Deepwater Horizon: When financial imperfections are not the problem, but the solution

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Abstract

The BP Deepwater Horizon oil spill of 2010 has focused considerable attention on the potential liability and the operating conduct of big oil companies. This paper shows that limiting the ability of a company to insure and diversify its risks creates incentives to internalize the welfare effects of catastrophic events, leading to a welfare improvement. We model an economy with complete financial markets where one agent’s actions impose an externality on other agents by altering the probability distribution of their risks. Then, a Pareto improved allocation can be reached via an asset reallocation, essentially restricting this agent’s choice of his portfolio of assets. Hence, in the presence of externalities, disturbing the functioning of perfect financial markets can be socially beneficial.

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Then, as now, nobody gave a rap for ethics. The almighty dollar was the only goal, and its possession placed a person beyond criticism for any breach of ethics incidental to the acquisition of it.

Charles Ponzi

Deepwater Horizon was an oil extraction rig owned by drilling company Transocean, which operates the largest fleet of deep-water offshore rigs in the world, and leased to BP, one of the world’s leading energy companies, part of a conglomerate with interests that cover a very large range of products, from oil and gas transportation fuels to diverse types of petrochemicals. On April 20th, 2010, Deepwater Horizon was deployed some 66km off the coast of Louisiana, in the US portion of the Gulf of Mexico, when a concrete seal in the Macondo well burst as a consequence of a surge of natural gas. During the installation of that seal, contractor Halliburton had used nitrogen to accelerate the “curing” of the concrete, a technique that is known to weaken the pressure that this type of seal withstands.

The most immediate consequence of the explosion was the death of eleven members of the rig’s crew, and the injuries caused to 17 more. After that, the oil spill that occurred during the next 87 days constitutes the largest one in the history of petroleum production, and the worst environmental catastrophe in US history. The damages caused to the ecosystem, to the local economy and to the population of the area remain largely unquantified, but some indicators are known.

During the six months following the explosion, some 8,000 dead animals were collected, with causes of death attributed directly to oil contamination. Later studies have shown that the number of local sea turtles becoming stranded onshore has multiplied five-fold, and that one half of the dolphins in the area exhibit serious health disorders known to be linked to oil exposure. In the long run, the number of seabird deaths attributed by scientists to the explosion ranges from 600,000 to 800,000.

Besides oil drilling, the economies of Louisiana, Mississippi and Alabama depend heavily on their fishing and tourism industries. Immediately after the explosion, the moratorium imposed on further drilling activities left an estimated 8,000–12,000 local workers unemployed. In response to the spill, government agencies deemed it necessary to close parts of the Gulf to all fishing activities, both commercial and recreational. At its peak, this ban covered 88,522 square miles, about one third of the US portion of the Gulf. Also, with 22% of the total news coverage in the US devoted to the incident during its first 100 days, the perception that seafood sourced in the gulf would be polluted is likely to have affected its

1 The rise of Mr. Ponzi, Inkwell Pub (November 2001).
2 The use of which, in turn, extends to products as diverse as clothes and packaging.
3 This figure refers to a sub-area of the Gulf of Mexico, the Barataria Bay, in Louisiana.
demand, as it did the demand for tourism: according to a poll, in the first four months after the spill, almost one third of travellers to the coast of Louisiana had cancelled or postponed their visits because of it.

And while much more difficult to measure, especially in the short run, the incident had significant consequences for the lives of people local to the area. Again in the four months following the explosion, the clinical diagnoses of depression increased by about 25%, and more than half of the people responding a local survey reported to have been worried “almost constantly” about the spill.

By the time the well was successfully re-sealed, 4.9 million barrels of oil had been released to the waters of the Gulf. Three quarters of this oil could not be intercepted during the recovery efforts, and while a significant part of this oil has naturally dispersed or evaporated, at least one third of it remained as residue, either at the water surface, floating as tar balls, washed ashore, or buried in sediment.

Shortly after the explosion, at the request of the US government BP placed in escrow a fund of $20bn to help address the financial losses caused by it. Later, it was also required to pay the $14bn cost of the clean-up efforts led by the US Coast Guard and other agencies, and on October 2015 the company reached a settlement agreement with the US Department of Justice for $20.8bn, which is meant to be comprehensive of all past and future damage at federal, state and local levels. One year after the incident, BP’s market capitalization had decreased by about 23% (valued in Sterling) and during the first three quarters of 2010 the company stopped paying dividends to its shareholders. But by the fourth quarter of that year the dividends were reinstated at about one half the value of the last quarter of 2009, and between then and the first quarter of 2016 they have grown by almost 60%. Interestingly, while the company estimates the cost it has had to incur, up to the third quarter of 2015, in $55bn, its market capitalization has dropped only by some $32bn.

In January, 2011, American oil producer Marathon split its refining activities from its exploration arm, after which its shares increased by 23%, gaining the company over $14bn in capital. The third-largest oil company in the country, ConocoPhillips, followed that strategy in May, 2012, and, even in the aftermath of the Deepwater Horizon spill, financial analysts stressed that doing the same would increase BP’s market value by about $100bn. It is apparent that the strategy of oil companies, in response to the lessons learnt during

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4 The methods used to attempt interception, which included burning some of the oil at the surface and the application of chemical dispersers, are themselves likely to have a detrimental impact on eco-systems.

5 See www.bloomberg.com/news/2011-07-24/bp-breakup-worth-100-billion-to-jpmorgan.html, which reports on a JPMorgan’s statement on this matter, and on similar positions held by analysts at UBS and Bank of America.
the Gulf of Mexico catastrophe, aims at isolating the revenue they make in other businesses from the potential losses caused by an exploration incident.\(^6\) More in general, their intent appears to be to further diversify away the risk they bear from such incidents.

This paper studies whether limiting the ability of an agent to insure, or in general to participate in financial trading to diversify his risks, can induce a Pareto improvement in the allocation of resources in the economy. That this may be the case is not obvious: the literature on financial economics shows that financial markets where not all the agents can trade freely fail to deliver Pareto efficiency. But this result is established under the assumption that no agent in the market causes an externality on others.

In 2011, a report commissioned by the US government concluded that the explosion of the well could have been prevented, and that its causes could be traced directly to decisions made by BP, Halliburton and Transocean. When these companies decided to accelerate the curing of the concrete on the seal of the Macondo Well, they increased the probability that an environmental catastrophe could occur in the Gulf. When they made their decisions, presumably they considered such increased probability, but only up to the losses that they themselves would have to bear in the event of that catastrophe. The increased probability of losses for other agents is, in the language of Public Economics, an externality that the three companies did not internalize.

If the agent that causes an externality through the probability distribution of risks is limited in his access to financial markets, he certainly will respond by altering those probabilities in a way that is optimal to him. It seems easy to conclude that such policy can then be used to effect a Pareto improvement: in a partial equilibrium argument, this is immediately the case. The general equilibrium trade-off that we study, and which makes the conclusion less immediate and more interesting, is that, in the presence of this kind of externalities, disturbing the functioning of perfect financial markets may be socially beneficial. The more obvious effect that a partial equilibrium argument ignores is the reaction of asset prices to the perturbation of the agent’s financial portfolio. More subtle, but equally important, is the fact that, after such price change, other agents in the economy will change their portfolios too.

We study an economy where, initially, all agents participate in the trading of a complete set of financial assets. The only market failure in this economy is the externality that one

\(^6\) This response would resemble the one that the industry had after the \textit{Exxon Valdez} oil spill of March 24th, 1989. That incident occurred when the vessel of that name, through a series of human and operational errors, ran aground on Prince William Sound, Alaska. Before Deepwater Horizon, the spill that followed was the largest one ever, and its consequences affect the area even today: there still is oil residue affecting the Sound’s beaches, waters and ecosystems. In 1993, the vessel’s owner, Exxon, spun off all its shipping activities to a subsidiary.
agent imposes on the others via the probability distribution of their risks. We show that, in a generic sense, everybody can be made better-off if the agent that causes the externality is not allowed to choose his portfolio of financial assets in an optimal manner (while everybody else continues to trade without constraints). Besides the restriction of that agent’s financial decisions, lump-sum transfers are used to make sure that all agents are, indeed, made better-off. We first allow for a lump-sum transfer to the agent who imposes the externality. Being lump-sum, this transfer does not affect that agent’s incentives, only his welfare. Still, it may be that for institutional reasons such transfer is not possible. Under the assumption that there is an external source of funding for the rest of the agents, which we call “relief aid,” we show that the Pareto improvement is still generically possible, even when the agent that imposes the externality is excluded from any lump-sum aid. We also check the robustness of our results to a more sophisticated behavior by agent 0, where he now recognizes that his decisions with respect to probabilities affect the willingness to pay of other agents for the existing financial assets and, therefore, the equilibrium level of assets prices. Again, we find that the equilibrium allocation is (weakly) constrained inefficient in this case. To conclude the analysis, we extend the results to economies with uninsurable idiosyncratic risk.

While the spirit of our results, and the techniques we use, are similar to those in the literature on incomplete financial markets, it is worthwhile noticing that in our economy there is only one commodity per state, so that it is not via relative commodity prices that the Pareto improvement is induced by our financial policies.

The paper is organized as follows. A short literature review is presented in Section 1. The following three sections describe the general kind of economies with externalities for which our analysis holds and define competitive equilibrium and Pareto efficiency for this kind of economies. Section 5 introduces notions of weak and strong constrained inefficiency for those economies and states the main theorem of our paper. Section 6 then gives the proof for that result: the genericity of strong constrained inefficiency. In Section 7 we study the case of a more sophisticated behavior by agent 0 while in section 8 we extend the results to economies with uninsurable idiosyncratic risk. A technical appendix completes the paper.

1. **Existing Literature**

It is well known that any equilibrium allocation is Pareto efficient in competitive economies with no externalities and perfect financial markets. Constrained efficiency is a weaker version of Pareto efficiency that takes into account the financial structure and is appropriate for the case of incomplete markets. As Stiglitz [9], Greenwald and Stiglitz [7] and Geanakoplos and Polemarchakis [4] have argued, in a numéraire asset model with incomplete markets,
generically, every equilibrium is constrained inefficient, provided that there are more than one commodities and there is an upper bound in the number of individuals: reallocations of existing assets support superior allocations. Following Geanakoplos and Polemarchakis [4], Citanna, Kajii and Villanacci [3] show that equilibria are generically constrained inefficient even without an upper bound on the number of households. They also show that perfectly anticipated lump-sum transfers in a limited number of goods are typically effective.

The welfare improving policies that we analyze have been the subject of the restricted participation literature. As Polemarchakis and Siconolfi [8] point out, incomplete markets are just a special case of an asset market with restricted participation. In this setting, they prove the generic existence of competitive equilibria when agents face asymmetric linear constraints on their portfolio incomes. Cass, Siconolfi and Villanaci [2] extend the literature by accommodating a wide range of portfolio constraints, including any smooth, quasi-concave inequality constraint on households' portfolio holdings. Gori, Pireddu and Villanacci [6] focus on price-dependent borrowing restrictions. After proving existence of equilibrium, they show that equilibria associated with a sufficiently high number of strictly binding participation constraints in the financial markets can be Pareto improved upon by a local change in these constraints.

For the case of abstract economies with externalities and multiple commodities, Geanakoplos and Polemarchakis [5] show that the competitive equilibrium is, generically, constrained inefficient: there exists an anonymous taxation policy that leaves all agents better off. While the concept of equilibrium used in that paper is the same as ours, neither the setting nor the type of externality are the same, and the mechanism through which the Pareto improvement is induced differs too.

Of course, any situation in which an externality induces a departure from economic efficiency can be studied from the perspective of the literature on mechanism design. We do not take that approach, which we consider complementary to our results, and instead apply the ideas of the literature on incomplete financial markets. Relatedly, Braido [1] presents a general equilibrium model as a two-stage game where agents act as producers, as consumers and as financial intermediaries with intermediation costs. Each individual is allowed to design a financial structure that consists of specifying securities pay-offs in each state and transaction constraints that restrict the participation of some agents in some markets, while allowing for non-exclusivity. He shows that an equilibrium exists and he offers examples and a reasoning as to why the equilibrium might be constrained inefficient and the markets incomplete. In particular, he offers an example with two agents where only one of them is risk averse. This agent faces a production risk where the probability of the "good" state is increasing in the costly unobservable effort he exerts. In this setup, Braido shows that an incomplete financial structure, in the form of trading constraints, is Pareto
superior to complete markets. Our framework is more general: all agents are risk-averse and face aggregate and/or idiosyncratic risk. In our setup, the agent who exerts effort creates an externality. Moreover, by imposing trading constraints, we are limiting the insurance opportunities of the whole society and, since the rest of the agents are risk averse too, it is therefore more demanding to show that a Pareto improving policy exists.

2. An Externality Via Probabilities

Consider a two-period economy populated by $I + 1$ individuals, who are denoted by $i = 0, \ldots, I$. In period 0, agents receive an endowment of a single consumption good. In period 1, there is uncertainty regarding endowments: there are (only) two states of the world, which we denote by $s = 1, 2$, and agents receive an endowment $e_i^s$ of the consumption good in state $s$.

Rather than being exogenous, the probabilities of the two states depend on some action of agent 0. In period 0, he has the choice of choosing a costly action, $\alpha$, which makes state 1 more likely. The cost of this action is that it subtracts from the agent’s consumption at date 0.

The preferences of the agents, who derive utility only from the single consumption good, are represented by

$$c_0^i + \pi(\alpha) \cdot u_1^i(c_1^i) + [1 - \pi(\alpha)] \cdot u_2^i(c_2^i),$$

where $\pi(\alpha) \in (0, 1)$ is the probability of state 1. We allow for negativity of present consumption, $c_0^i$, so it is not necessary for us the specify date-0 endowments.

We assume that $\pi$ is an increasing and concave function. Each state-dependent cardinal utility index $u_i^s$ is assumed to be continuous, strictly increasing and strictly concave, and to satisfy standard Inada conditions.

The assumption that all individuals have preferences that are quasilinear in present consumption simplifies the mathematical arguments, but, of course, implies a loss of generality. The gained simplicity is that, since utility is transferable in this case, we can express our arguments in terms of aggregate social welfare functions.

We assume that $\alpha \in [\alpha, \infty)$, for a lower bound on the action $\alpha$. Now, in order to have interior solutions, we impose the following condition. For each state $s$, let $(\hat{c}_s^i)_{i=0}^I$ solve the following maximization problem:

$$\max_{(c^i)_{i=0}^I} \left\{ \sum_{i=0}^I u_s^i(c^i) : \sum_{i=0}^I c_s^i = \sum_{i=0}^I e_s^i \right\}. \quad (2)$$

The results below will hold true if there are multiple commodities in the second period. In fact, results of the type we are studying are easier to argue in that case, only at the cost of heavier notation.
The following assumptions will be technically useful later on:

**Assumption 1 (Interiority).** At the consumption plan $\hat{c}^0$, agent 0 prefers state 1 to state 2, in the sense that $u_1^0(\hat{c}_1^0) > u_2^0(\hat{c}_2^0)$, and there exists a level of the action $\hat{\alpha} \in (\alpha, \infty)$ such that

$$\pi'(\hat{\alpha}) = \frac{1}{u_1^0(\hat{c}_1^0) - u_2^0(\hat{c}_2^0)}.$$ 

**Assumption 2 (Concavity).** Around the consumption plan $\hat{c}^0$ and action $\hat{\alpha}$, agent 0 has locally concave preferences: matrix

$$
\begin{pmatrix}
\pi(\hat{\alpha}) \cdot \partial^2 u_1^0(\hat{c}_1^0) & 0 & \pi'(\hat{\alpha}) \cdot \partial u_1^0(\hat{c}_1^0) \\
0 & [1 - \pi(\hat{\alpha})] \cdot \partial^2 u_2^0(\hat{c}_2^0) & -\pi'(\hat{\alpha}) \cdot \partial u_2^0(\hat{c}_2^0) \\
\pi'(\hat{\alpha}) \cdot \partial u_1^0(\hat{c}_1^0) & -\pi'(\hat{\alpha}) \cdot \partial u_2^0(\hat{c}_2^0) & \pi''(\hat{\alpha}) \cdot [u_1^0(\hat{c}_1^0) - u_2^0(\hat{c}_2^0)]
\end{pmatrix}
$$

is negative definite.

**Assumption 3 (Heterogeneity).** At consumption allocation $\hat{c}$, agents other than 0 have heterogeneous preferences: for all $i, j \geq 1$, $\partial^2 u_s^i(\hat{c}_s^i) \neq \partial^2 u_s^j(\hat{c}_s^j)$, if $i \neq j$, for both $s = 1, 2$.

### 3. Competitive Equilibrium

Financial markets are assumed to be complete: there is an elementary (Arrow) security for each state, with asset $s$ paying one unit of the consumption good in state $s$. Holdings of these securities are denoted by $\vartheta_s^i$.

#### 3.1. The problem of agents $i \geq 1$

All agents other than 0 have only one decision to make in period 0: they have to choose their holdings of securities, and therefore their consumption in that period and in both states in period 1. We assume that they take the prices of the securities and the probabilities of the states as given.

Letting $q_1$ and $q_2$ denote the prices of the securities, the problem of individual $i$ is, simply,

$$\max_{\vartheta_1^i, \vartheta_2^i} \{ -q_1 \cdot \vartheta_1^i - q_2 \cdot \vartheta_2^i + \pi(\alpha) \cdot u_1^i(e_1^i + \vartheta_1^i) + [1 - \pi(\alpha)] \cdot u_1^i(e_2^i + \vartheta_2^i) \}.$$  

(4)

The first-order conditions of this problem are standard:

$$q_1 = \pi(\alpha) \cdot \partial u_1^i(e_1^i + \vartheta_1^i) \quad \text{and} \quad q_2 = [1 - \pi(\alpha)] \cdot \partial u_2^i(e_2^i + \vartheta_2^i).$$  

(5)

These conditions are necessary and sufficient to characterise the solutions of Program (4).

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8 We use $\partial u_s^i$ to denote the first derivative of function $u_s^i$; this is in lieu of the awkward notation $u_s'^i$. 
3.2. The problem of agent $i = 0$

Agent 0 has an extra decision to make in period 0: apart from choosing his holdings of securities, he must choose his action. His problem is, then:

$$\max_{\alpha, q_0^1, q_0^2} \{ -\alpha - q_0^1 \cdot \theta_1^0 - q_2 \cdot \theta_2^0 + \pi(\alpha) \cdot u_0^0(e_1^0 + \theta_1^0) + [1 - \pi(\alpha)] \cdot u_2^0(e_2^0 + \theta_2^0) \}. \quad (6)$$

Assuming that this agent, too, takes asset prices as given, the following are the first-order conditions for any solution with an interior action:

$$1 = \pi'(\alpha) \cdot [u_0^0(e_1^0 + \theta_1^0) - u_2^0(e_2^0 + \theta_2^0)], \quad (7)$$

while

$$q_1 = \pi(\alpha) \cdot \partial u_1^0(e_1^0 + \theta_1^0) \quad \text{and} \quad q_2 = [1 - \pi(\alpha)] \cdot \partial u_2^0(e_2^0 + \theta_2^0). \quad (8)$$

Unfortunately, these conditions are only necessary, as we cannot guarantee the overall concavity of the individual’s objective function. We will come back to this issue later, but, for the moment, note that the first-order condition with respect to the action, Eq. (7), implies that when agent 0 prefers state 1 to state 2 sufficiently, he is willing to choose a high action to raise the likelihood of state 1. In his choice of action, however, he does not internalize the effect of a more likely state 1 on the well-being of the society as a whole. Since we have made no assumptions on aggregate endowments and social welfare in each state, all agents other than agent 0 could be worse-off or better-off in state 1. It is this feature that reflects the non-alignment of interest across agents.

3.3. Nash-Walras Equilibrium

Competitive equilibrium is defined by individual rationality and market clearing requirements. We write competitive equilibria as a tuple $\{\tilde{\vartheta}, \tilde{\alpha}, \tilde{q}\}$, where $\tilde{\vartheta} = [(\tilde{\vartheta}_s^i)_{s=1,2}]_{i=0}$ is the allocation of the two assets and $\tilde{q} = (\tilde{q}_1, \tilde{q}_2)$ is the vector of asset prices, such that:

1. action $\tilde{\alpha}$ and portfolio $(\tilde{\vartheta}_s^0)_{s=1,2}$ solve Program (6) when the prices are $q = \tilde{q}$;

2. for each $i \geq 1$, portfolio $(\tilde{\vartheta}_s^i)_{s=1,2}$ solves Program (4) when the prices are $q = \tilde{q}$ and the probability of state 1 is $\pi(\tilde{\alpha})$; and

3. both of the securities markets clear: $\sum_{i=0}^{1} \tilde{\vartheta}_s^i = 0$, for $s = 1, 2$.

For agents $i \geq 1$, individual rationality is characterised by the first-order conditions, Eq. (5). Importantly, we follow the spirit of Nash-Walras equilibrium in assuming that these agents take as given not only prices, but also the probabilities of the two states.
In the case of agent 0, in principle, the first-order conditions, Eqs. (7) and (8), are only necessary. Now, the combination of Eqs. (8) and (5) suffices to imply that, for each state of the world, the equilibrium allocation of consumption, given by \( c_{s}^i = \bar{c}_{s}^i + \bar{\theta}_{s}^i \), solves Program (2). Since the latter has a unique solution, by strict concavity of preferences, it follows that \( \bar{\theta}_{s}^i = \hat{c}_{s}^i - \bar{c}_{s}^i \). By Assumptions 1 and 2, it follows that Program (6) is concave in the action too, and hence that the first-order conditions are both necessary and sufficient. Note, then, that for agent 0, who causes an externality via his choice of the action, our assumption is that he takes prices as given, and considers the effects of the action on his own well-being only.

For later usage, let us define the function

\[
F(q, \vartheta, \alpha) = \begin{pmatrix}
\pi(\alpha) \cdot \partial u_0^0(e_1^0 + \bar{\theta}_1^0) - q_1 \\
[1 - \pi(\alpha)] \cdot \partial u_0^0(e_2^0 + \bar{\theta}_2^0) - q_2 \\
[\partial u_0^0(e_1^0 + \bar{\theta}_1^0) - \partial u_0^0(e_2^0 + \bar{\theta}_2^0)] \cdot \pi'(\alpha) - 1 \\
\pi(\alpha) \cdot \partial u_1^1(e_1^1 + \bar{\theta}_1^1) - q_1 \\
[1 - \pi(\alpha)] \cdot \partial u_1^1(e_2^1 + \bar{\theta}_2^1) - q_2 \\
\vdots \\
\pi(\alpha) \cdot \partial u_I^I(e_I^I + \bar{\theta}_I^I) - q_1 \\
[1 - \pi(\alpha)] \cdot \partial u_I^I(e_I^I + \bar{\theta}_I^I) - q_2 \\
\sum_{i=0}^I \bar{\theta}_1^i \\
\sum_{i=0}^I \bar{\theta}_2^i
\end{pmatrix}
\]

Note that the roots of this function characterise the competitive equilibrium of the economy.

The following assumption is generically true, under Assumption 2.

**Assumption 4 (Determinacy and trade).** At equilibrium, agent 0 participates in the market for elementary security 1 and the Jacobean of function \( F \) is invertible: \( \bar{\theta}_1^0 \neq 0 \) and matrix \( D F(\bar{q}, \bar{\vartheta}, \bar{\alpha}) \) is non-singular.

### 4. Pareto Efficiency

#### 4.1. Definition

The definition of Pareto efficient allocation is as usual: a feasible allocation of agent 0’s action and the consumption of the unique commodity both at date 0 and in the two future states across all agents is efficient, if it is impossible to find an alternative, feasible allocation of these same variables that makes at least one agent better-off without making any other agent worse-off. Given that all individuals have quasilinear preferences, Pareto efficiency
amounts to the choice of an allocation of consumption and an action so as to solve the program

$$\max_{\alpha, (c_1^i, c_2^i)_{i=0}^I} \left\{ -\alpha + \sum_{i=0}^I \left[ \pi(\alpha) \cdot u_1^i(c_1^i) + [1 - \pi(\alpha)] \cdot u_2^i(c_2^i) \right] \mid \sum_{i=0}^I c_1^i = \sum_{i=0}^I e_s^i, s = 1, 2 \right\}. \quad (9)$$

Here, the date-0 consumption levels are left undetermined, but any allocation that exhausts the remaining aggregate endowment, net of $\alpha$, will be Pareto efficient.

The first-order conditions that characterize Pareto efficiency are, therefore, that

$$\pi'(\alpha) \cdot \sum_{i=0}^I [u_1^i(c_1^i) - u_2^i(c_2^i)] = 1; \quad (10)$$

and that, for each pair of agents $i, j = 0, \ldots, I$,

$$\partial u_s^i(c_1^i) = \partial u_s^j(c_1^j) \quad (11)$$

for each state $s = 1, 2$; as well as the feasibility condition.

4.2. Inefficiency of competitive equilibrium

Comparison of the first-order conditions of the individual, for the competitive choices of consumption, Eqs. (5) and (8), with the first-order condition defining efficiency of consumption plans, Eq. (11), shows that the allocations of consumption prescribed by the competitive equilibrium are efficient in the sense that the marginal rates of substitutions would be equalised across agents in each state.

On the other hand, from the first-order conditions for the choice of action, Eqs. (7) and (10), it is immediate that action implied by the competitive equilibrium solution is generically not Pareto optimal: while agent 0 takes into account only the effects on his own welfare, the social planner considers the effects on social welfare when choosing the optimal action.

5. Constrained inefficiency of competitive equilibrium

5.1. Two definitions

We have shown that competitive equilibria need not yield the Pareto efficient action. This result says that if a planner could choose the action exerted by agent 0, he would choose a different level, and then would reallocate date-0 consumption to make sure that all agents, including 0 himself, are made better-off.
As is usual in the General Equilibrium literature, the latter observation does not mean that a social planner who faces constraints in terms of the policies he can apply would indeed be able to effect a welfare improving policy. Here, we consider the case in which the planner is constrained in the sense that he can only distort the asset holdings of agent 0, but cannot directly choose the action that the agent chooses. We also allow the planner to effect lump-sum transfers of revenue across all agents, perhaps including 0 himself. If such a policy exists that leaves everybody better-off, we shall say that the competitive equilibrium is constrained inefficient. Whether a lump sum transfer to agent 0 is required determines how strong the definition is.

Formally, we say that an allocation \((\alpha, c)\), where \( c = [(c_0^i)_{s=0,1,2}]_{i=0}^1 \),\(^9\) is weakly constrained inefficient if there exist an alternative level of the action, \(\tilde{\alpha}\); asset prices, \(\tilde{q}\); a profile of asset holdings, \(\tilde{\vartheta}\); and a profile of date-0 lump-sum transfers, \((\tau^i)_{i=0}^1\), such that:

1. \(\tilde{\alpha}\), the action of agent 0, solves

   \[
   \max_{\alpha} \left\{ \alpha - \pi(\alpha) \cdot u_1^0(e_1^0 + \tilde{\vartheta}_1^0) + [1 - \pi(\alpha)] \cdot u_2^0(e_2^0 + \tilde{\vartheta}_2^0) \right\}; \tag{12}
   
   \]

2. for each \(i \geq 1\), portfolio \((\tilde{\vartheta}_i^s)_{s=1,2}\) solves Program (4) when the prices are \(q = \tilde{q}\) and the probability of state 1 is \(\pi(\tilde{\alpha})\);

3. both of the securities markets clear: \(\sum_{i=0}^1 \tilde{\vartheta}_s^i = 0\), for \(s = 1, 2\);

4. the profile of lump-sum transfers is balanced: \(\sum_{i=0}^1 \tau^i = 0\);

5. agent 0 is better-off, in that

   \[
   -\tilde{\alpha} - \tilde{q}_1 \cdot \tilde{\vartheta}_1^0 - \tilde{q}_2 \cdot \tilde{\vartheta}_2^0 + \tau^0 + \pi(\tilde{\alpha}) \cdot u_0^0(e_1^0 + \tilde{\vartheta}_1^0) + [1 - \pi(\tilde{\alpha})] \cdot u_0^2(e_2^0 + \tilde{\vartheta}_2^0)
   
   \]

   is higher than

   \[
   -\alpha + c_0^0 + \pi(\alpha) \cdot u_1^0(c_1^0) + [1 - \pi(\alpha)] \cdot u_2^0(c_2^0);
   
   \]

6. every agent \(i \geq 1\) is better-off, in that

   \[
   -\tilde{q}_1 \cdot \tilde{\vartheta}_1^i - \tilde{q}_2 \cdot \tilde{\vartheta}_2^i + \tau^i + \pi(\tilde{\alpha}) \cdot u_1^i(e_1^i + \tilde{\vartheta}_1^i) + [1 - \pi(\tilde{\alpha})] \cdot u_2^i(e_2^i + \tilde{\vartheta}_2^i)
   
   \]

   is higher than

   \[
   c_i^0 + \pi(\alpha) \cdot u_1^i(c_i^0) + [1 - \pi(\alpha)] \cdot u_2^i(c_i^2).
   
   \]

---

\(^9\) The feasibility condition that \(\sum_{i=0}^1 c_i^s = \sum_{i=0}^1 e_i^s\), for both \(s = 1, 2\), will hold throughout our analysis. Also, note that in the following analysis \(c_0^0\) is taken to be gross of the action.
Intuitively, the allocation is constrained inefficient if a planner can effect a Pareto improvement by forcing agent 0 out of the financial markets: instead of letting him choose optimal holdings of the two assets, a portfolio $\tilde{\vartheta}^0$ is allocated to him. Then competitive assets markets open for every other agent in the economy, and assets prices are determined endogenously. Agent 0’s choices are limited to his action. Yet, at the allocation induced (endogenously) by the policy, every agent is strictly better-off.

What makes this definition weak is that we are allowing for agent 0 to receive a lump-sum transfer beyond the resulting price value of the portfolio imposed on him. It is important to note that this transfer does not affect the agent’s incentives in choosing his action. Yet, for institutional reasons it may be impossible for the planner to effect such transfer. We shall say that the allocation is constrained inefficient in the strong sense, if such Pareto improvement is possible even when $\tau^0$ is required to be null.

5.2. Genericity of weak constrained inefficiency

Fix the competitive equilibrium, $\{\bar{\vartheta}, \bar{\alpha}, \bar{q}\}$. Our first goal is to show that, generically on the date-1 endowments of individuals, the allocation $(\bar{\alpha}, \bar{c})$, where $c^i_s = e^i_s + \bar{\vartheta}^i_s$ for $s = 1, 2$, is constrained inefficient in the weak sense.

Since the weak definition of constrained inefficiency allows for transfers across all agents, we can again use the fact that preferences are quasilinear to write an expression for social welfare,

$$W = -\alpha + \pi(\alpha) \cdot \sum_{i=0}^1 u^1_i(c^1_i) + [1 - \pi(\alpha)] \cdot \sum_{i=0}^1 u^2_i(c^2_i).$$ (13)

Recall that the problem arises from the inefficient action chosen by agent 0 when maximizing his utility in the competitive setting. We consider an exogenous perturbation in the holdings of securities of agent 0. The idea is to restrict his insurance opportunities to make him more vulnerable to the risks associated with each state. Such perturbation, $(d\vartheta^0_1, d\vartheta^0_2)$, around the competitive equilibrium values induces changes in all other endogenous variables: from Eq. (7), it follows that the action chosen by agent 0 will change; this will induce different consumption and investment decisions by all agents $i \geq 1$; and, in order to guarantee market clearing, the prices of assets will need to accommodate too. If, starting from the equilibrium allocation we can find that there exists a perturbation such that $dW > 0$, we can conclude that the allocation is constrained inefficient in the weak sense: the higher value of the social welfare function $W$ implies the existence of the required profile of lump-sum transfers after which every agent in the economy is made strictly better-off.

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10 So, the planner is not bribing him to choose a higher action.
The total change on the social welfare function, dW, is given by the aggregate of three effects:

1. the direct effect through the change in the action taken by agent 0, −dα;

2. an indirect effect through the different likelihood of the two states that is induced by that change,
\[
\sum_{i=0}^{I} [u^i_1(\bar{c}^i_1) - u^i_2(\bar{c}^i_2)] \cdot \pi'(\bar{\alpha}) \cdot d\alpha;
\]

3. and an indirect effect through the changes in the consumption plans of all individuals, in response to the different likelihoods, the change in prices, and, for agent 0, his exogenously perturbed portfolio:
\[
\pi(\bar{\alpha}) \cdot \sum_{i=0}^{I} \partial u^i_1(\bar{c}^i_1) \cdot dc^i_1 + [1 - \pi(\bar{\alpha})] \cdot \sum_{i=0}^{I} \partial u^i_2(\bar{c}^i_2) \cdot dc^i_2.
\]

Since the only source of inefficiency in our model is agent 0’s action, the third effect vanishes. Indeed, using Eqs. (5) and (8), the total effect can be written simply as
\[
dW = \sum_{i=1}^{I} [u^i_1(\bar{c}^i_1) - u^i_2(\bar{c}^i_2)] \cdot \pi'(\bar{\alpha}) \cdot d\alpha + \bar{q}_1 \cdot \sum_{i=0}^{I} dc^i_1 + \bar{q}_2 \cdot \sum_{i=0}^{I} dc^i_2.
\]

By the feasibility constraint, \( \sum_{i=1}^{I} dc^i_s = 0 \), so that, finally,
\[
dW = \sum_{i=1}^{I} [u^i_1(\bar{c}^i_1) - u^i_2(\bar{c}^i_2)] \cdot \pi'(\bar{\alpha}) \cdot d\alpha. \tag{14}
\]

Now, we can solve for dα by differentiating the first-order condition of agent 0 with respect to his action, Eq. (7), to obtain\(^\text{11}\)
\[
d\alpha = \frac{\pi'(\bar{\alpha}) \cdot [\partial u^0_2(e^0_2 + \bar{\vartheta}^0_2) \cdot d\vartheta^0_2 - \partial u^0_1(e^0_1 + \bar{\vartheta}^0_1) \cdot d\vartheta^0_1]}{\pi''(\bar{\alpha}) \cdot [u^0_1(e^0_1 + \bar{\vartheta}^0_1) - u^0_2(e^0_2 + \bar{\vartheta}^0_2)]}.
\tag{15}
\]

The direction of the Pareto improving policy depends on the sign of expression
\[
\sum_{i=1}^{I} [u^i_1(\bar{c}^i_1) - u^i_2(\bar{c}^i_2)].
\]
Positive values imply that the society, excluding agent 0, is better-off in state 1 than in state 2. As a result, dα must be positive: the competitive action is too low and the Pareto

\(^{11}\) Note that the interiority condition imposed above, namely Assumption 1, implies that
\[
u^0_1(e^0_1 + \bar{\vartheta}^0_1) - u^0_2(e^0_2 + \bar{\vartheta}^0_2) > 0,
\]

since \( e_s^0 + \bar{\vartheta}^0_s = e_s^0 \).

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improving policy involves inducing agent 0 to increase the action chosen at equilibrium. Looking at Eq. (15), this can be achieved with \( \mathrm{d} \theta_2^0 < 0 \) and \( \mathrm{d} \theta_1^0 > 0 \). That is, in order to induce agent 0 to take a higher action at equilibrium, a planner would like to restrict his insurance opportunities in a way that he is better-off in the state associated with a higher action, namely state 1, and worse-off in the other state. The agent who causes the externality is, therefore, “forced” to internalise the externality through considerations of his own welfare.\(^{12}\)

It is important to note that, generically in the space of endowments,

\[
\sum_{i=1}^{I} [u_i^1(c_i^1) - u_i^2(c_i^2)] \neq 0, \tag{16}
\]

so that there almost always is a Pareto improving policy. The proof of this result is shown in the appendix. This analysis implies, then, our first main result.

**Theorem 1.** *If the economy satisfies Assumptions 1 and 2, then, except on a negligible set of individual endowments, the competitive equilibrium allocation is weakly constrained inefficient.*

### 6. Catastrophe, Relief Aid and Genericity of Strong Constrained Inefficiency

If the policy that aims at effecting a Pareto improvement is restricted to not include agent 0 in the profile of date-0 transfers, we can no longer use the social welfare function \( W \). For agent 0, the policy needs to increase

\[
U = - \alpha - q_1 \cdot \bar{\theta}_1^0 - q_2 \cdot \bar{\theta}_2^0 + \pi(\alpha) \cdot (u_1^0(e_1^0 + \bar{\theta}_1^0) + (1 - \pi(\alpha)) \cdot u_2^0(e_2^0 + \bar{\theta}_2^0), \tag{17}
\]

while, simultaneously, increasing

\[
-q_1 \cdot \sum_{i=1}^{I} \bar{\theta}_1^i - q_2 \cdot \sum_{i=1}^{I} \bar{\theta}_2^i + \pi(\alpha) \cdot \left( \sum_{i=1}^{I} u_1^i(e_1^i + \bar{\theta}_1^i) + (1 - \pi(\alpha)) \cdot \sum_{i=1}^{I} u_2^i(e_2^i + \bar{\theta}_2^i). \right.
\]

This would suffice, as the higher value of the latter sub-aggregate implies the existence of the required sub-profile of lump-sum transfers, \((\tau^i)_{i=1}^{I}\), after which all the agents \( i \geq 1 \) are made strictly better-off.

\(^{12}\) Of course, in the case where \( \sum_{i=1}^{I} [u_i^1(c_i^1) - u_i^2(c_i^2)] < 0 \), the society excluding agent 0 is better-off in state 2, so that the competitive action is inefficiently high: in this case, Pareto optimality prescribes \( \mathrm{d} \alpha < 0 \) which can be achieved with \( \mathrm{d} \theta_2^0 > 0 \) and \( \mathrm{d} \theta_1^0 < 0 \). The intuition is, again, that one would like to make agent 0 better-off in the state associated with a lower action and worse-off in the other state so that it is optimal for him to choose a lower action than before.
The following two assumptions will allow us to prove that, in a generic sense, the competitive equilibrium allocation is strongly constrained inefficient. As for the techniques used in obtaining this result, we apply the methods developed by Citanna, Kajii and Villanacci [3].

Assumption 5 (Catastrophe). At the consumption allocation \((\hat{c}^i)_{i=0}^1\), the whole society prefers state 1 to state 2, in the sense that

\[ \mu^0 = u_i^0(\hat{c}^0_i) - u_i^0(\hat{c}^0_2) > 0 \]

and

\[ \mu^{-0} = \sum_{i=1}^I [u_i^1(\hat{c}^1_i) - u_i^2(\hat{c}^2_i)] > 0. \]

Assumption 6 (Relief aid). There exists an external source of funding that aids agents \(i = 1, \ldots, I\) in their purchases of the elementary security for state 2. This fund covers a total of \(\rho > 0\) units of the asset.

We assume that this aid takes the form of lump-sum transfers (of the correct value at equilibrium), so that they have no impact on the characterization of equilibrium we have used. Under the latter assumption, we can write the aggregate utility of agents \(i = 1, \ldots, I\) as

\[ V = -q_1 \sum_{i=1}^I \vartheta_i^1 - q_2 \left( \sum_{i=1}^I \vartheta_i^2 - \rho \right) + \pi(\alpha) \sum_{i=1}^I u_i^1(\hat{c}^i_1 + \vartheta_i^1) + [1 - \pi(\alpha)] \sum_{i=1}^I u_i^2(\hat{c}^i_2 + \vartheta_i^2). \]

6.1. Local subspaces of functions

Under the quasi-linearity assumption, the equilibrium allocation is \(\hat{c}\) and the equilibrium action is \(\hat{\alpha}\). In order to perform genericity analysis on the spaces of utilities and the probability function, we parameterize a local subspace of functions as follows. Given some \(\beta > 0\), let \(b : \mathbb{R} \to [0, 1]\) be a \(C^1\) function such that

\[ b(x) = \begin{cases} 0, & \text{if } |x| > \beta; \\ 1, & \text{if } |x| < \beta/2. \end{cases} \]

We refer to this function as a bump. Also for \(\bar{\delta} > 0\), consider the perturbed mappings

\[ (\alpha, \delta_0) \mapsto \pi(\alpha) + b(\alpha - \bar{\alpha}) \cdot \delta_0 \cdot (\alpha - \bar{\alpha})^2, \]

and, for each \(i \geq 1\) and \(s = 1, 2\),

\[ (c^i_s, \delta^i_s) \mapsto u^i_s(c^i_s) + b(c^i_s - \bar{c}^i_s) \cdot \delta^i_s \cdot (c^i_s - \bar{c}^i_s)^2. \]

13 See, also, Villanacci et Al [10].
For simplicity of notation, we will write these perturbed functions as \( \pi(\cdot; \delta_0) \) and \( u^i_s(\cdot; \delta^i_s) \).

Note that if \( \beta \) and \( \bar{\delta} \) are small enough, these mappings are increasing and concave, so long as \( |\delta_0| < \bar{\delta} \) and \( |\delta^i_s| < \bar{\delta} \) for all \( i = 1, \ldots, I \) and \( s = 1, 2 \). This implies that we can use each of these parameters to perturb the corresponding function on a one-dimensional open neighborhood of the original function. For technical reasons, we restrict attention to the open subset

\[
\Delta = \{ \delta \in (-\bar{\delta}, \bar{\delta})^6 \mid \partial^2 u^1_1(\hat{c}^1_1) + \delta^1_1 \neq \partial^2 u^2_2(\hat{c}^2_2) + \delta^2_2 \text{ and } \partial^2 u^1_1(\hat{c}^1_1) + \delta^1_1 \neq \partial^2 u^2_2(\hat{c}^2_2) + \delta^2_2 \}. \quad (21)
\]

Under Assumption 3, it is immediate that \( 0 \in \Delta \).

The following lemma is immediate.

**Lemma 1 (Invariance of equilibrium to perturbations).** Perturbations to the probability function and preferences of agents \( i = 1, \ldots, I \) do not affect the equilibrium: since \( F(\bar{q}, \bar{\theta}, \bar{\alpha}) = 0 \), tuple \( (\bar{q}, \bar{\theta}, \bar{\alpha}) \) continues to be an equilibrium when the probability function is \( \pi(\cdot; \delta_0) \) and preferences are \( u^i_s(\cdot; \delta^i_s) \), as long as \( |\delta_0| < \bar{\delta} \) and \( |\delta^i_s| < \bar{\delta} \) for all \( i = 1, \ldots, I \) and \( s = 1, 2 \).

The lemma exploits one key property of our construction: that the perturbations do not affect the first derivatives of the perturbed functions at the equilibrium values. What they do, however, is to affect their second derivatives, which is the second key property — one that we will use below.

### 6.2. A characterization of strong constrained inefficiency

In order to make the concept of strong constrained inefficient easier to analyse, define the following function, where, for simplicity, we assume that \( I = 2 \),

\[
(q, \theta, \alpha, \delta) \mapsto \begin{pmatrix}
U \\
V \\
[u^0_1(e^0_2 + \theta^0_2) - u^0_2(e^0_2 + \theta^0_2)] \cdot \pi'(\alpha; \delta_0) - 1 \\
\pi(\alpha; \delta_0) \cdot \partial u^1_1(e^1_1 + \theta^1_1; \delta^1_1) - q_1 \\
[1 - \pi(\alpha; \delta_0)] \cdot \partial u^1_2(e^1_1 + \theta^1_1; \delta^1_1) - q_2 \\
\pi(\alpha; \delta_0) \cdot \partial u^2_2(e^2_1 + \theta^2_1; \delta^2_1) - q_1 \\
[1 - \pi(\alpha; \delta_0)] \cdot \partial u^2_2(e^2_1 + \theta^2_1; \delta^2_1) - q_2 \\
\sum_{i=0}^1 \theta^i_1 \\
\sum_{i=0}^1 \theta^i_2
\end{pmatrix}, \quad (22)
\]
where $U$ and $V$ are as in Eqs. (17) and (20), respectively, with $\pi(\cdot; \delta_0)$ and $u^i_s(\cdot; \delta^i_s)$ instead of $\pi(\cdot)$ and $u^i_s(\cdot)$.$^{14}$

Denoting this mapping by $H(q, \vartheta, \alpha; \delta)$, the following lemma follows as an implication of the definition of strong constrained inefficiency, and of the construction of functions $U$ and $V$.

**Lemma 2.** If $\{\bar{q}, \bar{\alpha}, \bar{\vartheta} \}$ is a competitive equilibrium and the partial Jacobean of $H$ with respect to $(q, \vartheta, \alpha)$ has full row rank, then the equilibrium allocation $(\bar{\alpha}, \bar{\vartheta})$, where $\bar{c}^i_s = e^i_s + \bar{\vartheta}^i_s$ for $s = 1, 2$, is constrained inefficient in the strong sense.

**Proof.** If the Jacobean

$$D_{q, \vartheta, \alpha}H(q, \vartheta, \alpha; \delta)$$

has full row rank, by the inverse function theorem it follows that $H(\cdot; \delta)$ maps a neighbourhood of $(q, \vartheta, \alpha)$ onto a neighbourhood of $H(q, \vartheta, \alpha; \delta)$. It then follows that, for a small enough $d > 0$, we can find $(\bar{q}, \bar{\vartheta}, \bar{\alpha})$ such that

$$H(q, \vartheta, \alpha; \delta) = H(q, \vartheta, \alpha; \delta) + (d, d, 0, \ldots, 0)^T.$$

At equilibrium, by definition, the third to last entries of $H(q, \vartheta, \alpha; \delta)$ equal 0. Substituting, this means that

$$
\begin{pmatrix}
U(q, \alpha, \vartheta; \delta) \\
V(q, \alpha, \vartheta; \delta) \\
[u^2_1(e^0_2 + \bar{\vartheta}^0_2) - u^2_2(e^0_2 + \bar{\vartheta}^2_2)] \cdot \pi'(\alpha; \delta_0) - 1 \\
\pi(\alpha; \delta_0) \cdot \partial u^1_1(e^1_1 + \bar{\vartheta}^1_1; \delta^1_1) - \bar{q}_1 \\
[1 - \pi(\alpha; \delta_0)] \cdot \partial u^1_2(e^1_2 + \bar{\vartheta}^1_2; \delta^1_2) - \bar{q}_2 \\
\pi(\alpha; \delta_0) \cdot \partial u^2_1(e^2_1 + \bar{\vartheta}^2_1; \delta^2_1) - \bar{q}_1 \\
[1 - \pi(\alpha; \delta_0)] \cdot \partial u^2_2(e^2_2 + \bar{\vartheta}^2_2; \delta^2_2) - \bar{q}_2 \\
\sum_{i=0}^1 \bar{\vartheta}^i_1 \\
\sum_{i=0}^1 \bar{\vartheta}^i_2
\end{pmatrix}
= 
\begin{pmatrix}
U(q, \alpha, \vartheta; \delta) + d \\
V(q, \alpha, \vartheta; \delta) + d \end{pmatrix}
$$

The first entry of this equality and the fact that $d > 0$ imply that agent 0 is made better-off, which is the fifth requirement of the definition of weak constrained inefficiency. The second entry implies that the aggregate $V$ of utilities is higher too, which in turn implies that every other agent can be made better-off by an appropriate choice of lump-sum transfers, hence guaranteeing the sixth requirement in the definition (without violating the fourth

$^{14}$ When writing a function, the notation $f(x; \omega)$ is used to emphasize that $x$ is the function's argument and $\omega$ is just a parameter. Hopefully, this makes it clear that $\partial f(x; \omega)$ refers to the derivative of the function with respect to argument $x$, when the parameter takes the value $\omega$. 

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requirement). Noting that the third entry implies the first-order condition of Program (12),
the first requirement is also satisfied. The second requirement follows from the fourth to the
ante-penultimate entries, as each successive pair of them implies the first-order conditions of
Program (4), for agents $i = 1, \ldots, I$. Finally, the last two entries imply the third requirement
of the definition.

6.3. Genericity of strong constrained inefficiency

The characterization of constrained inefficiency by Lemma 2 seemingly allows us to prove
a second generic result.

**Theorem 2.** If the economy satisfies Assumptions 1 to 6, then, on an open and dense
set of individual endowments, probability functions and individual preferences, the
competitive equilibrium allocation is strongly constrained inefficient.

Fixing all preferences and endowments such that the assumptions hold true, the chal-
lenge is to show that mapping

$$(q, \vartheta, \alpha, \delta, \gamma) \mapsto \begin{pmatrix}
\mathcal{F}(q, \vartheta, \alpha; \delta) \\
D_{q, \vartheta, \alpha} \mathcal{H}(q, \vartheta, \alpha; \delta)^T \gamma \\
\frac{1}{2} (\gamma \cdot \gamma - 1)
\end{pmatrix},$$

where $\gamma \in \mathbb{R}^{2+2(I+1)+1}$, is transverse to 0. Let $\mathcal{M}$ denote the previous mapping.

Note that the entries of vector $\gamma$ can be named according to the rows of the Jacobean
matrix $D_{q, \vartheta, \alpha} \mathcal{H}$. Recalling the definition of mapping $\mathcal{H}$, these rows are:

1. the utility level of agent 0, $U$;
2. the aggregate utility of all other agents, $V$;
3. the first-order condition with respect to his action, in the maximisation of agent 0;
4. for each $i$ other than 0 and each $s$, the first-order condition with respect to $\vartheta_i^s$, in the
   maximization of agent $i$; and
5. for each $s$, the market clearing condition of the corresponding elementary security.

It is then convenient to use a mnemonic to denote the entries of the vector, as follows:

$$\gamma = (\gamma_U, \gamma_V, \gamma^0_1, \gamma^1_1, \gamma^2_1, \gamma_2^1, \gamma_2^2, \gamma_1, \gamma_2).$$

**Lemma 3** (Simplifying the system). Suppose that $\mathcal{M}(q, \vartheta, \alpha, \delta, \gamma) = 0$. Then,
1. \((q, \bar{\theta}, \alpha) = (\bar{q}, \bar{\bar{\theta}}, \bar{\alpha})\);

2. the second component of \(M\), namely \(D_{q,\bar{\theta},\alpha}H(\bar{q}, \bar{\bar{\theta}}, \bar{\alpha}; \bar{\delta})^T\gamma\), simplifies to

\[
\begin{pmatrix}
\pi' \cdot \sum_{i=1}^{2} [(u_i^1 - u_i^2) \cdot \gamma_V + (\partial u_i^1 \cdot \gamma_1 - \partial u_i^2 \cdot \gamma_2)] + (\pi'' + \delta_0) \cdot (u_1^0 - u_2^0) \cdot \gamma^0 \\
\pi' \cdot \partial u_i^0 \cdot \gamma^0 + \gamma_1 \\
-\pi' \cdot \partial u_i^2 \cdot \gamma^0 + \gamma_2 \\
(\partial^2 u_i^1 + \delta_i^1) \cdot \gamma_i^1 + \gamma_1 \\
(\partial^2 u_i^2 + \delta_i^2) \cdot \gamma_i^2 + \gamma_2 \\
(\partial^2 u_i^2 + \delta_i^2) \cdot \gamma_i^2 + \gamma_2 \\
-\delta_i^0 \cdot (\gamma_1 - \gamma_2) - \gamma_i^1 - \gamma_i^2 \\
-\delta_i^0 \cdot (\gamma_1 - \gamma_2) + \rho \cdot \gamma_V - \gamma_i^2 - \gamma_i^2
\end{pmatrix}
\]

\((*)\)

3. \(\gamma_1 \neq 0\) and \(\gamma_2 \neq 0\);

4. \(\gamma^0 \neq 0\), and for both \(i = 1, 2\) and both \(s = 1, 2\), \(\gamma_s^i \neq 0\); and

5. either \(\gamma_1^1 \neq \gamma_2^1\) or \(\gamma_1^2 \neq \gamma_2^2\).

**Proof.** The first statement follows immediately from Lemma 1, as \(M = 0\) implies that \(F = 0\). For the second statement, it follows by direct computation that \(D_{q,\bar{\theta},\alpha}H(\bar{q}, \bar{\bar{\theta}}, \bar{\alpha}; \bar{\delta})^T\gamma\) equals the sum of expression \((*)\) and

\[
\begin{pmatrix}
[1 - \pi' \cdot (u_1^0 - u_2^0)] \cdot \gamma_u \\
(-q_1 + \pi \cdot \partial u_1^0) \cdot \gamma_u \\
[q_2 + (1 - \pi) \cdot \partial u_2^0] \cdot \gamma_u \\
(-q_1 + \pi \cdot \partial u_1^1) \cdot \gamma_V \\
[q_2 + (1 - \pi) \cdot \partial u_2^1] \cdot \gamma_V \\
(-q_1 + \pi \cdot \partial u_1^2) \cdot \gamma_V \\
[q_2 + (1 - \pi) \cdot \partial u_2^2] \cdot \gamma_V \\
0 \\
0
\end{pmatrix}
\]

At \((\bar{q}, \bar{\bar{\theta}}, \bar{\alpha})\), this latter vector vanishes.

For the third statement, suppose, by way of contradiction, that \(\gamma_1 = 0\). It follows from the second equation of the system \(D_{q,\bar{\theta},\alpha}H(\bar{q}, \bar{\bar{\theta}}, \bar{\alpha}; \bar{\delta})^T\gamma = 0\) that \(\gamma^0 = 0\). In the third equation, this implies that \(\gamma_2 = 0\) too. The fourth to seventh equations then imply that

\[
\gamma_1^1 = \gamma_2^1 = \gamma_1^2 = \gamma_2^2 = 0.
\]
By Assumption 4, the previous-to-last equation implies that $\gamma_U = \gamma_V$. Then, Assumption 6 implies, via the last equation, that $\gamma_V = 0$. Summing up, $\gamma = 0$, which is impossible since $\gamma \cdot \gamma = 1$, as $M = 0$.

For the fourth statement, note that if $\gamma^0 = 0$, the second equation implies that $\gamma_1 = 0$, which we just showed to be impossible. The same occurs if $\gamma_3^i = 0$: then, from the fourth to seventh equations it would follow that $\gamma_s = 0$, contradicting the third statement.

For the last statement, suppose to the contrary that $\gamma_1^1 = \gamma_2^1$ and $\gamma_1^2 = \gamma_2^2$. The fourth and fifth equations imply, given Assumption 6, that $\gamma_U = \gamma_V = 0$. From the last four equations we then have that

$$\partial^2 u_1^1 + \delta_1^1 = \partial^2 u_1^2 + \delta_1^2$$
and
$$\partial^2 u_2^1 + \delta_1^1 = \partial^2 u_2^2 + \delta_1^2,$$
which is impossible by the definition of the space $\Delta$ of perturbations, Eq. (21). \( \square \)

**Lemma 4** (Transversality). *Mapping $M$ is transverse to 0.*

*Proof.* Suppose that $M = 0$. It follows from Lemma 1 that its Jacobean can be written as

$$\begin{pmatrix} D_{q,\theta,\alpha}F & 0 \\ M & \Omega \end{pmatrix},$$
where $\Omega$ is the partial Jacobean of mapping

$$(q, \theta, \alpha, \delta, \gamma) \mapsto \begin{pmatrix} D_{q,\theta,\alpha}H(q, \theta, \alpha; \delta)^T \gamma \\ \frac{1}{2}(\gamma \cdot \gamma - 1) \end{pmatrix}$$
with respect to $(\delta, \gamma)$.\(^{15}\)

Then, given Assumption 4, all we need to show is that matrix $\Omega$ has full row rank. To make it easier to see that this is indeed the case, it is convenient to re-organize this matrix. Note that the first rows of this matrix correspond to the entries of the product $D_{q,\theta,\alpha}H(q, \theta, \alpha)^T \gamma$.\(^{16}\) By construction, these rows correspond to the arguments with respect to which mapping $H$ has been differentiated, which we have been writing in the order

$$(\alpha, \theta^0_1, \theta^0_2, \theta^1_1, \theta^1_2, \theta^2_1, \theta^2_2, q_1, q_2).$$
It is now convenient, in fact, to write the last two rows of the matrix in the fourth and fifth positions, which amounts to taking the derivatives of $H$ in the order

$$(\alpha, \theta^0_1, \theta^0_2, q_1, q_2, \theta^1_1, \theta^1_2, \theta^2_1, \theta^2_2).$$

\(^{15}\) Matrix $M$ is the partial Jacobean of this same mapping with respect to $(q, \theta, \alpha)$. We need not concern ourselves with its computation.

\(^{16}\) These rows are followed by only one other: the derivatives of the last row of $M$. 

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Of course, the rank of $\Omega$ is not affected by this operation. As for its columns, it will also be convenient to write them in an unusual order:

$$(\delta^0, \gamma_1, \gamma_2, \gamma_u, \gamma_v, \delta^1_1, \delta^1_2, \delta^2_1, \delta^2_2, \gamma_1^1, \gamma_2^1, \gamma_1^2, \gamma_2^2).$$

One remaining argument, $\gamma^0$, will not be necessary for our differentiations.

Permuted in this way, when $M = 0$ matrix $\Omega$ reads as

$$
\begin{pmatrix}
\mu^0 \gamma_0 & 0 & 0 & 0 & \pi^0 \mu^{-0} & 0 & 0 & 0 & 0 & \pi^0 \partial u_1^1 & -\pi^0 \partial u_1^2 & \pi^0 \partial u_2^1 & -\pi^0 \partial u_2^2 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\theta^0_1 & \theta^0_2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & -\theta^0_2 & \theta^0_2 + \rho & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & \pi \gamma_1^1 & 0 & 0 & 0 & 0 & \pi \gamma_1^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (1-\pi) \gamma_2^1 & 0 & 0 & 0 & 0 & \pi (1-\pi) \gamma_2^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (1-\pi) \gamma_1^2 & 0 & 0 & 0 & 0 & \pi (1-\pi) \gamma_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \pi \gamma_1^1 & 0 & 0 & 0 & 0 & \pi \gamma_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \pi \gamma_2^1 & 0 & 0 & 0 & 0 & \pi \gamma_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \pi (1-\pi) \gamma_1^1 & 0 & 0 & 0 & \pi (1-\pi) \gamma_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \pi (1-\pi) \gamma_2^1 & 0 & 0 & 0 & \pi (1-\pi) \gamma_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
$$

where $\mu^0$ and $\mu^{-0}$ come from Eqs. (18) and (19), respectively, and $h^1_s$ is used to denote $\partial^0 u^1_s + \delta^1_s$, for brevity.

Consider first the leading principal minor of order 5, namely the matrix

$$
\begin{pmatrix}
\mu^0 \cdot \gamma_0 & 0 & 0 & 0 & \pi^0 \cdot \mu^{-0} \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -\theta^0_1 & \theta^0_2 \\
0 & 0 & 0 & -\theta^0_2 & \theta^0_2 + \rho \\
\end{pmatrix}.
$$

Under Assumptions 5 and 6, this matrix is non-singular, thanks to the fact that $\gamma^0 \neq 0$, as per Lemma 3. Now, add the next four columns and rows. This adds, the columns

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\pi \cdot \gamma_1^1 & 0 & 0 & 0 & 0 \\
0 & (1-\pi) \cdot \gamma_2^1 & 0 & 0 & 0 \\
0 & 0 & \pi \cdot \gamma_1^2 & 0 & 0 \\
0 & 0 & 0 & \pi \cdot \gamma_2^2 & 0 \\
0 & 0 & 0 & 0 & (1-\pi) \cdot \gamma_1^1 \\
0 & 0 & 0 & 0 & (1-\pi) \cdot \gamma_2^1 \\
\end{pmatrix}.
$$

Given that for both $i = 1, 2$ and both $s = 1, 2$, $\gamma^1_s \neq 0$, again as per Lemma 3, it follows that this whole $9 \times 9$ leading principal minor is non-singular.
It only remains to show that when we add the last row of the matrix and the four remaining columns, the whole matrix maintains its full row rank. Given the last statement in Lemma 3, we can assume, with no loss of generality, that \( \gamma_1 \neq \gamma_2 \). Now, perform the following operations:

1. add the columns corresponding to \( \gamma_1 \);
2. subtract the column corresponding to \( \gamma_2 \);
3. subtract \( h_1/\gamma_1 \) times the column corresponding to \( \delta_1 \);
4. add \( h_2/\gamma_2 \) times the column corresponding to \( \delta_2 \); and
5. subtract \( \frac{\pi^\prime \cdot (\partial u_1 - \partial u_2)}{(u_0^0 - u_0^2) \cdot \gamma_0} \) times the column corresponding to \( \delta_0 \), which can be done thanks to Assumption 5 and the fourth statement in Lemma 3.

Note that the result of these operations is vector

\[
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\gamma_1 - \gamma_2
\end{pmatrix}
\]

Since \( \gamma_1 \neq \gamma_2 \), the last entry of this resulting vector is non-zero. This implies that the matrix, as a whole, can also span the last dimension of its co-domain, and, given the previous results, that it has full row rank. \( \square \)

We are now ready to prove that, generically on endowments, preferences and the probability function, the competitive equilibrium allocation is constrained inefficient in the strong sense too, under the extra assumptions.

**Proof of Theorem 2.** Since \( \mathcal{M} \cap 0 \), by the transversality theorem we will have that \( \mathcal{M}(\cdot, \delta) \cap 0 \), generically on \( \delta \in \Delta \). Now, \( \mathcal{M}(\cdot, \delta) \) has

\[
3 + 2I + 2 + 2(I + 1) + 1 + 1
\]
entries, and only
\[ 2 + 2(I + 1) + 1 + 2 + 2(I + 1) + 1 \]
arguments. It follows that \( D_{q,0,\alpha,\gamma}M \) cannot have full row rank, and hence it must be that, generically on \( \delta \in \Delta \), \( M(q, \bar{\theta}, \alpha, \gamma, \delta) \neq 0 \).

Now, if that is the case, it is immediate that whenever
\[ \mathcal{F}(q, \bar{\theta}, \alpha, e, \gamma) = 0 \text{ and } D_{q,0,\alpha}H(q, \bar{\theta}, \bar{\alpha}, e)^{\mathsf{T}} \gamma = 0, \]
we also have that \( \gamma = 0 \). This implies, by construction, that when \( \mathcal{F}(q, \bar{\theta}, \alpha, e, u, \pi) = 0 \), matrix \( D_{q,0,\alpha}H(q, \bar{\theta}, \bar{\alpha}, e) \) has full row rank. By Lemma 2, it follows that the equilibrium allocation is constrained inefficient, in the strong sense, generically on \( \delta \in \Delta \).

This result implies the claim, as long as we endow the space of preference and probability functions with a metrisable topology: for any given initial functions, any open neighbourhood of these functions will intersect the lower-dimensional subspace \( \Delta \), since \( 0 \in \Delta \). Since the equilibrium allocation is strongly constrained inefficient generically on \( \Delta \), we can find an array of functions in the intersection of the open neighbourhood and \( \Delta \) where this is the case. This implies denseness, as needed. \( \square \)

7. More Sophisticated Behaviour by Agent 0

Our analysis so far has assumed that all agents, including 0, take assets prices as given. This is the application to our context of the standard definition of Nash-Walras equilibrium from Public Economics. In Financial Economics, on the other hand, when an agent has the ability to affect the probability distribution of shocks in the economy, it is usually considered that she recognizes the effect that her decisions \( \text{with respect to those probabilities} \) affect the willingness to pay of other agents for the existing financial assets and, therefore, the equilibrium level of assets prices. In a sense, the point is that the kind of externality that agent 0 induces in our economy is \textit{too salient} for him to take everybody else’s actions as given.

7.1. Stackelberg-Walras Equilibrium

We now recognise this possibility and study whether our previous results are robust to this more sophisticated behaviour by agent 0. In order to do this, we need to decompose asset prices into the part of them that depends on the probability distribution and the part that is determined by trade, given the probabilities. Effectively, it is as if decisions were made sequentially in the first period: individual 0 first chooses the action he is to take, and then,
taking that action as given, he and all the other agents trade financial assets in a competitive manner.

Luckily, our formulation allows us to nest these two decisions in just one problem for agent 0. In order to do this, we first introduce two functions that are of common use in financial economics. For each state \( s \), and each level of trading of the corresponding elementary security for that state by agent 0, \( \tilde{\vartheta}^0_s \), let \( \tilde{\zeta}^i_s \) denote the solution to the following maximisation problem:

\[
\max_{\{c^i\}_{i=1}^I} \left\{ \sum_{i=1}^I u^i_s (c^i) : \sum_{i=1}^I c^i_s = \sum_{i=0}^I e^i_s - \tilde{\vartheta}^0_s \right\}.
\] (23)

Obviously, the solution to this problem depends on \( \tilde{\vartheta}^0_s \), and so we can define \( \kappa_s(\tilde{\vartheta}^0_s) = \partial u^1_s (\tilde{\zeta}^1_s) \), a function to which we will refer as the (ex-post) pricing kernel for state \( s \).

The utility of this function is that If \( \{\bar{\vartheta}, \bar{\alpha}, \bar{q}\} \) is a competitive equilibrium (in the sense defined before), it follows from Eq. (5), that the equilibrium prices decompose into the product of the pricing kernel for each state and its corresponding probability:

\[
\begin{pmatrix}
\bar{q}_1 \\
\bar{q}_2
\end{pmatrix} = \begin{pmatrix}
\pi(\bar{\alpha}) \cdot \kappa_1(\bar{\vartheta}^0_1) \\
(1 - \pi(\bar{\alpha})) \cdot \kappa_2(\bar{\vartheta}^0_2)
\end{pmatrix}
\]

Since we want to maintain the assumption that all agents are competitive in the financial markets, we will now consider the situation in which agent 0 recognises the direct effect that his choice of action has on the vector of asset prices, via the probabilities, but acts as if his choice of portfolio did not affect the pricing kernels. The behaviour of all other agents in the economy remains unchanged, as well as the market clearing condition. For the sake of definiteness, we refer to this situation as Stackelberg-Walras equilibrium, defined as follows: it is a tuple \( \{\bar{\vartheta}, \bar{\alpha}, \bar{q}\} \), where \( \bar{\vartheta} = [(\bar{\vartheta}^i)_s]_{s=1,2} \) is the allocation of the two assets and \( \bar{q} = (\bar{q}_1, \bar{q}_2) \) is the vector of asset prices, such that:

1. action \( \bar{\alpha} \) and portfolio \( (\bar{\vartheta}^0_s)_{s=1,2} \) solve the following Program:

\[
\max_{\alpha, \bar{\vartheta}^0_1, \bar{\vartheta}^0_2} \left\{ -\alpha - q_1(\alpha; \bar{\vartheta}^0_1)\bar{\vartheta}^0_1 - q_2(\alpha; \bar{\vartheta}^0_2)\bar{\vartheta}^0_2 + \pi(\alpha)u^0_1(e^0_1 + \bar{\vartheta}^0_1) + (1 - \pi(\alpha))u^0_2(e^0_2 + \bar{\vartheta}^0_2) \right\}
\] (24)

where \( q_1(\alpha; \bar{\vartheta}^0_1) = \pi(\alpha) \cdot \kappa_1(\bar{\vartheta}^0_1) \) and \( -q_2(\alpha; \bar{\vartheta}^0_2) = (1 - \pi(\alpha)) \cdot \kappa_2(\bar{\vartheta}^0_2) \);

2. for each \( i \geq 1 \), portfolio \( (\bar{\vartheta}^i_s)_{s=1,2} \) solves Program (4) when the prices are \( q = \bar{q} \) and the probability of state 1 is \( \pi(\bar{\alpha}) \); and

3. both of the securities markets clear: \( \sum_{i=0}^I \bar{\vartheta}^i_s = 0 \), for \( s = 1, 2 \).
Importantly, as agent 0 recognises the effect of his action on prices, Eq. (7) is no longer a valid first-order condition of his problem; instead, now his behaviour is characterized by the requirement that
\[
1 = \pi'(\alpha) \cdot \{-[\kappa_1(\bar{\theta}_0) - \kappa_2(\bar{\theta}_2)] + [u_1^0(e_1^0 + \bar{\theta}_1^0) - u_2^0(e_2^0 + \bar{\theta}_2^0)]\} \tag{25}
\]
along with Eqs. (8) evaluated at \((\bar{\alpha}, \bar{\theta}_1^0, \bar{\theta}_2^0)\).

### 7.2. Weak constrained inefficiency

As before, if we fix a Stackelberg-Walras equilibrium, \((\bar{\theta}, \bar{\alpha}, \bar{q})\), our goal is again to show that the allocation \((\bar{\alpha}, \bar{c})\), where \(\bar{c}_i = e_i^s + \bar{\theta}_s^i\), is constrained inefficient in the weak sense, generically on date-1 endowments.

With the different first-order condition with respect to agent 0’s action, Eq. (25), the results of §5.2 imply that, instead of Eq. (14), we obtain
\[
dW = \left\{[\kappa_1(\bar{\theta}_0) \cdot \bar{\theta}_1^0 - \kappa_2(\bar{\theta}_2) \cdot \bar{\theta}_2^0] + \sum_{i=1}^I[u_i^1(\bar{c}_i^1) - u_i^2(\bar{c}_i^2)]\right\} \cdot \pi'(\bar{\alpha}) \cdot d\alpha \tag{26}
\]
while
\[
d\alpha = \frac{\pi'(\bar{\alpha}) \cdot [\partial_2(\bar{\theta}_2^0) \cdot \partial u_2^0(e_1^0 + \bar{\theta}_1^0) \cdot d\bar{\theta}_2^0 - \partial_1(\bar{\theta}_2^0) \cdot \partial^2 u_2^0(e_1^0 + \bar{\theta}_1^0) \cdot d\bar{\theta}_1^0]}{\pi''(\bar{\alpha}) \cdot \left\{[\kappa_1(\bar{\theta}_0^0) \cdot \bar{\theta}_1^0 - \kappa_2(\bar{\theta}_2^0) \cdot \bar{\theta}_2^0] + [u_1^0(e_1^0 + \bar{\theta}_1^0) - u_2^0(e_2^0 + \bar{\theta}_2^0)]\right\}}. \tag{27}
\]
Generically on date-1 endowments, a policy can induce \(d\alpha\) such that \(dW > 0\).

### 8. Uninsurable Idiosyncratic Risk

As an alternative framework, and in order to show that these results extend to economies with idiosyncratic risk, we now study a model where the agents are subject to uninsurable idiosyncratic shocks. For the sake of brevity, we restrict attention to the weaker definition of constrained inefficiency.

Individuals are of different types, \(i = 0, \ldots, I\), and within each type there is a continuum of individuals of mass \(m_i^0\). For simplicity, we assume that \(m_0^0 = 1\).

Individuals of different types differ in their period-1 preferences and in their endowments, but they face the same idiosyncratic shocks. For simplicity, let us assume that there are only three personal states, denoted by \(s = 1, 2, 3\). In \(s = 1\) there is no shock in the endowment of the consumption good, while in \(s = 2, 3\) there is a positive and negative shock, respectively, of size \(z\). In period 1, a fraction \(\pi(\alpha)\) of the individuals find themselves in state 1, while equal fractions of size \(\frac{1}{2}(1 - \pi(\alpha))\) find themselves in states 2 and 3. However, there is no aggregate uncertainty: the aggregate endowment of the economy in period 1 is
\[ \sum_{i=0}^{1} m^i \cdot \bar{e}^i, \text{ where } \bar{e}^i \text{ is the endowment of individuals of type } i \text{ in state } 1, \text{ where there is no shock.} \]

Once again, the probabilities of each personal state depend on the aggregate action that agents of type 0 will choose. In particular, while the expected value of the endowment remains unchanged, a higher action increases the probability of observing no shock and decreases the probability of positive and negative shocks. A lower action, in other words, induces a mean-preserving spread in the distributions of the agents’ wealth. Therefore, a risk averse agent would prefer a distribution that second-order stochastically dominates, and this is why he would choose a non-zero action.

8.1. Competitive equilibrium

Suppose there is only a risk-less asset that can be traded: it pays one unit of the consumption good at date 1. Holdings of the asset are \( b^i \) and its price is \( q \).

8.1.1. The problem of agents of type \( i \geq 1 \)

Once again, all agents of types other than 0 have to choose their holdings of the riskless bond, and therefore the consumption in period 0 and in every personal state in period 1. The problem that an agent of type \( i \geq 1 \) faces is to choose \( b^i \) so as to maximize

\[ -q \cdot b^i + \pi(\alpha) \cdot u^i(e^i + b^i) + \frac{1}{2} \cdot [1 - \pi(\alpha)] \cdot [u^i(e^i + z + b^i) + u^i(e^i - z + b^i)]. \]

These agents take the price of the riskless bond and the probabilities of the personal states as given. The first-order condition of this problem is, then, that

\[ q = \pi(\alpha) \cdot \partial u^i(e^i + b^i) + \frac{1}{2} \cdot [1 - \pi(\alpha)] \cdot [\partial u^i(e^i + z + b^i) + \partial u^i(e^i - z + b^i)]. \] (28)

8.1.2. The problem of agents of type 0

On the other hand, each agent of type 0 has to maximize

\[ -\alpha - q \cdot b^0 + \pi(\alpha) \cdot u^0(e^0 + b^0) + \frac{1}{2} \cdot [1 - \pi(\alpha)] \cdot [u^0(e^0 + z + b^0) + u^0(e^0 - z + b^0)] \]

by his choice of savings, \( b^0 \), and action, \( \alpha \).

It is important to note that, although we have assumed that the probability of each state depends on the aggregate action exerted by all agents of type 0, when an agent \( j \) of type 0 solves his maximisation problem, he sees the probability of each state as depending
only on his own action.\footnote{That is, strictly speaking: each agent \( j \) of type 0 chooses an action \( \alpha_j \), considering the probability \( \pi(\alpha_j) \); agents of types \( i \geq 1 \), on the other hand, take as given the probability \( \pi(\int \alpha \, dj) \).} In equilibrium, this does not matter as all agents of type 0 choose the same action.

Here, the first-order conditions are that\footnote{As with the case of complete markets, we can make interiority assumptions so that we do not have to look for boundary solutions.}

\[
1 = \pi'(\alpha) \cdot \left[ (u^0(e^0 + b^0) - \frac{1}{2} \cdot u^0(e^0 + z + b^0) - \frac{1}{2} \cdot u^0(e^0 - z + b^0)) \right]
\] (29)

and

\[
q = \pi(\alpha) \cdot \partial u^0(e^0 + b^0_j) + \frac{1}{2} \cdot [1 - \pi(\alpha)] \cdot [\partial u^0(e^0 + z + b^0) + \partial u^0(e^0 - z + b^0)] .
\] (30)

8.1.3. Nash-Walras equilibrium

As before, the first-order conditions of individual rationality and the market clearing requirement characterise competitive equilibrium. We denote competitive equilibria by \( \{\bar{b}, \bar{\alpha}, \bar{q}\} \), where \( \bar{b} = (\bar{b}_i)_{i=0} \) is a profile of savings. These values solve the first-order conditions, Eqs. (28), (29) and (30), as well as the equality \( \sum_{i=0}^I m^i \cdot b^i = 0 \).

8.2. Constrained inefficiency of competitive equilibrium

Now, consider a policy intervention that perturbs by \( d b^0 \) the holdings of the riskless bond of all agents of type 0. The welfare effects of such policy around the competitive equilibrium point, \( dW \) are, as before, the sum of:

1. the direct loss due to a different action, \(-d\alpha\);

2. the indirect effect due to the change in probabilities,

\[
\sum_{i=0}^I m^i \cdot \left\{ u^i(\bar{c}_1^i) - \frac{1}{2} \cdot [u^i(\bar{c}_2^i) + u^i(\bar{c}_3^i)] \right\} \cdot \pi'(\bar{\alpha}) \cdot d\alpha,
\]

where \( \bar{c}_s^i \) represents the equilibrium consumption of agents of type \( i \) in state \( s \); and

3. the indirect effect due to the reallocation of savings,

\[
\pi(\bar{\alpha}) \cdot \sum_{i=0}^I m^i \cdot \partial u^i(\bar{c}_1^i) \cdot d\bar{c}_1^i + \frac{1}{2} \cdot [1 - \pi(\bar{\alpha})] \cdot \sum_{i=0}^I m^i \cdot [\partial u^i(\bar{c}_2^i) \cdot d\bar{c}_2^i + \partial u^i(\bar{c}_3^i) \cdot d\bar{c}_3^i] .
\]
As before, taking into account the first-order conditions of the agents at the equilibrium point and the market clearing conditions, this expression simplifies to

$$dW = \sum_{i=1}^{1} m^i \cdot \left\{ u^i(\bar{c}^i_1) - \frac{1}{2} \cdot [u^i(\bar{c}^i_2) + u^i(\bar{c}^i_3)] \right\} \cdot \pi'(\bar{\alpha}) \cdot d\alpha, \quad (31)$$

where we can substitute

$$d\alpha = -\frac{\pi'(\bar{\alpha}) \cdot \partial u^0(\bar{c}^0_1) \cdot dc^0_1 - \frac{1}{2} \cdot [\partial u^0(\bar{c}^0_2) \cdot dc^0_2 + \partial u^0(\bar{c}^0_3) \cdot dc^0_3]}{u^0(\bar{c}^0_1) - \frac{1}{2} \cdot [u^0(\bar{c}^0_2) + u^0(\bar{c}^0_3)]}. \quad (32)$$

By strong concavity,

$$\sum_{i=1}^{1} m^i \cdot u^i(e^i + \bar{b}^i) - \frac{1}{2} \cdot \sum_{i=1}^{1} m^i \cdot u^i(e^i + z + \bar{b}^i) - \frac{1}{2} \cdot \sum_{i=1}^{1} m^i \cdot u^i(e^i - z + \bar{b}^i) > 0,$$

which implies that the competitive action is inefficiently low: an increase in the action would be welfare improving for the whole society. Then, Eq. (15) implies that the direction of the perturbation of the bond holdings of agent 0 that implements a higher action depends on the sign of expression

$$\partial u^0(e^0 + \bar{b}^0) \cdot dc^0_1 - \frac{1}{2} \cdot [\partial u^0(e^0 + z + \bar{b}^0) \cdot dc^0_2 + \partial u^0(e^0 - z + \bar{b}^0) \cdot dc^0_3].$$

If we now assume that agents of type 0 are prudent, we conclude that a perturbation $db^0 < 0$ induces an improvement in social welfare: if these agents save below the equilibrium level, with convex marginal utility they will exert a higher action since, by doing so, they make the future look less “volatile”. This reduction in volatility makes the aggregate welfare higher, in spite of the fact that a higher action subtracts from the aggregate welfare functions and even though it implies an imperfect operation of the financial market.

9. **Concluding Remarks**

The BP Deepwater Horizon oil spill of 2010 has focused considerable attention on the potential liability and the operating conduct of big oil companies. In this case, BP’s unusually “deep pockets” made full compensation to the victims feasible, drawing attention away from the glaring safety failures in both the private and the public sector. Big oil companies make an annual purchase of roughly $500 millions of liability insurance, an amount much lower than the $20 billion fund that BP was able to self-raise. It seems that big oil companies, the same ones that are more likely to cause a catastrophic oil spillage, do not need to buy liability insurance at all: they have a sufficiently diversified investment portfolio that allows them to self-insure for the vast majority of their potential catastrophic liability. Future
spills, however, may not follow this pattern revealing the need to create incentives for the parties involved in activities with potential catastrophic environmental consequences, to take actions to prevent such events.

In this spirit, this paper suggests that limiting the ability of a company to insure and diversify its risks, creates incentives to internalise the welfare effects of catastrophic events, leading to a welfare improvement for society as whole. We model an economy where, initially, all agents participate in the trading of a complete set of financial assets. The only market failure in this economy is the externality that one agent imposes on the others via the probability distribution of their risks. We first show that as in all economies with market failures, the competitive equilibrium is Pareto inefficient. A more interesting question, however, is whether a social planner, who faces constraints in terms of the policies he can apply, would indeed be able to effect a welfare improving policy. Here, we consider the case in which the social planner can only induce an asset reallocation and lump-sum transfers. We find that, in a generic sense, everybody can be made better-off if the agent that causes the externality is not allowed to choose his portfolio of financial assets in an optimal manner, while everybody else continues to trade without constraints. We first allow for a lump-sum transfer to the agent causing the externality, getting a weak definition of constrained inefficiency. For strong constrained inefficiency, we only require that lump-sum transfers are financed through “relief aid”, an external source of funding. In this way, we account for the case when, for institutional reasons and despite not altering his incentives, a transfer to the agent causing the externality cannot be implemented.

One might argue that the kind of externality that agent 0 induces in our economy is too salient for him to take everybody else’s actions as given. To check the robustness of our results to a more sophisticated behaviour by agent 0, we now allow for him to recognise that his decisions with respect to probabilities affect the willingness to pay of other agents for the existing financial assets and, therefore, the equilibrium level of assets prices. Effectively, it is as if decisions were made sequentially in the first period: individual 0 first chooses the action, and then, taking that action as given, he and all the other agents trade financial assets in a competitive manner. Our results survive this added level of sophistication, and the equilibrium allocation is shown to be weakly constrained inefficient, generically.

Finally, our results extend to economies with uninsurable idiosyncratic risk. Assuming that the agents causing the externality are prudent, we find that a restriction in their equilibrium level of savings induces a distribution of risks that second-order stochastically dominates and therefore an improvement in social welfare. Hence, adding another distortion to an already imperfect financial market turns out to be socially beneficial.
**APPENDIX A**

**Lemma 5.** Eq. (16) is true at competitive equilibrium, generically in the space of endowments.

**Proof.** At competitive equilibrium, the allocation of commodities satisfies the first-order conditions of Pareto efficiency with respect to the allocation of commodities — compare Eqs. (5) and (8) with Eq. (11). Thus, we start by defining the following function

\[
\mathcal{K}(c_1, c_2, \lambda_1, \lambda_2, e_1, e_2) = \begin{pmatrix}
\frac{\partial u_1}{\partial c_1} - \lambda_1 \\
\frac{\partial u_1}{\partial c_2} - \lambda_1 \\
\vdots \\
\frac{\partial u_1}{\partial c_1} - \lambda_1 \\
\sum_{-o}[u_1 - u_2(c_1)] \\
\frac{\partial u_2}{\partial c_1} - \lambda_2 \\
\vdots \\
\frac{\partial u_2}{\partial c_1} - \lambda_2 \\
\sum_{-o}[e_2 - e_1(c_2)]
\end{pmatrix}
\]

(33)

which includes those conditions, the resource constraints for each state in period 1 and the utility sub-aggregate \(\sum_{i=1}[u_1(c_1) - u_2(c_2)]\). Then, at any competitive equilibrium of the economy, the 4-tuple \((c_1, c_2, \lambda_1, \lambda_2, e_1, e_2)\) is such that all entries of \(\mathcal{K}\) other than the one corresponding to \(\sum_{i=1}[u_1(c_1) - u_2(c_2)]\), are equal to 0. With arguments in the order

\((c_1^0, \ldots, c_1^1, c_1^0, \lambda_1, c_2^0, \ldots, c_2^1, \lambda_2, e_1^1, \ldots, e_1^0, e_2^1, \ldots, e_2^1)\),

the Jacobian at the Pareto efficient allocation at each state is

\[
\begin{pmatrix}
\delta^2 u_1^0 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \delta^2 u_1^0 & \cdots & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \delta^2 u_1^0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
-1 & -1 & \cdots & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
0 & \delta u_1^1 & \cdots & \delta u_1^1 & 0 & 0 & 0 & 0 & 0 & \delta u_1^2 & \cdots & \delta u_1^2 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \delta u_2^1 & \cdots & -1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \delta^2 u_2^1 & \cdots & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \delta^2 u_2^1 & \cdots & -1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \delta^2 u_2^1 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1
\end{pmatrix}
\]

(34)

We now argue that this \(2(I+1)+3 \times 4(I+1)+2\) matrix has full rank, in the following four steps:

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\(^{19}\) We denote by \(\lambda_1\) and \(\lambda_2\) the Lagrange multipliers associated with the resource constraints in states 1 and 2 respectively. These can be constructed by taking the ratio of the price of each elementary security and the probability of the corresponding state, at equilibrium.
1. Consider the submatrix without the last $2I + 1$ columns, which we denote with $\mathcal{V}$, and partition this submatrix as

\[
\begin{pmatrix}
\partial^2 u^0_i & 0 & \ldots & 0 & 0 & -1 & 0 & 0 & \ldots & 0 & 0 \\
0 & \partial^2 u^1_i & \ldots & 0 & 0 & -1 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \partial^2 u^I_i & 0 & -1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & -1 & \ldots & -1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & \partial u^1_i & \ldots & \partial u^I_i & 0 & 0 & 0 & \partial u^1_1 & \ldots & \partial u^1_2 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & \partial^2 u^0_2 & 0 & \ldots & 0 & -1 \\
0 & 0 & \ldots & 0 & 0 & 0 & \partial^2 u^1_2 & \ldots & 0 & -1 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \partial^2 u^I_2 & \ldots & 0 & -1 \\
0 & 0 & \ldots & 0 & 0 & 0 & -1 & -1 & \ldots & -1 & 0 \\
\end{pmatrix}
\]

(35)

2. We now show that the top-left $(I + 3) \times (I + 3)$ submatrix, which we denote $\mathcal{V}_{11}$, has full row rank. First notice that, by strong concavity, the submatrix without the last row has full row rank. Then, for $\mathcal{V}_{11}$ to have full row rank, it suffices to show that there exists a column vector $\zeta \in \mathbb{R}^{I + 3}$ such that $\mathcal{V}_{11} \zeta = (0, \beta)^T$ for some scalar $\beta \neq 0$. Let

\[
\zeta = \begin{pmatrix}
\partial^2 u^0_2 (\mathcal{c}^0_2)^{-1} \\
\partial^2 u^1_2 (\mathcal{c}^1_2)^{-1} \\
\vdots \\
\partial^2 u^I_2 (\mathcal{c}^I_2)^{-1} \\
\sum_{i=0}^{I} \partial^2 u^i_1 (\mathcal{c}^i_1)^{-1} \\
\end{pmatrix}
\]

Then, $\mathcal{V}_{11} \zeta$ is 0 everywhere, except from the last row where it is

\[
\beta = \sum_{i=1}^{I} \frac{\partial u^i_1 (\mathcal{c}^i_1)}{\partial^2 u^i_1 (\mathcal{c}^i_1)} \neq 0.
\]

3. Noting that $\mathcal{V}_{11}$ is invertible, $\mathcal{V}_{21}$ is a zero matrix and $\mathcal{V}_{22}$ has full row rank (by strong concavity), and since

\[
|\mathcal{V}| = |\mathcal{V}_{11}| |\mathcal{V}_{22} - \mathcal{V}_{21} \mathcal{V}_{11}^{-1} \mathcal{V}_{12}|
\]

we conclude that $\mathcal{V}$ has full row rank.

4. Finally, adding the last $2I + 1$ columns the row rank of the matrix will not change, so $D\mathcal{K}$ has full row rank.

Now, since $D\mathcal{K}$ has full row rank, we conclude that $\mathcal{K} \triangleright 0$. Then, the set of endowments at which $\mathcal{K}(\cdot, e_1, e_2) \triangleright 0$ has full measure. Since $D\mathcal{K}(\cdot, e_1, e_2)$ has one fewer column than it has rows, $\mathcal{K}(\cdot, e_1, e_2) \triangleright 0$ implies that, whenever all conditions for competitive equilibrium are true, Eq. (16) is also true. \[\square\]
REFERENCES


