succ_E gFf \). Because \( E_pFf = E_pf \), we have that \( E_p(af + (1 - a)gF) = E_pf \), which implies that \( f \succeq_E af + (1 - a)gFf \). From Axiom 1 we have \( f \succ_E gFf \), a contradiction.

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3Interestingly, if we do not assume that the agent’s information is represented by a partition, DC does not imply that the information is valuable. An example, in the context of unawareness, is provided in Galanis (2014).
4 Applications

4.1 Speculative trade

In this Section we show that weak DC (Axioms 12 and 13) and Consequentialism (Axiom 9) are enough to ensure the absence of speculative trade. By discarding Axiom 13, we can show the absence of weak speculative trade, where the allocation is only required to be weakly efficient. An allocation is weakly efficient if there does not exist another feasible allocation that everyone strictly prefers.

Each agent $i$ is endowed with a partition $\Pi^i$ of $S$. Let $\Pi^i(s)$ be the cell of the partition containing state $s \in S$. At $s$, agent $i$ considers states in $\Pi^i(s)$ to be possible and has conditional preferences $\gtrsim_{\Pi^i(s)}^i$. Given an event $E \subseteq S$, let $K^i(E) = \{s \in S : \Pi^i(s) \subseteq E\}$ be the set of states where $i$ knows $E$. Event $E$ is self evident if $E \subseteq K^i(E)$ for all $i \in I$. That is, an event is self evident if whenever it happens, everyone knows it. An event $F$ is common knowledge at $s$ if and only if there exists a self evident event $E$ such that $s \in E \subseteq F$ (Aumann (1976)).

**Proposition 3.** Suppose convex preferences $\{\gtrsim_E^i\}_{i \in I, E \in \mathcal{E}}$ satisfy Axioms 9, 12 and let $f$ be an interior, ex ante weakly efficient allocation. Then, there exist no state $s$ and a feasible allocation $g$ such that $G = \{s \in S : g^i \gtrsim_{\Pi^i(s)}^i f^i \text{ for all } i \in I\}$ is common knowledge at $s$. If, in addition, preferences $\{\gtrsim_E^i\}_{i \in I, E \in \mathcal{E}}$ satisfy Axiom 13 and $f$ is an interior, ex ante efficient allocation, then there exist no state $s$ and a feasible allocation $g$ such that $H = \{s \in S : g^i \gtrsim_{\Pi^i(s)}^i f^i \text{ for all } i \in I\}$, where for some $s \in H$ and $i \in I$, $g^i \gtrsim_{\Pi^i(s)}^i f^i$, is common knowledge at $s$.

**Proof.** Because $f$ is an ex ante weakly efficient allocation, there does not exist a feasible allocation $h$ such that $h^i \gtrsim f^i$ for all $i \in I$. Suppose that there exists feasible allocation $g$ such that $G$ is common knowledge at $s \in S$. Let $F$ be a self evident event such that $s \in F \subseteq G$. Note that each $\Pi^i$ partitions $F$. Then, we have that for each $i \in I$, for each $s' \in F$, $g^i \gtrsim_{\Pi^i(s')}^i f^i$. From Axiom 9, $g^i\Pi^i(s')f^i \gtrsim_{\Pi^i(s')}^i f^i$. From Axiom 12, we have that for all $p \in \pi^u(f)$, $\mathbb{E}_p(g^i\Pi^i(s')f^i) > \mathbb{E}_p f^i$. Because this is true for all $s' \in S$ such that $\Pi^i(s') \subseteq F$ and $\Pi^i$ partitions $F$, we have that $\mathbb{E}_p(g^iFf^i) > \mathbb{E}_p f^i$, for all $p \in \pi^u(f)$. Define $h^i = g^iFf^i$ and $h = \{h^i\}_{i \in I}$.

Allocation $f$ is interior, hence $\pi^u(f) = \pi^w(f) = \pi(f)$. This implies that for small enough $\epsilon$, $\epsilon h^i + (1 - \epsilon)f^i \gtrsim f^i$. By taking $\epsilon < \epsilon^i$ for all $i \in I$, we have that $\epsilon h^i + (1 - \epsilon)f^i \gtrsim f^i$ for all $i \in I$. Moreover, $\epsilon h + (1 - \epsilon)f$ is feasible because both $f$ and $h$ are feasible. Hence, we have arrived at a contradiction.

For the second claim, by using the same arguments and Axiom 13 we get that $\mathbb{E}_p h^j \geq \mathbb{E}_p f^i$, for all $p \in \pi^j(f)$, for all $i \in I$, strict inequality for some $j$. By reducing $h^j$ for all $s \in S$ by a sufficiently small amount and distributing it to the other agents, we can find a new feasible allocation $h$ such that $\mathbb{E}_p h^i > \mathbb{E}_p f^i$, for all $p \in \pi^j(f)$, for all $i \in I$. Using the same arguments as in the previous paragraph, we reach a contradiction.

Employing the results of Section 3, we can provide a new way of interpreting the absence of speculative trade, in the presence of Consequentialism. First, if beliefs
revealed by unwillingness to trade are updated using Bayes’ rule, then weak speculative trade is impossible. Adding Axiom 13, which is weaker than assuming that information is valuable for each agent, is enough to preclude speculative trade. As we show in the Appendix, Axiom 13 is consistent with Ellsberg preferences, which implies that agents with these preferences will not engage in speculative trade.

Halevy (2004) shows the absence of speculative trade by assuming DC but relaxing Consequentialism to the following two properties: Resolute Conditional preferences and Conditional Decomposability. In the Appendix, we show that weak DC and Resolute Conditional preferences are enough to preclude speculative trade. Hence, neither Consequentialism nor Conditional Decomposability are necessary for the result.

4.2 Value of information in a competitive risk sharing environment

In this Section we examine under which conditions more information is not valuable in a competitive risk sharing environment. Schlee (2001) showed in the SEU model with common prior, risk aversion and no aggregate uncertainty, that more information makes everyone weakly worse off. As we show in the Appendix, this is not necessarily the case if we allow for ambiguity averse agents. However, the following Proposition shows that if ambiguity averse agents prefer more information to less, then the result will still hold.

Consider a set $I$ of agents, with $|I| = m$ and typical element $i$. Each agent’s consumption set is the set of acts $F$. He is endowed with a collection of convex preferences $\{\succsim^i_E\}_{E \in \mathcal{E}}$.

We assume that there is symmetric information among all agents, represented by a partition $\Pi$ of $S$. At each element $E$ of $\Pi$, the agents trade with each other, using their conditional preferences $\succsim^i_E$. Trading at each $E \in \Pi$ generates an act for each agent, which is evaluated using preference relation $\succsim^S_i$, which we denote by $\succsim_i$.

In order to rule out pure indifference to betting, we assume the strict versions of Convexity and Monotonicity (Axioms 5 and 6). We also assume Axiom 7, which was proposed by Rigotti et al. (2008). They showed that it is satisfied by most classes of ambiguity averse preferences and it implies that subjective beliefs are constant across constant acts: $\pi^E_i(x) = \pi^E_i(x')$ for all constant acts $x, x' > 0$. We henceforth write $\pi^E_i$ instead of $\pi^E_i(x)$ for all constant acts $x > 0$.

There are two periods, 0 and 1. In period 0, the agents’ common information structure about period 1 is represented by partition $\Pi$. In period 1, all agents are informed that some event $E \in \Pi$ has occurred and they trade. The resulting economy is a tuple $(E, \succsim^1_E, \ldots, \succsim^m_E, e)$, where $e \in \mathbb{R}^S_{++}$ is the aggregate endowment. We assume that there is no aggregate uncertainty, so $e$ is constant across all states in $E$. Agent $i$’s endowment is given by $e^i \in F$.

An allocation for this economy is a tuple $f^E = (f^1_E, \ldots, f^m_E) \in F^m$. It is feasible if $\sum_{i=1}^m f^i_E = e$. It is interior if $f^i_E(s) > 0$ for all $s \in S$ and for all $i$. A feasible allocation $f^E$ is a full insurance allocation if $f^i_E$ is constant across all states in $S$, for all $i$. It is Pareto optimal if there is no feasible allocation $g^E$ such that $g^i_E \succsim^i_E f^i_E$ for all $i \in I$ and $g^j_E \succsim^j_E f^j_E$ for some $j \in I$.
Let $\mathcal{M} = (S, \Pi, e, \{\succeq^i_{E}\}_{i \in I, E \in \mathcal{E}})$ be an aggregate decision problem, where $S$ is the state space, $\Pi \subseteq \mathcal{E}$ is a partition of $S$, $e \in \mathbb{R}^S_+$ is the aggregate endowment, assumed constant across states, and $\{\succeq^i_{E}\}_{i \in I, E \in \mathcal{E}}$ is a collection of convex preferences, one for each agent.

Given an event $E \in \Pi$, define $f_E = \{f^i_E\}_{i \in I}$ to be an equilibrium allocation if it is feasible and there are prices $p \in \mathbb{R}^E_+$ such that, for each $i \in I$, $\mathbb{E}_p f^i_E \leq \mathbb{E}_p e^i$ and $f^i_E \succeq^i_E g$ for all $g$ such that $\mathbb{E}_p g \leq \mathbb{E}_p e^i$. We say that $\{f^i\}_{i \in I}$ is admissible for aggregate decision problem $\mathcal{M} = (S, \Pi, e, \{\succeq^i_{E}\}_{i \in I, E \in \mathcal{E}})$ if, for each agent $i \in I$, for each $E \in \Pi$, $f^i = f^i_E \mathcal{M}$, where $f_E = \{f^i_E\}_{i \in I}$ is an equilibrium allocation of economy $(E, \succeq^1_{E}, \ldots, \succeq^m_{E}, e)$.

We compare aggregate decision problems by evaluating the admissible acts they generate.

**Definition 5.** Aggregate decision problem $\mathcal{M}_1 = (S, \Pi_1, e, \{\succeq^i_{E}\}_{i \in I, E \in \mathcal{E}})$ is not more valuable than $\mathcal{M}_2 = (S, \Pi_2, e, \{\succeq^i_{E}\}_{i \in I, E \in \mathcal{E}})$ if, whenever $\{g^i\}_{i \in I}$ is admissible for $\mathcal{M}_2$, there exists $\{f^i\}_{i \in I}$ which is admissible for $\mathcal{M}_1$ and $g^i \succeq f^i$, for each $i \in I$.

We say that public information is not valuable if it causes all agents to be weakly worse off.

**Definition 6.** Public information is not valuable for $\{\succeq^i_{E}\}_{i \in I, E \in \mathcal{E}}$ if, for all endowments $e$ and partitions $\Pi_1, \Pi_2$ of $S$, $\Pi_1$ finer than $\Pi_2$, aggregate decision problem $\mathcal{M}_1 = (S, \Pi_1, e, \{\succeq^i_{E}\}_{i \in I, E \in \mathcal{E}})$ is not more valuable than $\mathcal{M}_2 = (S, \Pi_2, e, \{\succeq^i_{E}\}_{i \in I, E \in \mathcal{E}})$.

Finally, we show that if information is valuable for each agent, then public information is not valuable, just like in the standard expected utility model.

**Proposition 4.** Suppose convex preferences $\{\succeq^i_{E}\}_{i \in I, E \in \mathcal{E}}$ satisfy Axioms 5, 6, 7, 9 and information is valuable for each $i \in I$. Then, public information is not valuable for $\{\succeq^i_{E}\}_{i \in I, E \in \mathcal{E}}$.

**Proof.** Let $e$ be the endowment and suppose partition $\Pi_1$ is finer than partition $\Pi_2$. Let $\{g^i\}_{i \in I}$ be admissible for $\mathcal{M}_2 = (S, \Pi_2, e, \{\succeq^i_{E}\}_{i \in I, E \in \mathcal{E}})$, defined as follows. Let $\{g^i_{E_2}\}_{E_2 \in \Pi_2}$ be a tuple where, for each $E_2 \in \Pi_2$, $g^i_{E_2}$ is an equilibrium allocation for economy $(E_2, \succeq^1_{E_2}, \ldots, \succeq^m_{E_2}, e)$ with (normalized) prices $p^E_2 \in \Delta E_2$. For each $E_2 \in \Pi_2$, let $g^i = g^i_{E_2}$. From the first welfare theorem, $g^i_{E_2}$ is Pareto optimal. From Proposition 9 in Rigotti et al. (2008), $g^i_{E_2}$ is a full insurance allocation. From Axiom 5, $p^E_2 \gg 0$ and $g^i_{E_2}(s) = \mathbb{E}_{p^E_2} e^i$, for all $s \in E_2$ and all $i \in I$. Hence, $g^i_{E_2}$ is also an interior allocation. Proposition 9 in Rigotti et al. (2008) implies that $p^E_2 \in \bigcap_{i \in I} \pi^i_{E_2}$.

Proposition 1 in Rigotti et al. (2008) shows that $\pi^i(f) = \pi^i_u(f)$ for all strictly positive acts. Axiom 5 implies Axiom 8. From Lemma 2 we have that $p^{E_2}(E_1) > 0$ and $p^{E_2}_{E_1} \in \bigcap_{i \in I} \pi^i_{E_1}$, for each $E_1 \subseteq E_2$, where $E_1 \in \Pi_1$. Construct allocation $f^{E_1} = (f^i_{E_1}, \ldots, f^m_{E_1})$ such that, for each $i \in I$, $f^i_{E_1}(s) = \mathbb{E}_{p^{E_2}} e^i$ for all $s \in E_1$. Because $p^{E_2}_{E_1} \in \pi^i_{E_1}$, $f^i_{E_1}$ is weakly preferred to each act $h$ that is affordable given prices $p^{E_2}_{E_1}$. Because $f^{E_1}$ is feasible, it is an equilibrium allocation of economy $(E_1, \succeq^1_{E_1}, \ldots, \succeq^m_{E_1}, e)$.

Define, for each $i \in I$, act $f^i$ as follows. If $E_1 \subseteq E_2$, where $E_1 \in \Pi_1$ and $E_2 \in \Pi_2$, then $f^i = f^i_{E_1}$. By construction, $\mathbb{E}_{p^{E_2}} f^i = \mathbb{E}_{p^{E_2}} g^i_{E_2}$. Because $p^{E_2} \in \bigcap_{i \in I} \pi^i_{E_2}$, we have
that $g_i^{E_2} \succeq f_i$, for all $i \in I$ and all $E_2 \in \Pi_2$. Axiom 9 implies that $g_i^{E_2} \succeq f_i$ for each $E_2 \in \Pi_2$ and each $i \in I$.

Enumerate the partition cells of $\Pi_2 = \{E_1, \ldots, E_n\}$. If $n = 1$ then $\Pi_2 = \{S\}$ is the uninformative partition and $g_i \succeq f_i$ for each $i \in I$, so we are done. Suppose that $n \geq 2$. For cell $1 \leq k \leq n$ define act $h^i_k$ as follows. Let $h^i_k(s) = g^i(s)$ if $s \in E_j$, where $1 \leq j \leq k$, and $h^i_k(s) = f^i(s)$ otherwise. Note that $h^i_n = g^i$ and let $h^i_0 = f^i$. For each $1 \leq k \leq n$, from Axiom 9, we have that $g^i \succeq f_i$ implies $h^i_k \succeq h^i_{k-1}$. Proposition 2 implies that $\{ \succeq^i_E \} \in \mathcal{E}$ satisfy Axiom 10, for each $i \in I$. Applying Axiom 10 we have $h^i_k \succeq h^i_{k-1}$. By Axiom 1, we have that $g^i \succeq f_i$, for each $i \in I$, which implies that $\mathcal{M}_1$ is not more valuable than $\mathcal{M}_2$. Therefore, public information is not valuable for $\{ \succeq^i_E \} \in I, E \in \mathcal{E}$.

\[\square\]

A Appendix

A.1 Weakening Consequentialism

In this Section we show that Consequentialism (Axiom 9) is not necessary for some of our results, namely the characterization of the Bayesian updating of subjective beliefs and the absence of speculative trade.

A weakening of Consequentialism, considered by Machina (1989) and defined by Halevy (2004), is that of Resolute Conditional preferences. The idea is that an agent chooses an act $h$ in the ex ante stage that serves as the “status quo”. Once the agent learns that event $E$ has occurred, he compares two acts, $f$ and $g$, by looking not only at what they prescribe inside $E$, but also by looking at what the status quo act would prescribe outside of $E$.

Definition 7. Agent $i$’s Resolute Conditional preferences at event $E$ relative to act $h$ (denoted $\succeq^i_{E,h}$) are defined by: for any two acts $f, g$, $f \succeq^i_{E,h} g$ if and only if $fEh \succeq^i_{E} gEh$.

If, for each event $E \in \mathcal{E}$ and each act $h$, $\succeq^i_E$ is Resolute Conditional at $E$ relative to $h$, then we say that $\{ \succeq^i_E \} \in \mathcal{E}$ are resolute. If $\{ \succeq^i_E \} \in \mathcal{E}$ are resolute for each $i \in I$, then we say that $\{ \succeq^i_{E,i} \} \in I, E \in \mathcal{E}$ are resolute.

Note that Resolute Conditional preferences retain part of Consequentialism, but only for the status quo act $h$. That is, $f \succeq^i_{E,h} g$ if and only if $fEh \succeq^i_{E,h} gEh$. Moreover, Resolute Conditional preferences imply Consequentialism if, for example, the agent always uses the same status quo act $h$ to compare any two acts, but weaken it otherwise.

A.1.1 Bayesian updating of subjective beliefs

We first show that Bayesian updating of subjective beliefs does not require Consequentialism. In what follows, preferences conditional on event $E$ are the “ex ante” preferences, where the status quo act $f$ is chosen. When event $F \subseteq E$ is revealed,
the agent has Resolute Conditional preference, \( \succ_{F,f} \). We define the resolute subjective beliefs revealed by unwillingness to trade to be \( \pi_E^{u}(f) = \{ p \in \Delta S : f \succ_{F,f} g \text{ for all } g \text{ such that } \mathbb{E}_p g = \mathbb{E}_p f \} \). We also redefine Weak Consistency of Implementation (Axiom 12).

**Axiom 14.** (Resolute Consistency of Implementation) For all acts \( f, g \in F \) and events \( E, F \subseteq \mathcal{E} \), if \( g \in \mathcal{R}_E(f) \) and \( f =_{F^c} g \) then \( f \succ_{F,f} g \).

Bayesian updating is redefined in the obvious way, but we write the definition for clarity.

**Definition 8.** The subjective beliefs of \( \{ \succ_{E} \}_{E \in \mathcal{E}} \) are updated using Bayes’ rule if, for all \( E, F \in \mathcal{E} \), \( F \subseteq E \), and all \( f \in F \), if \( p \in \pi_E(f) \) and \( p(F) > 0 \) then \( p_F \in \pi_{F,f}(f) \). Bayesian updating of subjective beliefs revealed by unwillingness to trade, \( \pi^{u}_E(f) \), is similarly defined.

Finally, we show that Bayesian updating of subjective beliefs is equivalent to Axiom 14.

**Proposition 5.** Suppose that convex preferences \( \{ \succ_{E} \}_{E \in \mathcal{E}} \) are resolute and satisfy Axiom 8. Then, subjective beliefs revealed by unwillingness to trade are updated using Bayes’ rule if and only if Axiom 14 is satisfied.

**Proof.** Fix events \( E, F \in \mathcal{E} \), \( F \subseteq E \). Suppose \( g \in \mathcal{R}_E(f) \) and \( f =_{F^c} g \). Then, \( \mathbb{E}_p f \geq \mathbb{E}_p g \) for some \( p \in \pi_E(f) \). Axiom 8 and Lemma 1 imply that \( p(F) > 0 \). Because subjective beliefs are updated using Bayes’ rule and \( f =_{F^c} g \), we have \( \mathbb{E}_{p_F} f \geq \mathbb{E}_{p_F} g \) and \( p_F \in \pi_{F,f}(f) \). Axiom 3 implies \( f \succ_{F,f} g \).

Conversely, suppose that there exists \( p \in \pi^{u}_E(f) \) with \( p(F) > 0 \) and \( p_F \notin \pi^{u}_E(f) \). Then, there exists act \( g \) such that \( g \succ_{F,f} f \) and \( \mathbb{E}_{p_F} g = \mathbb{E}_{p_F} f \). Because preferences are resolute, \( g \succ_{F,f} f \) implies \( gF f \succ_{F,f} f \). Because \( \mathbb{E}_{p_F}(gF f) = \mathbb{E}_{p_F} f \) and \( gF f =_{F^c} f \), we have that \( \mathbb{E}_p(gF f) = \mathbb{E}_p f \), hence \( gF f \in \mathcal{R}_E(f) \). From Axiom 14 we have \( f \succ_{F,f} gF f \), a contradiction.

\( \square \)

### A.1.2 Speculative trade

We next show that Resolute Conditional preferences are enough to preclude speculative trade. It is important to note that we do not require Conditional Decomposability, unlike Halevy (2004).

**Proposition 6.** Suppose convex preferences \( \{ \succ_{E} \}_{E \in \mathcal{E}} \) are resolute and satisfy Axiom 12 and let \( f \) be an interior, \( ex \) ante weakly efficient allocation. Then, there exist no state \( s \) and a feasible allocation \( g \) such that \( G = \{ s \in S : g^i \succ_{I(s),f} f^i \text{ for all } i \in I \} \) is common knowledge at \( s \). If, in addition, preferences \( \{ \succ_{E} \}_{E \in \mathcal{E}} \) satisfy Axiom 13 and \( f \) is an interior, \( ex \) ante efficient allocation, then there exist no state \( s \) and a feasible allocation \( g \) such that \( H = \{ s \in S : g^i \succ_{I(s),f} f^i \text{ for all } i \in I \} \), where for some \( s \in H \) and \( i \in I, g^i \succ_{I(s),f} f^i \), is common knowledge at \( s \).
Proof. Because \( f \) is an ex ante weakly efficient allocation, there does not exist a feasible allocation \( h \) such that \( h^i >^i f^i \) for all \( i \in I \). Suppose that there exists feasible allocation \( g \) such that \( G \) is common knowledge at \( s \in S \). Let \( F \) be a self evident event such that \( s \in F \subseteq G \). Note that each \( \Pi^i \) partitions \( F \). Then, we have that for each \( i \in I \), for each \( s' \in F \), \( g^i >^i_{\Pi'(s')} f^i \). From the definition of Resolute Conditional preference, we have \( g^i_{\Pi'(s')} f^i >^i_{\Pi'(s')} f^i \). From Axiom 12, we have that for all \( p \in \pi^i(f) \), \( \mathbb{E}_p(g^i_{\Pi'(s')} f^i) > \mathbb{E}_p f^i \). Because this is true for all \( s' \in S \) such that \( \Pi'(s') \subseteq F \) and each \( \Pi^i \) partitions \( F \), we have that \( \mathbb{E}_p(g^i_{\Pi'(s')} f^i) > \mathbb{E}_p f^i \), for all \( p \in \pi^i(f) \). Define \( h^i = g^i_{\Pi'(s')} f^i \) and \( h = \{ h^i \}_{i \in I} \).

Allocation \( f \) is interior, hence \( \pi^{iu}(f) = \pi^{iu}(f) = \pi(f) \). This implies that for small enough \( \epsilon' \), \( \epsilon^i h^i + (1 - \epsilon^i) f^i >^i f^i \). By taking \( \epsilon < \epsilon' \) for all \( i \in I \), we have that \( \epsilon h^i + (1 - \epsilon) f^i >^i f^i \) for all \( i \in I \). Moreover, \( \epsilon h + (1 - \epsilon) f \) is feasible because both \( f \) and \( h \) are feasible. Hence, we have arrived at a contradiction.

For the second claim, by using the same arguments and Axiom 13 we get that \( \mathbb{E}_p h^i \geq \mathbb{E}_p f^i \), for all \( p \in \pi^i(f) \), for all \( i \in I \), strict inequality for some \( j \). By reducing \( h^j \) for all \( s \in S \) by a sufficiently small amount and distributing it to the other agents, we can find a new feasible allocation \( h \) such that \( \mathbb{E}_p h^i > \mathbb{E}_p f^i \), for all \( p \in \pi^i(f) \), for all \( i \in I \). Using the same arguments as in the previous paragraph, we reach a contradiction.

\[ \square \]

### A.2 Ellsberg and the value of information

Using an example, we show that, under Consequentialism, behavior consistent with Ellsberg can lead to information not being valuable. This is because one direction of DC (Axiom 10) is violated. In Proposition 2 we show that Axiom 10 is not only necessary but also sufficient for valuable information. Hence, if one wants to retain the value of information under Ellsberg preferences, Consequentialism must be relaxed. However, the weak version of Axiom 10, which is 13, is consistent with Ellsberg preferences.

Consider the following dynamic Ellsberg’s three-color problem, taken from Hanany and Klibanoff (2007). An urn contains 120 balls, 40 of which are known to be black (B), whereas the remaining 80 are somehow divided between red (R) and yellow (Y). Hence, the state space is \( S = \{ B, R, Y \} \). A bet is an act from \( S \) to \( \mathbb{R} \), specifying a payoff at each state.

Consider the following acts \( f_1 = (1, 0, 0), f_2 = (0, 1, 0), f_3 = (0, 1, 1) \) and \( f_4 = (1, 0, 1) \), where, for example, \( f_1 \) specifies a payoff of 1 if the state is \( B \) and 0 otherwise. Typical Ellsberg preferences are \( f_1 > f_2 \) and \( f_3 > f_4 \). We also assume that the agent is endowed with preferences \( \succeq_{(B,R)} \) conditional on learning that event \( \{ B, R \} \) has occurred, and preferences \( \succeq_{\{Y\}} \), conditional on learning that event \( \{ Y \} \) has occurred.

Note that, conditional on \( \{ Y \} \), \( f_1 \) is identical to \( f_2 \) and \( f_3 \) is identical to \( f_4 \). From Consequentialism, we have that \( f_1 \sim_{\{Y\}} f_2 \) and \( f_3 \sim_{\{Y\}} f_4 \). Moreover, because conditional on \( \{ B, R \} \) \( f_1 \) is identical with \( f_4 \) and \( f_2 \) is identical with \( f_3 \), Consequentialism requires that \( f_1 \sim_{\{B,R\}} f_4 \) and \( f_2 \sim_{\{B,R\}} f_3 \).

Information is represented by a partition \( \Pi \) of \( S \). Following Geanakoplos (1989), we compare two partitions by the acts they generate. Let \( A \) be the set of available
acts. At each element $E \in \Pi$, the agent has conditional preferences $\succsim_E$ and chooses the optimal act in $\mathcal{A}$. Then, partition $\Pi$ is associated with act $g^{\Pi}$, defined as follows. For each $E \in \Pi$, $g^{\Pi}(s) = f(s)$ for all $s \in E$, where $f \succsim_E h$ for all $h \in \mathcal{A}$. We say that partition $\Pi_1$ is more valuable than $\Pi_2$ if $g^{\Pi_1} \succsim g^{\Pi_2}$.

We consider two cases. First, suppose that $f_1 \succsim_{\{B,R\}} f_2$, which implies $f_1 \succsim_{\{B,R\}} f_3$. Let $\mathcal{A} = \{f_3, f_4\}$ be the set of available acts. First, suppose that the agent has no information, so his partition is $\Pi_1 = \{S\}$. Then, he chooses $f_3$, because $f_3 \succ f_4$. In other words, partition $\Pi_1$ generates act $f_3$.

Suppose now that the agent has more information. In particular, his partition is $\Pi_2 = \{\{B,R\}, Y\}$, meaning that he is informed whether $Y$ has occurred or not, before making his choice. If he learns that $\{B,R\}$ has occurred, he chooses $f_4$ because $f_4 \succeq_{\{B,R\}} f_3$, whereas if he learns that $\{Y\}$ has occurs, he again chooses $f_4$ because $f_4 \asymp_{\{Y\}} f_3$. From an ex ante point of view, $\Pi_2$ generates the act $f_4$.

Because $f_3 \succ f_4$, the agent strictly prefers partition $\Pi_1$ to partition $\Pi_2$. In other words, information is not valuable. If we assume that $f_2 \succ_{\{B,R\}} f_1$, we can obtain negative value of information using a similar example with $\mathcal{A} = \{f_1, f_2\}$.

Note that the Ellsberg preferences violate one direction of DC, Axiom 10: if $f_4 \succeq_{\{B,R\}} f_3$ then $f_4 \succeq f_3$. From Proposition 2, we know that this Axiom characterizes valuable information, hence it is inescapable that Ellsberg preferences lead to information not being valuable. On the other hand, they do not violate the weak version of Axiom 10, which is Axiom 13, neither the weakening of the other direction of DC, Axiom 12. From Proposition 3, this means that Ellsberg preferences can preclude speculative trade.

References


