Indeterminacy and fiscal policies in a growing economy

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Abstract

This paper studies the dynamic properties of a standard one-sector model of endogenous growth with inelastic labor supply, in which capital taxes are used to finance public production and consumption services. The government is benevolent and chooses its tax policy by taking into account the decentralized competitive equilibrium. Within this second-best setup, we establish conditions for multiple balanced growth paths, in conjunction with indeterminacy of transitional dynamics. The mechanism for global and local indeterminacy is a simple combination of public production services (that provide production externalities to private firms, and hence cause a wedge between private and social returns), public consumption services (that provide direct utility to households) and endogenously chosen distorting taxes.

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1. Introduction

Dynamic general equilibrium theory has investigated the possibility of indeterminacy of equilibrium paths.\textsuperscript{1} In models of endogenous growth, indeterminacy can take the...
form of multiple balanced growth paths (BGPs) along which the economy can persistently grow in the long run, and/or multiple transition paths all of which converge to the same BGP. Indeterminacy may explain why fundamentally similar economies can exhibit the same per capita income but grow at different rates, or why economies with the same growth rate can exhibit different per capita levels of income. The present paper explores the role of endogenously chosen fiscal policy in generating indeterminacy. We focus on fiscal policy because we believe it is difficult to understand multiplicity at macroeconomic level without studying the exact role of fiscal policy (see e.g. Rodrik, 1995; Schmitt-Grohe and Uribe, 1997; Guo and Lansing, 1998; Krusell and Rios-Rull, 1999).

In the literature on endogenous growth, the general idea is that the same mechanism that can generate long-term growth may also open the door for indeterminacy (see Benhabib and Farmer, 1999). One can organize this literature around three strands or mechanisms: increasing social returns, monopolistic competition, and positive external effects that lead to constant social returns. Studies of the first type include Caballero and Lyons (1992), Benhabib and Farmer (1994), Benhabib and Perli (1994), Xie (1994) and Cazzavillan et al. (1998). Studies of the second type are, for instance, Farmer and Guo (1994), Benhabib and Farmer (1994), Gali (1994) and Evans et al. (1998). The third type includes Cazzavillan (1996), Schmitt-Grohe and Uribe (1997), Benhabib et al. (2000) and Park and Philippopoulos (2003).

The present paper investigates the possibility of multiplicity of BGPs, in conjunction with indeterminacy of transitional dynamics, when it is fiscal policy that provides external effects that lead to long-term growth. Since there are constant returns to scale at social level (as in the well-known paper by Barro (1990)), it belongs to the third strand of the literature mentioned above. However, our work differs from the existing literature, because here we investigate the link between economic growth and endogenously chosen second-best policy. We show that when we include the two main types of government expenditure, namely public production and public consumption services, the introduction of economic policy, although well intentioned, triggers an expectations coordination problem. That is, the private and government sectors may fail to coordinate their decisions in the presence of external effects from these two types of government expenditure.

We use a standard one-sector general equilibrium model of endogenous growth with inelastic labor supply, in which the government uses capital taxes to finance public production services (that provide positive externalities to private firms) and public consumption services (that provide direct utility to households). These services are treated as public goods by firms and households. We then solve for second-best policy. Namely, a benevolent government chooses its distorting tax policy by taking into account the decentralized competitive equilibrium (DCE). In doing so, the government attempts to internalize externalities and collect revenues to finance the provision of public services. The solution to this second-best policy problem can give multiple general equilibria.

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2 See the seminal contributions of Shell (1967) and Arrow and Kurz (1972).
We show that there are two long-run general equilibria, in each of which consumption and capital can grow at a common constant rate, known as BGP. That is, with the same economic fundamentals and with constant (rather than increasing) returns at social level, there can be two different second-best tax policies associated with two different BGPs (a low-tax and high-growth equilibrium, and a high-tax and low-growth one). This is a case of global indeterminacy. The analysis shows that multiplicity of long-run equilibria arises from a simple combination of public production services, public consumption services and endogenously chosen distorting taxes. It is public production services that generate long-term growth, and also cause a wedge between the rate of capital return that determines the stream of income (called social or realized return) and the rate of capital return that drives consumption/saving decisions (called private or perceived return). On the other hand, it is public consumption services that generate transitional dynamics for fiscal policies.

The same combination can also lead to an infinite number of transitional paths, all of which converge to the same long-run equilibrium. In other words, the BGPs can be locally indeterminate. As a result, the long-run levels of output and consumption are also indeterminate. The possibility of local indeterminacy depends heavily on the magnitude of the above wedge. In other words, the very same wedge between social and private capital returns that can lead to multiple steady states can also lead to local indeterminacy.

Numerical simulations confirm the possibility of our theoretical results. In particular, they show that there are two long-run equilibria with different properties. The “good” long-run equilibrium (i.e. low-tax and high-growth) is locally indeterminate, while the “bad” long-run equilibrium (i.e. high-tax and zero-growth) is locally determinate. In other words, there is an infinite number of ways to reach the persistently growing long-run equilibrium, but there is a unique way to reach the non-growing long-run equilibrium.

The interpretation of our results is as follows. In our model, indeterminacy is due to a failure to coordinate expectations. Namely, economic agents’ actions depend on their expectations about future prices, and also these prices depend on actions. We therefore have multiple self-fulfilling rational expectations equilibria. This is independent of initial conditions or other fundamentals. Note that although the benevolent government takes into account, and also affects, the DCE when it chooses its tax policy, it cannot resolve this expectations coordination failure. That is, it cannot select the tax policy yielding the highest possible social utility. This is because the choice of tax policy affects private agents’ decisions and, at the same time, government expenditures (public production and public consumption services) yield external benefits to private agents. To put it differently, this is not a social planner’s problem. It is a Ramsey-type tax policy problem and, in addition, the government tries to internalize the production externalities that have been ignored by private actors.3

The rest of the paper is as follows. Section 2 presents the competitive equilibrium. Section 3 solves for second-best policy. Section 4 closes the paper.

3 See Chari and Kehoe (1999) for a survey of the literature on Ramsey optimal tax policy.
2. The economy and DCE

This section sets up a decentralized economy with a private sector and a government sector. The private sector consists of a representative household and a representative firm, who both act competitively. The household consumes, rents out its assets to the firm and supplies inelastically one unit of labor services per period of time. The firm produces output by using capital and labor. The government taxes the firm’s installed capital\(^4\) to provide consumption and production public services. This section will solve for a DCE, for given government policy. We assume continuous time, infinite horizons and perfect foresight. We also assume zero population growth and zero capital depreciation.

2.1. The problem of the representative household

The representative household maximizes intertemporal utility:

\[
\int_{0}^{\infty} u(c, h)e^{-\rho t} dt; \tag{1}
\]

where \(c\) is consumption per capita, \(h\) is public consumption services per capita and the parameter \(\rho > 0\) is the rate of time preference. The function \(u(c, h)\) is increasing and concave in \(c\) and \(h\), and satisfies the Inada conditions. For simplicity, we assume that \(u(c, h)\) is additively separable and logarithmic, so that:

\[
u(c, h) = \log c + v \log h; \tag{2}\]

where the parameter \(v \geq 0\) measures the non-negative utility effect of public services. See e.g. Turnovsky (2000, Eq. (4)) for a similar preference specification. As Baxter and King (1993, p. 317) say, this specification “denotes basic government purchases, defined as those that absorb resources without directly altering the marginal utility of private consumption or the marginal product of private factors of production”. Barro (1989, Eq. (5.13)) gives a similar interpretation for those components of government expenditure that enter (2) in a separable way.\(^5\)

The household saves in the form of capital. If \(k\) denotes capital per capita, the household receives interest income \(rk\), where \(r\) is the market return to capital. Also, the household supplies inelastically one unit of labor services per unit of time so that wage income per capita equals \(w\). It also receives net profits, \(\pi\), from the firm. Thus, the flow constraint of the household is

\[
c + \dot{k} = rk + w + \pi, \tag{3}\]

where a dot over a variable denotes time derivative and the initial stock \(k_0 > 0\) is given.

\(^4\)Our main results do not change if we use income taxes on households, or output taxes on firms. This is because our model (at aggregate level) is a variant of the AK model (see below).

\(^5\)This is a constant intertemporal elasticity of substitution function and thus allows for a BGP. In contrast, Cazzavillan (1996) considers a felicity function exhibiting increasing returns in public consumption services, which is necessary for multiple BGPs.
The representative household acts competitively by taking prices, policy instruments and public consumption services as given. The control variables are $c$ and $k$, so that the first-order conditions are Eq. (3) and the standard Euler condition:

$$
\dot{c} = c(r - \rho).
$$

(4)

The transversality condition is

$$
\lim_{t \to \infty} \left( \frac{1}{c} \right) ke^{-\rho t} = 0.
$$

2.2. The problem of the representative firm

As in Benhabib et al. (1996), we assume that the government taxes the firm’s capital stock carried over from the previous period. Thus, if $0 < \theta < 1$ is the tax rate on installed capital, $(1 - \theta)k$ is the net stock used in production in the current period. Also, as in Barro and Sala-i-Martin (1995, p. 153), we assume that technology at the firm’s level takes a Cobb–Douglas form:

$$
y = g^{1-z}[(1 - \theta)k]^z,
$$

(5)

where $y$ is the output-labor ratio, $k$ is the capital-labor ratio, $g$ is the flow of public production services, and $0 < z < 1$.

The representative firm acts competitively by taking prices, policy instruments and public production services as given. It maximizes profits, $\pi$, written as

$$
\pi = y - rk - w.
$$

(6)

This problem is well defined because the production function is increasing and concave and satisfies the Inada condition. The first-order condition for $k$ is simply:

$$
r = zg^{1-z}(1 - \theta)^z k^{z-1}.
$$

(7a)

Also, by the zero profit condition, the wage rate is

$$
w = (1 - z)g^{1-z}(1 - \theta)^z k^z,
$$

(7b)

so that factor prices equal their marginal products.

2.3. Government budget constraint

We assume that the government runs a balanced budget at each point of time. Thus,

$$
g + h = \theta k,
$$

(8a)

where $g + h$ is total government expenditures and $\theta k$ is total tax revenues.\(^8\)

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\(^6\)Specifically, the production function for the firm is $Y = [(1 - \theta)K]^z L^{1-z} g^{1-z}$, so that dividing both sides by labor input $L$, we get (5), where $y \equiv Y/L, k \equiv K/L$.

\(^7\)We assume that government services are not subject to congestion.

\(^8\)Recall that there is one representative private agent.
Concerning the decomposition of government expenditures between public production services, \( g \), and public consumption services, \( h \), we assume:

\[
g = b0k, \tag{8b}
\]

\[
h = (1 - b)0k, \tag{8c}
\]

where the parameter \( 0 < b < 1 \) is the share of tax revenues used to finance public production services and \( 1 - b \) is the share used to finance public consumption services. We will assume that \( b \) is institutionally fixed.\(^9\) In our setup, exogeneity of \( b \) is without loss of generality.\(^10\) It will also allow us to make our results easily comparable to those of the literature by showing how the decomposition of tax revenues between consumption and production public services affects the dynamics of growth and tax policy.

2.4. Decentralized competitive equilibrium

We will now characterize a DCE for any fiscal policy. As (8a)–(8c) show, only one of the three policy instruments (\( g, h, \theta \)) can be independently set. Here, we choose to express the DCE in terms of the tax rate, \( \theta \). Then, for any \( \theta \), in a DCE: (i) Private decisions maximize households’ utilities and firms’ profits. (ii) The government budget constraint is satisfied. (iii) All markets clear (in the labor market, this means \( L = 1 \)).

By using (8b) into (5), we get the economy-wide output:

\[
y = (b\theta)^{1-z}(1 - \theta)^2k. \tag{9a}
\]

That is, at social level, \( y \) is linear in \( k \). Therefore, for any level of employment, the aggregate technology exhibits constant returns, rather than increasing returns. This is a variant of the well-known \( AK \) model (see e.g. Barro and Sala-i-Martin, 1995, pp. 153–154).

Notice that Eq. (9a) implies that the social or realized return to capital, denoted by \( R \), is \( R \equiv \hat{\gamma}_k = (b\theta)^{1-z}(1 - \theta)^2 \). On the other hand, by using (8b) into (7a), the return to capital perceived by private agents in a DCE is

\[
r = z(b\theta)^{1-z}(1 - \theta)^2. \tag{9b}
\]

\(^9\)See also e.g. Devarajan et al. (1996) and Baier and Glomm (2001) for similar modeling.

\(^10\)Our main results do not change if we also endogenize \( b \). This is because \( \theta \) and \( b \) play a symmetrically opposite role. Specifically, in each time period, when the government allocates a larger share of tax revenues to public production services, it can afford a lower tax rate because public production services stimulate growth and increase tax bases. Technically, the dynamics of \( \theta \) are also the dynamics of \( b \) and vice versa. See Park and Philippopoulos (2002, 2003) for models where both \( \theta \) and \( b \) are chosen. These models differ from the one used here. In particular, in Park and Philippopoulos (2003), we use a model with state-contingent redistributive transfers, and examine how moral hazard behavior leads to local indeterminacy. For differences between the present model and that in Park and Philippopoulos (2002), see footnote 11 below.
which is the return that drives private consumption/saving decisions. Since private agents have ignored the externalities generated by public production services, \( r < R \) (where, \( 0 < \alpha < 1 \)).

By using all the above, we get for (3) and (4), respectively:

\[
\dot{k} = \frac{\Delta(\theta)}{\alpha} k - c, \tag{10}
\]

\[
\dot{c} = c[\Delta(\theta) - \rho], \tag{11}
\]

where, from (9b), \( r = \alpha (b \theta)^{1-\alpha}(1 - \theta)^{\alpha} \equiv \Delta(\theta) \). That is, \( R = \Delta(\theta)/\alpha \).

Eqs. (10) and (11) summarize the private agents’ decision rules in a DCE. Observe two things in (10) and (11). First, for given policy (summarized by the tax rate, \( \theta \)), there is a unique DCE. Specifically, as in e.g. Barro (1990), the consumption growth rate in (11) is an inverted U-shaped function in \( \theta \), with a maximum growth rate at \( \theta = 1 - \alpha \). This is a standard result. Second, due to production externalities (i.e. \( 0 < \alpha < 1 \)), it is the realized, or social, return to capital (\( R \)) that matters in the budget constraint (10), while it is the perceived, or private, return to capital (\( r \)) that matters in the Euler equation (11). Thus, there is a wedge between the rate of return that determines the stream of income in (10) and the rate of return that drives consumption/saving decisions in (11). \(^{11}\)

In sum, this section has solved for a DCE. This is summarized by Eqs. (10), (11), (8b) and (8c), and holds for any tax policy, \( \theta \). The next section will endogenize the choice of \( \theta \).

3. Second-best policy and general equilibrium

The government chooses economic policy, \( \theta \), to maximize the utility of the representative household subject to the DCE. Thus, it is also constrained by the private agent’s optimal decision rules given by (10) and (11). We assume commitment technologies on behalf of the government, so that decisions are made once-and-for-all and become an open-loop equilibrium. In solving this optimization problem, the benevolent government will attempt to internalize externalities and collect revenues to finance public services.

\[^{11}\] In Park and Philippopoulos (2002), there is \( r \) in both (10) and (11). This is because, in that model, workers get the “realized” wage rate being calculated as a residual of what actually goes to capital, i.e. \( w \equiv y - Rk = 0 \) (with zero profits). This is as in the basic AK model, where labor’s share in income converges to zero. As a result, there is not a wedge. By contrast, in the present paper, both capital and labor are chosen so as to get their perceived marginal products, as shown by (7a) and (7b). This is the usual way of modeling (see e.g. Alesina and Rodrik (1994), and Glomm and Ravikumar (1994, 1997) in similar models) that also remedies the “troublesome property” that labor’s share converges to zero (see Jones and Manuelli, 1997, pp. 83–84). As a result, in a DCE, the private agent obtains the stream of lifetime income (see (10)), while his saving decisions are determined by the perceived return (see (11)). The presence of a wedge is crucial to our results (see below).
3.1. Solution

The government chooses \( \theta \) to maximize (1) and (2) subject to (10), (11) and (8c). The current-value Hamiltonian, \( H(c, k, \theta, \lambda, \gamma) \), is

\[
H \equiv \log c + v \log[(1 - b)\theta k] + \lambda c[A(\theta) - \rho] + \gamma \left[ \frac{A(\theta)}{\alpha} k - c \right], 
\]

where \( \gamma \) and \( \lambda \) are dynamic multipliers associated with (10) and (11), respectively.

The necessary conditions with respect to \( c, k, \lambda, \gamma \) and \( \theta \) are given by (13a), (13b), (13c), (13d) and (13e), respectively:

\[
\begin{align*}
\dot{v} + \frac{\lambda c + \gamma k}{\theta} A_\theta(\theta) &= 0, \\
\dot{\lambda} &= \rho \lambda - \frac{1}{c} - \lambda[A(\theta) - \rho] + \gamma, \\
\dot{c} &= c[A(\theta) - \rho], \\
\dot{\gamma} &= \rho \gamma - \frac{A(\theta)}{\alpha} - \frac{v}{k}, \\
\dot{k} &= \frac{A(\theta)}{\alpha} k - c,
\end{align*}
\]

where \( A_\theta(\theta) = (1 - \alpha - \theta)A(\theta)/\theta(1 - \theta) \).

Also, the transversality condition is

\[
\lambda(\theta) - \rho < \rho
\]

which says that the long-run growth rate of consumption, \( A(\theta) - \rho \geq 0 \), is less than the discount rate, \( \rho > 0 \).\(^{12}\)

Eqs. (13a)–(13e) constitute a system of five equations in \( \theta, c, \lambda, k, \gamma \). Following usual practice, we will reduce its dimensionality to facilitate analytical tractability. Define \( z \equiv c/k \) and \( \phi \equiv c\gamma \) to be auxiliary variables. By taking logarithms on both sides of (13a), differentiating with respect to time, and using (13b)–(13e), we get after some algebra:

\[
\begin{align*}
\dot{z} &= [z - \dot{\gamma} A(\theta) - \rho]z, \\
\dot{\phi} &= -z \dot{\lambda} A(\theta) - v z, \\
\dot{\theta} &= \left[ \frac{[\theta A_\theta(\theta)]^2}{v[A_\theta(\theta) + \theta A_{\theta\theta}(\theta)]} \right] \left[ -1 - \frac{v}{\alpha} - \frac{v \rho}{\theta A_\theta(\theta)} - \dot{z} \phi \right],
\end{align*}
\]

\(^{12}\)Since the utility function and the constraints are continuous and bounded, and the utility function is strictly concave in the controls \( c, k, \theta \) and the constraints are linear in \( c, k \) and strictly concave in \( \theta \), existence is assured. Further, since \( H(c, k, \lambda, \gamma) = \max_\theta H(c, k, \theta, \lambda, \gamma) \) is linear in \( c, k \), for given \( \lambda, \gamma \), conditions (13a)–(13f) are also sufficient for existence of a solution. This is based on Arrow’s sufficiency theorem in the optimal control theory.
where $\Delta_{\psi}\theta = -\varpi(1 - \varpi)\Delta(\theta)/[\theta(1 - \theta)]^2 < 0$ and $\hat{\varpi} \equiv ((1/\varpi) - 1) > 0$. Thus, the five-dimensional system \((13a)-(13e)\) in $\theta, c, \lambda, k, \gamma$ has been reduced to the three-dimensional system \((14a)-(14c)\) in $z, \phi, \theta$. The dynamics of the latter are equivalent to those of the former. The next two subsections will study the steady state and transitional dynamic properties of \((14a)-(14c)\).

### 3.2. Long-run equilibrium and global indeterminacy

This subsection focuses on a long-run general equilibrium in which the economy can grow at a constant rate, the so-called BGP. An interior BGP is a trajectory $(\tilde{z}, \tilde{\phi}, \tilde{\theta})$ that solves $\dot{z} = \dot{\phi} = \dot{\theta} \equiv 0$ in \((14a)-(14c)\) and satisfies the initial condition $k_0 > 0$ and the terminal condition \((13f)\). Thus, $(\tilde{z}, \tilde{\phi}, \tilde{\theta})$ satisfies:

\begin{align}
\dot{z} &= \rho + \hat{\varpi}A(\tilde{\theta}), \\
\dot{\phi} &= -v - \frac{v\rho}{\hat{\varpi}A(\tilde{\theta})}, \\
A(\tilde{\theta}) &= \frac{-zv\rho}{(1 + v)(1 - \varpi - \tilde{\theta})}.
\end{align}

The main task is to solve Eq. \((15c)\) for the long-run tax rate, $\tilde{\theta}$. Once we solve \((15c)\) for $\tilde{\theta}$, \((15a)\) can give the long-run consumption-to-capital ratio $\tilde{z}$, and \((15b)\) can give the long-run value of the auxiliary variable, $\tilde{\phi}$.\footnote{Since $\tilde{\phi} \equiv \dot{\tilde{c}} = \dot{\gamma}$ is negative in the long run. Note that $\gamma$ can also be thought as the social marginal value of distortionary taxation (see also proof of Theorem 1 in Chamley, 1986).} Observe that \((15c)\) implies that for the return to capital, $A(\tilde{\theta})$, to be positive, $1 - \varpi < \tilde{\theta} < 1$.\footnote{This implies $A(\tilde{\theta}) > 0$.}

Appendix A studies the shapes of the left-hand side, $LHS(\tilde{\theta}) \equiv \varpi(b\tilde{\theta})^{1-\varpi}(1 - \tilde{\theta})^2$, and the right-hand side, $RHS(\tilde{\theta}) \equiv -zv\rho/(1 + v)(1 - \varpi - \tilde{\theta})$, in \((15c)\). These shapes are also shown in Fig. 1.

Then, we have

**Proposition 1.** There exist values of the parameters $(v, \varpi, b, \rho)$ such that for some $0 < \theta^* < 1$, we have $LHS(\theta^*) \geq RHS(\theta^*)$. Then, (i) when $LHS(\theta^*) > RHS(\theta^*)$, there exist two solutions, $\tilde{\theta}_1$ and $\tilde{\theta}_2$, where $1 - \varpi < \tilde{\theta}_1 < \tilde{\theta}_2 < 1$, that solve Eqs. \((15a)-(15c)\); (ii) when $LHS(\theta^*) = RHS(\theta^*)$,\footnote{Throughout the paper, subscripts denote derivatives.} there exists a unique solution, where $1 - \varpi < \tilde{\theta} < 1$, that solves Eqs. \((15a)-(15c)\).

**Proof.** See Appendix A.

These results are also shown in Fig. 1. Therefore, we can have multiple (two) long-run tax rates. In turn, each tax rate supports a different BGP, at which consumption and capital can grow at a common constant rate. The growth rate associated with the low tax rate, $\tilde{\theta}_1$, is higher than the growth rate associated with the high tax rate, $\tilde{\theta}_2$. Numerical simulations below will confirm this possibility. Notice that the initial capital
stock and Eqs. (15a)–(15c) cannot determine whether the economy will be at the low tax rate, \( \tilde{\theta}_1 \), or the high tax rate, \( \tilde{\theta}_2 \). In fact, the initial condition, \( k_0 \), is irrelevant here. Instead, it is the first choice of the non-predetermined variables, \( c \) and \( \theta \), that will put the economy at either one of the two BGPs. Thus, we have:

**Corollary 1.** Under the conditions in Proposition 1, case (i), there exist two distinct BGPs and hence there is global indeterminacy.

To understand what causes multiplicity, it is convenient to consider two special cases. First, when public services provide no direct utility, i.e. \( v = 0 \) in (2) above. Then, (13a) implies that the optimal tax rate is unique and equal to \( \tilde{\theta} = 1 - \zeta \) in all time periods (see Appendix B). This is Barro’s (1990) popular model. Second, when public services do not provide any production externalities, i.e. \( \zeta = 1 \). Then, the long-run optimal tax rate is again unique and equal to \( \tilde{\theta} = \rho v/(1 + v) \) (see Appendix C).

Therefore, for a range of parameter values, multiplicity of long-run equilibria can arise when: (i) public services provide production externalities to private firms; and (ii) public services provide direct utility to households. It is condition (i) that generates long-term growth, as well as a wedge between the rate of return that determines the stream of income and the rate of return that drives consumption/saving decisions. On the other hand, it is condition (ii) that generates transitional dynamics.
3.3. Transitional dynamics and local indeterminacy

We will now investigate local stability properties around BGPs. Linearizing (14a)–(14c) around the BGPs in (15a)–(15c) implies the following linear system:

\[
\begin{pmatrix}
\dot{z} \\
\dot{\theta} \\
\dot{\phi}
\end{pmatrix} =
\begin{pmatrix}
\hat{z} & -\hat{z} \hat{\Delta}_\theta & 0 \\
0 & \rho & -\hat{z} \left[ \hat{\theta} \dot{\Delta}_\theta \right] \\
-v & -\hat{z} \hat{\Delta}_\theta \dot{\phi} & -\hat{z} \hat{\Lambda}
\end{pmatrix}
\begin{pmatrix}
z \\
\theta \\
\phi
\end{pmatrix}
\]

(16)

where \( \hat{\Delta} \equiv \Delta(\hat{\theta}) \) and \( \hat{\Delta}_\theta \equiv \Delta_\theta(\hat{\theta}) \).

Let \( J \) define the 3 × 3 Jacobian matrix in (16). Also, we define:

\[
\Theta = -\hat{z} \hat{\Delta} \left[ 1 - \frac{\hat{\theta} (1 - z - \hat{\theta})^3}{(1 - \hat{\theta}) \left[ \hat{\theta} (1 - \hat{\theta}) (1 - z - \hat{\theta}) - z \hat{\theta} \right] } \right].
\]

(17)

We further define the characteristic equation of \( J \) as

\[
\omega^3 - [tr \, J] \omega^2 + \Omega \omega - \det J = 0,
\]

(18)

where \( tr \, J = 2\rho, \Omega = \rho^2 + \Theta \), and \( \det J = \rho \Theta \).

At this stage, it is necessary to recall the features of our dynamic system. There are three variables (\( z, \phi, \theta \)), and all of them are non-predetermined or jump. Therefore, for a BGP to be locally determinate, we need three unstable (i.e. positive) roots. If there is at least one stable (i.e. negative) root, a BGP is locally indeterminate. Notice that, since the trace of the Jacobian matrix is positive, all three characteristic roots cannot be negative at once. In addition, one observes that if the determinant of \( J \) is negative, there are one negative and two positive roots. In this case, a BGP is locally indeterminate. Hence, a negative determinant of \( J \) is sufficient for local indeterminacy. This can happen only if the sign of \( \Theta \) is negative in (17). On the other hand, if the determinant is positive, there are two possibilities: either there are three positive roots, or one is positive and the other two are negative. However, we can eliminate the latter case by applying Descartes’ rule of signs in polynomial equations. That is, when \( \Omega \) is positive, \( \Theta \) is positive and \( \det J \) is also positive. Then, when \( \Omega \) is positive, the signs of the coefficients of (17) alter three times and thereby all three roots must be positive. In this case, a BGP is locally determinate.

Since the sign of \( \Theta \) in (17) is ambiguous, the sign of \( \det J \) is also ambiguous, so that it is possible to have local indeterminacy. Numerical simulations below will confirm this possibility.

We sum up the above arguments:

**Proposition 2.** Under the conditions in Proposition 1, whenever \( \Theta \) in (17) is negative, a BGP is locally indeterminate. On the other hand, whenever \( \Theta \) in (17) is positive, a BGP is locally determinate.
What is crucial for local indeterminacy? As we said, it is the sign of $\Theta$ in (17) that determines local (in)determinacy. If $[1 - \hat{\Theta}(1 - x - \hat{\Theta})^2/(1 - \hat{\Theta})(1 - \hat{\Theta})(1 - x - \hat{\Theta}) - z\hat{\Theta}]$ is positive, $\Theta$ is negative and there is indeterminacy. Observe that this term includes the endogenous tax rate, $\hat{\Theta}$, and the parameter $x$ which measures the productivity of private capital vs. the productivity of externality-generating public services. Of course, since $\hat{\Theta}$ is endogenous, and so a function of $a, b, v, \rho$ according to (15c) above, all parameter values can matter.

The two special cases (i.e. $v = 0$ and $x = 1$), that helped us to understand the causes of global indeterminacy, can also help us to understand the causes of local indeterminacy. As shown above, when public services do not provide any direct utility (i.e. $v = 0$), there is a unique flat tax rate, which in turn gives unique solutions for the capital return, $A(\hat{\Theta})$, and the BGP. Actually, in this case, (13c) and (13e) imply that there are no transitional dynamics. Also, when public services do not provide any production externalities (i.e. $x = 1$), there is local determinacy (see Appendix C). Therefore, when $v = 0$ or $x = 1$, local indeterminacy cannot arise. In other words, the combined effect of the two conditions (namely, public production services and public consumption services) that opened the door of global determinacy can also open the door for local determinacy.

The magnitude of $x$ is therefore particularly critical to local indeterminacy. To see this, take for simplicity $\hat{\Theta}$ as given so as to focus on the direct effect of $x$ only. Then, as $x$ decreases, the positive value of $[\hat{\Theta}(1 - x - \hat{\Theta})^2/(1 - \hat{\Theta})(1 - \hat{\Theta})(1 - x - \hat{\Theta}) - z\hat{\Theta}]$ decreases, so that it is more possible for $\Theta$ to be negative. That is, as the magnitude of production externalities increases (i.e. $x$ decreases), the possibility of local indeterminacy increases.

Before we move on to numerical examples, notice that in Park and Philippopoulos (2002) we got uniqueness. Specifically, we got determinacy, both global and local. This is because in that paper (as explained in footnote 11 above), there was not a wedge between the rate of return that determines the stream of income and the rate of return that drives consumption/saving decisions. Therefore, our multiplicity result in the present paper is consistent with that in e.g. Benhabib and Nishimura (1998), Benhabib and Farmer (1999) and Benhabib et al. (2000), who have also emphasized the key role of a wedge between social and private returns in generating (local) indeterminacy. Thus, our results point out the importance of “pricing” in economies with externalities. Specifically, in our setup, a lot seems to depend on how factor prices are determined.

3.4. Numerical examples and indeterminacy

To solve (15a)–(15c) numerically, we work as follows. We first choose calibrated values for a set of benchmark parameters of the economy. Secondly, we find the steady-state values of the tax rate, $\hat{\Theta}$, and the return to capital, $A(\hat{\Theta})$. Thirdly, we determine the signs of the characteristic roots for each BGP. Finally, we check the growth condition, $A(\hat{\Theta}) - \rho \geq 0$, and the transversality condition (13f). Since the model is stylized, the aim of numerical examples is to show the possibility of our theoretical results, rather than to account for reality.
We choose the parameter values: \( v = 0.25, x = 0.67, b = 0.15 \) and \( \rho = 0.1 \).\(^{16}\) Using Matlab 6.0, we confirm that there are two equilibrium tax rates, and two associated capital returns. Their values are \( \hat{\theta}_1 = 0.40 \) and \( \Delta(\hat{\theta}_1) = 0.18 \) at the low tax rate equilibrium, and \( \hat{\theta}_2 = 0.95 \) and \( \Delta(\hat{\theta}_2) = 0.02 \) at the high tax rate equilibrium. At the BGP associated with \( \hat{\theta}_1 \), \( \Theta \) is negative and thereby this BGP is locally indeterminate, while the BGP associated with \( \hat{\theta}_2 \) is determinate because \( \Theta \) is positive. Also, both BGPs satisfy the transversality condition, but although the economy grows with \( \hat{\theta}_1 \), it cannot grow with \( \hat{\theta}_2 \).\(^{17}\)

Therefore, under the conditions in Proposition 1, our numerical results confirm that there are two long-run equilibria with different dynamic properties. The low tax rate is associated with a growing BGP and is locally indeterminate. That is, there is an infinite number of equilibrium paths, all of which are consistent with a given initial condition and with convergence to the same growing long-run equilibrium. There is also a high tax rate that is associated with a non-growing BGP and is locally determinate. That is, there is a unique equilibrium path to the non-growing long-run equilibrium.

The economy can equally end up being a growing or a non-growing one. In addition, one cannot predict the levels of output and consumption in the long run. Therefore, economies with the same fundamentals and endowments can grow at completely different rates, and also exhibit completely different levels of output, consumption and public services.

### 4. Conclusions and extensions

This paper has studied the role of second-best economic policy in the dynamics of growth. Economic policy took the form of public production and consumption services being financed by distorting capital taxes. The focus was on the possibility of global and local indeterminacy. Although standard and analytically tractable, our model produced a coordination failure that can account for the observed significant differences in growth rates, policy choices and the size of public sectors across countries with similar fundamentals.

We showed that there can be two long-run equilibria, and both of them can be locally determinate or indeterminate. Thus, we cannot exclude any of them on the grounds of stability. Namely, for any initial value, the economy can select any of the two long-run equilibria, and also any of a continuum of locally stable paths to them. However, if one is willing to add a learning process, it is possible to use stability as a criterion for selecting one of the many equilibria. Thus, learning can act as a selection device (see e.g. Evans et al. (1998) and for a survey Evans and Honkapohja (1999)). We leave this for future research.

\(^{16}\) We report that our main results are robust to small changes in parameter values.

\(^{17}\) Thus, the two long-run solutions have different properties. By (15c), the value of \((1 - v - z)/(1 - v)\) determines whether we have a growing or a non-growing economy; if the solution of \( \theta \) is such that \((1 - v - z)/(1 - v) \geq \theta \) (resp. \((1 - v - z)/(1 - v) < \theta \)), the economy will grow (resp. not grow) in the long run. The same value determines whether the BGPs are locally determinate or indeterminate; that is, it is a saddle-node bifurcation.
We close with two other extensions. First, it would be interesting to include endogenous labor supply decisions, as well as cross-effects between private and public consumption or between consumption and leisure. Second, we could introduce sunspots to multiple BGPs and/or to equilibrium trajectories.

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Appendix A.

Proof of Proposition 1. $LHS(\tilde{\theta}) \equiv \frac{a(b\tilde{\theta})^{1-x}(1-\tilde{\theta})^2}{V^{\tilde{\theta}}}$ is positive, continuous and strictly concave in $\tilde{\theta}$. It also approaches zero as $\tilde{\theta}$ approaches either 0 or 1, and reaches its maximum when $\tilde{\theta} = 1 - x$. When $\tilde{\theta} > 1 - x$, $RHS(\tilde{\theta}) \equiv -\frac{xv}{V^{\tilde{\theta}}}/(1+v)(1-x-\tilde{\theta})$ is positive, continuous, strictly convex and decreasing in $\tilde{\theta}$. It is easy to see that as $\tilde{\theta}$ goes to $1 - x$ from the right, the $RHS$ approaches plus infinity; while as $\tilde{\theta}$ goes to 1 from the left, the $RHS$ approaches $v/(1+v) > 0$. These shapes are shown in Fig. 1.

Although in Fig. 1 we have drawn the $LHS$ to lie above the $RHS$ over some range of $\tilde{\theta}$ between $1 - x$ and 1, this is not the only possibility. Actually, there are three possibilities: First, the $LHS$ to lie below the $RHS$ over the whole range between $1 - x$ and 1; in this case, there is no solution. Second, the $LHS$ to be tangent to the $RHS$; in this case, there is a single solution. Third, the $LHS$ to lie above the $RHS$ as shown in Fig. 1; in this case, there are two solutions (the properties of the functions—see above—allow for no more than two solutions). The first possibility is excluded if there exist values of the parameters $(v, x, b, \rho)$ such that for some $0 < \theta^* < 1$, we have $LHS(\theta^*) \geq RHS(\theta^*)$. Then, Proposition 1, case (ii), models the second possibility, and Proposition 1, case (i), models the third possibility.

Appendix B. The special case where $v = 0$

When $v = 0$, (13a) implies either $\Delta_\theta(\theta) = 0$, or $[\lambda c + (\gamma k/x)] = 0$. Differentiation of $[\lambda c + (\gamma k/x)] = 0$ with respect to time gives an expression that contradicts the other first-order conditions, (13b)–(13e). Therefore, it must be

$$\Delta_\theta(\theta) \equiv \frac{(1-x-\theta)\Delta(\theta)}{\theta(1-\theta)} = 0 \quad \text{or} \quad \theta = 1 - x.$$
Appendix C. The special case where $\alpha = 1$

When $\alpha = 1$, (13a) implies $v/\theta = \lambda c + \gamma k$. Differentiating it with respect to time and using (13b)–(13e), we get $\dot{\theta} = (1 + v)\theta^2/v - \rho \theta$. When $\theta \equiv 0$, this gives a unique long-run solution, $\bar{\theta} = \rho v / (1 + v)$. Next, concerning transitional dynamics around $\bar{\theta}$, we have $\partial \theta / \partial \bar{\theta} = \rho > 0$. Since $\theta$ is a jump variable, a positive (i.e. unstable) root implies local determinacy. That is, the jump variable takes its long-run value and there are no transitional dynamics (as in the basic AK model).

References


