Are Regime Shift Sources of Risk priced in the Market?

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Abstract

In this paper we suggest a discrete-time option pricing model of European calls when the log-return of the underlying asset (stock) is subject to discontinuous regime shifts in its mean or volatility. The risk of these shifts is allowed to be priced in the options market. The paper shows how to estimate the model and, then, it employs it to empirically examine if regime-shift risks are priced in the options market as separate sources of risk. The results of the paper clearly indicate that shifts from the bull to bear and from bear to crash regimes carry substantial prices of risk. Ignoring this risk will lead to substantial option pricing errors, across different moneyness levels.

Keywords: European call prices, stock market regime shifts, Markov regime switching model, risk neutral transition probabilities, occupation time of a regime.

JEL Classification: G10, G12, G13

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1 Introduction

Many recent studies have documented that discrete-time, Markov chain based of regime-switching (MRS) processes (see, e.g., Hamilton (1989) can capture the dynamics of asset prices and explain main features of their log-return distributions, e.g. negative skewness and excess kurtosis (see, e.g., Ceccheti et al (1990), Whitelaw (2000), Ang and Bekaert (2002), Dai et al (2007), Chabi-Yo et al (2008), and Ghosh and Constantinides (2010), *inter alia*). The intuition behind regime-switching asset pricing models is clear. A simple two regime MRS asset price model can reflect the bull and bear states of asset markets. The bull regime is often identified by stock market data as one which has the highest mean and lowest volatility, while the bear as that with the lowest mean and highest volatility. MRS asset pricing models allowing for a higher than two market regimes have been also considered in the literature. Some of these models allow for a crash regime related to collapsing asset price bubbles (see, e.g., Driffill and Sola (1998)), Brooks and Katsaris (2005)).

Modelling asset price dynamics based on MRS processes brings us to ask an important question for option pricing. Are regime shifts in asset (or stock) markets priced as independent sources of risk? Answering this question has important implications for option pricing and portfolio management, as regime shifts in asset markets can not be completely hedged out and, thus, should be priced in these markets. The price of a regime-shift source of risk must compensate risk averse market participants towards possible adverse regime shifts, i.e. from the bull to the bear, or the crash, regime. Assuming that this price of risk is zero (see, e.g., Bollen et al (2000), Chourdakis and Tzavalis (2000), Chourdakis (2004), Edwards (2005), Aingworth et al (2006)) may lead to serious option mispricing. To answer the above questions, in this paper we suggest a discrete-time MRS option pricing model for European calls. This model considers regime shifts in the mean or volatility of the log-return (or log-price) of the underlying asset.\(^1\) The discrete-time nature of the model facilitates its estimation and implementation in option pricing, by employing Hamilton’s (1989) EM method. To estimate the market price of a regime-shift risk, the paper puts forward a joint estimation procedure of the MRS process and option pricing model suggested, utilising both option and stock data.

\(^1\)Note that allowing for a regime shift in the conditional mean of stock return distribution can capture mean reversion in stock prices (see, e.g. Ceccheti et al (1990)). As aptly noted by Lo and Wang (1995), ignoring this mean reversion will lead to substantial biases in pricing options.
The option pricing model that the paper derives calculates a European call option price, analytically, as a weighted average of Black-Scholes (BS) European call prices written on stocks. This formula uses as weights the risk neutral probabilities of the number of periods that stock market will spend in each of its different regimes over the entire life of the option, referred to as occupation time. This weighting means that, if, for instance, stock market is currently in the bear regime and is expected to stay in this regime for some time, then our option pricing formula will put more weight on the BS option price corresponding to the bear regime, compared to that corresponding to the bull regime. This will lead to a higher option price of a European call written on a stock than that predicted by the standard BS formula, which does not allow for regime shifts. Intuitively, this can be attributed to the higher volatility or lower mean of the bear regime than those implied by the bull regime.

Based on option and aggregate stock market price data for US from year 1996 to 2010, the paper provides a number of interesting results. First, it shows that the stock market can be characterized by three different regimes: the bull, bear and crash. The joint estimation employed to obtain values of the parameters of the MRS option pricing model and underlying log-price MRS process helps to better identify the different regimes of the stock market over time and to trace out the regime-switching dates. For the crash regime, these dates are associated with those of the Asian and Russian financial crises, the terrorist attack of September 16, 2001, the collapses of Enron, WorldCom and Lehman brothers. Second, the paper clearly shows that regime shifts are priced in the options market, and thus should not be treated as risk neutral in the literature. These can magnify the steepness of the smirk and the curvature of the implied volatility smile, explained by MRS option pricing models.

In particular, the paper shows that the prices of regime shifts from bull to bear and from bear to crash markets, which mainly concern investors in the options market, are substantial. The paper also documents important regime-shift sources of risks from crash to bear and from bear to bull markets, despite the fact that transition probabilities of these regime switches are very small. Investors in the options markets price these regime-shifts by assigning smaller risk neutral transition probabilities to them than their physical counterparts. This can be also attributed to their risk aversion behaviour, which decreases the physical transition probability of a shift to a better market regime. Finally, both the physical and risk neutral transition probabilities indicate that the most likely to happen sequence of regime shifts is that from bull to bear and from that to crash, directly.

The paper is organized as follows. In Section 2, we present a simple version of the MRS option pricing model assuming that stock market consists of two regimes: the bull and bear, as often assumed in practice. This version of the
model will help us to better understand the option pricing formula implied by the MRS process of the underlying stock log-price process. In Section 3, this model is extended to the case of $N$ distinct stock market regimes. In Section 4, we implement the MRS option pricing model suggested by the paper to actual data with the main aim of estimating the prices of regime-shift sources of risks implied by options market participants. In this section, we also examine if the MRS model can explain patterns of the BS implied volatility observed in practice and we assess the magnitude of the option pricing errors if regime-shift risks are assumed that are not priced in option markets. Finally, we also compare the pricing performance of the MRS model with the discrete-time option pricing model of Nandi and Heston (2000), assuming a GARCH volatility function of the underlying log-price process. This model can be considered as competent to the MRS model, given evidence that GARCH effects can be generated by regime shifts in volatility (see, e.g., Morana and Beltratti (2004), Kramer and Tameze (2007), and Andreou and Ghysels (2009), for a survey), or they can approximate these shifts (see Duan et al (2002)). Section 5 concludes the paper. All proofs of the paper are given in its Appendix.

2 The MRS Option Pricing Model

2.1 The MRS data generating process

Let the logarithm of the stock price, defined as $y_t = \log Y_t$, obey the following Markov regime-switching (MRS) process under the physical probability measure, allowing for two regimes of stock market:

$$y_{t+1} = y_t + \mu (S_{t+1}) - \frac{1}{2} \sigma^2 (S_{t+1}) + \sigma (S_{t+1}) \epsilon_{t+1},$$

(1)

where

$$\mu (S_t) = \mu' S_t, \ \mu = (\mu_1, \mu_2)', \ S_t = (e_1, e_2)',$$

$$\sigma (S_t) = \sigma' S_t, \ \sigma = (\sigma_1, \sigma_2)'$$

and

$$\epsilon_{t+1} \sim \text{NIID}(0, 1).$$

In the above specification of the MRS process (1), $S_t$ is a binary vector process which follows a discrete-time state space Markov chain. That is, $S_t \in \{e_1, e_2\}$, where $\{e_i, i \in (1, 2)\}$ are unit vectors that span the two dimension space $R^2$. The $i$-th element of $e_i$ is 1 for the $i$-th regime (state) of stock market, and zero otherwise. The transition probability matrix between the two regimes of stock market considered
by (1), denoted as ‘1’ and ‘2’, from time $t$ to $t + 1$ is defined as follows:

$$P \equiv [p_{ij}] = \begin{bmatrix} 1 - p_{12} & p_{21} \\ p_{12} & 1 - p_{21} \end{bmatrix},$$

(2)

where $p_{ij}$ declares the transition probability of moving from regime ‘i’ to ‘j’. Regime ‘1’ is often identified by stock market data as the bull regime. This is characterized as one having the highest mean and lowest volatility of stocks returns distributions between the two regimes. On the other hand, regime ‘2’, referred to as bear, is often identified as one having the highest volatility and lowest mean.

### 2.2 Option pricing

To derive a European call option pricing formula for the MRS process (1), we will first define the risk neutralized equivalent process of (1) of the $\tau$-period ahead log-price $y_{t+\tau}$ (or its implied log-return, defined as $\Delta_{\tau}y_{t+\tau} \equiv y_{t+\tau} - y_{t}$). Then, we will obtain the risk neutral density of this process conditional on the current, $t$-time market information set, denoted as $I_t$. To this end, we make the following two assumptions: First, we will consider that information set $I_t$ includes, in addition to all historical stock prices, the present and past values of the state vector process $S_t$, i.e. $I_t = \{y_t, S_t, y_{t-1}, S_{t-1}, \ldots\}$. This is a standard assumption made in the literature (see, e.g., Garcia et al (2003) and Dai et al (2007)), which means that option market investors know the current regime of the stock market. Second, we will assume that vector process $S_t$ is independent of the innovation term $\epsilon_t$ of MRS process (1), for all $t$. This is also a standard assumption made in the literature which means that stock market investors are surprised by future regime changes (see, e.g., Turner et al (1989), Cecceti et al (1990), and Chabi-Yo et al (2008), more recently).

To derive the risk-neutral process of the $\tau$-period log-return $\Delta_{\tau}y_{t+\tau}$, write its physical counterpart, by solving forward MRS process (1), as follows:

$$\Delta_{\tau}y_{t+\tau} = \left(\mu_1 - \frac{1}{2}\sigma_1^2\right)\tau + \left[\frac{1}{2}(\mu_2 - \mu_1) - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)\right]O_{\tau}^2 + \sum_{i=1}^{\tau} \sigma (S_{t+i}) \epsilon_{t+i},$$

(3)

Recent studies (see, e.g., Barro and Ursua (2009)) associate the bull and bear stock market regimes to expansionary and recessionary conditions of the economy, respectively. Deep recessionary conditions of the economy are often associated with stock market crashes, which can be considered as a separate regime.
or, by substituting volatility function of $\sigma(S_{t+i})$ into (3), as

$$
\Delta_{\tau}y_{t+\tau} = \left( \mu_1 - \frac{1}{2} \sigma_1^2 \right) \tau + \left[ (\mu_2 - \mu_1) - \frac{1}{2} (\sigma_2^2 - \sigma_1^2) \right] O_\tau^2 + \sqrt{\sigma_1^2 \tau + (\sigma_2^2 - \sigma_1^2) O_\tau^2} \omega_{t,\tau},
$$

where $O_\tau^2$ is a random variable, defined as

$$
O_\tau^2 = \sum_{k=1}^{\tau} e_2^2 S_{t+k},
$$

which represents the number of times that the stock market will stay in regime ‘2’ over maturity horizon $\tau$, and $\omega_{t,\tau}$ is a normally distributed variable capturing the innovation terms $\varepsilon_{t+i}$ of process $\Delta_{\tau}y_{t+\tau}$. Given that $S_{t+i}$ is a binary vector process taking values $e_1 = (1, 0)'$ and $e_2 = (0, 1)'$, random variable $O_\tau^2$ takes values $\zeta_2 \in \{0, \ldots, \tau\}$. These will reflect the occupation time of regime ‘2’, over interval $\tau$. Thus, $O_\tau^2$ will be henceforth referred to as occupation time variable.

In the appendix, we present an algorithm calculating the probabilities of the set of values of $O_\tau^2, \zeta_2$. These probabilities are needed to calculate the expectation of log-return process $\Delta_{\tau}y_{t+\tau}$ conditional on current information set $I_t$ and, then, to obtain European call option prices with maturity intervals bigger than one period.

The risk-neutral counterpart of (3) can be derived under the equivalent risk neutral measure, denoted as $Q$, implying that the discounted stock price $Y_{t+\tau}$ will form a martingale process, i.e.

$$
E_t^Q(Y_{t+\tau}) = Y_t \exp \{ \tau r_t^f \},
$$

where $r_t^f$ is the $\tau$-period risk-free interest rate at time $t$ and $E_t^Q(\cdot)$ is the conditional expectation $E_t(\cdot)$ under risk neutral measure $Q$. Under this measure, the following risk neutralized measure of $\Delta_{\tau}y_{t+\tau}$ can be defined

$$
\frac{\Delta_{\tau}y_{t+\tau}}{r_t^f} - \frac{1}{\tau} \ln E_t^Q \left( \exp \left[ (\mu_2 - \mu_1) O_{\tau}^{2,Q} \right] \right) - \frac{1}{2} \sigma_1^2 + \left[ (\mu_2 - \mu_1) - \frac{1}{2} (\sigma_2^2 - \sigma_1^2) \right] O_{\tau}^{2,Q}

+ \sqrt{\sigma_1^2 \tau + (\sigma_2^2 - \sigma_1^2) O_{\tau}^{2,Q}} \omega_{t,\tau},
$$

where $\omega_{t,\tau}^Q$ and $O_{\tau}^{2,Q}$ constitute the risk neutral counterparts of random variables $\omega_{t,\tau}$ and $O_{\tau}^2$, respectively, under measure $Q$ (see Appendix). Under this measure,

\footnote{See, e.g. Pliska (1999).}
\( \omega^Q_{t, r} \) will be assumed that is a \( \text{NIID}(0, 1) \) random variable, while the transition probability matrix of the binary state vector \( S_t, P \), determining the probabilities of occupation time variable \( O^Q_{t, r} \), will be defined as follows:

\[
P^Q \equiv [p^Q_{ij}] = \begin{bmatrix} 1 - p^Q_{12} & p^Q_{21} \\ \frac{p^Q_{12}}{1 - p^Q_{21}} & 1 - p^Q_{21} \end{bmatrix},
\]

where

\[
p^Q_{ij} = \lambda_{ij} p_{ij}.
\]

These relationships between risk neutral transition probabilities \( p^Q_{ij} \) and their physical counterparts, \( p_{ij} \), reflect the risk averse behaviour of investors, and thus suggest that the risk of a regime-shift is priced in options market. Such an assumption are often made in the literature. For instance, Jarrow et al (1997) and Whitelaw (2000) assume that regime shifts from the bull to the bear regime must be priced in the stock and options markets, since they have adverse effects on the value and the income of investments.

Based on an equilibrium pricing approach, Dai et al (2007) show that even regime shifts from the bear to the bull regime should be priced by the markets, due to the risk aversion behavior of investors. This behavior implies that a regime shift from the bull to the bear regime will imply a value of the price of regime-shift risk coefficient \( \lambda_{ij} \) bigger than unity, implying that \( p^Q_{ij} > p_{ij} \), for \( i = 1 < j = 2 \). On the other hand, a regime shift from the bear to the bull regime will imply a value of \( \lambda_{ij} < 1 \), implying that \( p^Q_{ij} < p_{ij} \), for \( i = 2 < j = 1 \). A value of \( \lambda_{ij} \) bigger than unity, for \( \{i, j\} = \{1, 2\} \), will increase the risk neutral probability that the stock market will stay in the bear regime more than one times over the maturity interval \( \tau \), i.e. \( \zeta_2 > 0 \). This will be done at the expense of the probability that the market will stay in the bull regime over \( \tau \), i.e. \( \zeta_2 = 0 \). The opposite will happen for a value of \( \lambda_{ij} \) less than unity, for \( \{i, j\} = \{2, 1\} \). Finally, note that a value of \( \lambda_{ij} \) equal to unity, implying \( p^Q_{ij} = p_{ij} \), means that regime shifts are considered as risk risk neutral for investors in the options markets, and thus they are not priced in the options markets. That is, they have zero price.

The risk-neutral process of log-return \( \Delta_t y_{t+\tau} \), given by equation (7), implies that, in addition to the risk-free interest rate \( r^f_t \), its conditional on information set \( I_t \) mean also depends on the difference of the means across the two regimes \( \mu_2 - \mu_1 \), as well as the conditional mean of the occupation time random variable \( O^Q_{t, r} \). These terms affect the conditional mean of \( \Delta_t y_{t+\tau} \) and, hence, its risk neutral distribution, since regime shifts are not traded in the stock market. Thus, their effects can not be hedged out under the risk neutral measure \( Q \). The dependence of the risk neutral distribution of \( \Delta_t y_{t+\tau} \) on the difference of means \( \mu_2 - \mu_1 \) can
explain the skewness of this distribution and the smirk (skew) of the Black-Scholes implied volatility smile, observed in practice (see, e.g., Bakshi et al (1997)). The curvature of this volatility smile can be explained by regime shifts in the volatility function of MRS process (1), implying $\sigma_2^2 \neq \sigma_1^2$. These shifts can explain evidence of excess kurtosis of the risk neutral distribution of $\Delta_t y_{t+\tau}$. Based on the above risk-neutral specification of the MRS process of log-return $\Delta_t y_{t+\tau}$ (or log-price $y_{t+\tau}$), in the next proposition we derive a closed form solution of a European call option price based on Cox’s et al (1976) risk neutral pricing framework.

**Proposition 1** Let the log-price of the underlying stock, $y_t$, follow MRS process (1). Then, the price of a European call option with strike price $K$ and maturity interval $\tau$, denoted as $C_t(\tau)$, can be analytically derived by the following formula:

$$C_t(\tau) = \sum_{\zeta_2=0}^{\tau} C_t^{BS} (\sigma^2(\zeta_2), \nu(\zeta_2), \tau) \nu(\zeta_2) \Pr \left[ O_{\tau}^{2,Q} = \zeta_2 \big| I_t \right], \quad (8)$$

where

$$C_t^{BS} (\sigma^2(\zeta_2), \nu(\zeta_2), \tau) = Y_t \Phi (d_1(\zeta_2)) - \frac{K}{\nu(\zeta_2)} e^{-\tau r_f} \Phi (d_2(\zeta_2)), \quad (9)$$

$$d_1(\zeta_2) = - \ln \frac{K}{\nu(\zeta_2)} + \tau r_f + \frac{\sigma^2(\zeta_2)}{2},$$

$$d_2(\zeta_2) = - \ln \frac{K}{\nu(\zeta_2)} + \tau r_f - \frac{\sigma^2(\zeta_2)}{2},$$

$$\nu(\zeta_2) = \frac{e^{(\mu_2-\mu_1)\zeta_2}}{E_{I_t}^{Q} \left[ e^{(\mu_2-\mu_1)\zeta_2} \right]},$$

$$\sigma(\zeta_2) = \sqrt{\sigma_1^2 \tau + (\sigma_2^2 - \sigma_1^2) \zeta_2},$$

and $\Phi(\cdot)$ denotes the cumulative normal distribution. See Appendix, for the proof of the proposition.

Proposition 1 demonstrates that the European call option price implied by the MRS process (1) can be written as a weighted average of the Black-Scholes (BS) European call option prices $C_t^{BS} (\sigma^2(\zeta_2), \nu(\zeta_2), \tau)$ with strike prices $\frac{K}{\nu(\zeta_2)}$, defined by formula (9). These option prices are conditional on the values of the risk neutral equivalent measure of the occupation time variable $O_{\tau}^{2,Q}$, given by set $\zeta_2$, reflecting
the number of periods that stock market will stay in the bear regime, "2", over its maturity interval \( \tau \). Each of these conditional option prices is weighted by its corresponding risk neutral probability of the values of set \( \zeta_2 \) to occur, given as \( \Pr [O_{\tau}^{2,Q} = \zeta_2 | I_t] \). Calculating these probabilities is necessary to price European call options with a maturity horizon bigger than one period, \( \tau > 1 \).

The risk neutral probabilities \( \Pr [O_{\tau}^{2,Q} = \zeta_2 | I_t] \), by which the BS conditional option prices \( C^{BS}_t (\sigma^2 (\zeta_2), \nu (\zeta_2), \tau) \) are weighted, capture the effects of a possible adverse regime shift of the stock market from the bull to the bear regime on European call option prices, over maturity horizon \( \tau \). For instance, if the stock market lies in the bear regime at current time \( t \), they will tend to weight more the BS prices \( C^{BS}_t (\sigma^2 (\zeta_2), \nu (\zeta_2), \tau) \) considering that the stock market will stay in this regime for more than one times until the expiration date of the option, compared to those assuming that the stock market will stay more times in the bull regime, over \( \tau \). Obviously, the values of probabilities \( \Pr [O_{\tau}^{2,Q} = \zeta_2 | I_t] \) will depend on the persistency of each regime.

There are two interesting special cases where the MRS option pricing formula (8) can be reduced. The first is when there is no regime change, for all \( t \). Then, \( \zeta_2 = 0 \) and, thus, equation (8) reduces to the standard BS option pricing formula, which assumes no regime-switching. The second case is when there is no regime shift in the mean of log-return \( y_t \) (i.e., \( \mu_1 = \mu_2 = \mu \)), but only in its volatility function (i.e., \( \sigma_1 \neq \sigma_2 \)). In this case, (8) reduces to a formula which is a weighted average of the standard BS European call option prices conditional on the occupation time values \( \zeta_2 \), given as

\[
C^{BS}_t (\sigma^2 (\zeta_2), \nu (\zeta_2), \tau) = Y_t \Phi [d'_1 (\zeta_2)] - e^{-r \tau} K \Phi [d'_2 (\zeta_2)],
\]

where \( d'_1 (\zeta_2) = \frac{\log(Y_t/K) + r \tau}{\sigma (\zeta_2)} + \frac{1}{2} \sigma (\zeta_2) \) and \( d'_2 (\zeta_2) = d'_1 (\zeta_2) - \sigma (\zeta_2) \). This formula corresponds to that suggested by Hull and White’s (1987) assuming that log-return \( y_t \) follows a stochastic volatility model. This model can only explain BS implied volatility smiles which are symmetric, as it does not consider stochastic changes in the conditional mean of \( y_{t+1} \).

### 2.3 Generalization to \( N \) different market regimes

In this section, we generalize the MRS option pricing model provided in the previous section to the case where the number of stock market regimes is bigger than two, given by a finite number \( N \). In particular, we assume that the stock log-price process \( y_t \), given by MRS process (1), allows for \( N \) different regimes of the market. The mean and volatility functions of this process will be defined as the inner product of the \((N \times 1)\)-dimension vectors of the \( N \) different mean and volatility parameters \( \mu_i \) and \( \sigma_i \), respectively, with state vector \( S_t \), which now is defined as
\( S_t = (e_1, e_2, ..., e_N)' \), where \( \{e_i, i \in (1, 2, ..., N)\} \) are the unit vectors spanning the \( N \)-dimension space \( R^N \). That is, we will have \( \mu(S_t) = \mu'S_t \) and \( \sigma(S_t) = \sigma'S_t \), where \( \mu = (\mu_1, \mu_2, ..., \mu_N)' \) and \( \sigma = (\sigma_1, \sigma_2, ..., \sigma_N)' \). This multi-regime specification of MRS process (1) implies that the transition matrix of the physical probabilities \( P \) among all the different stock market regimes considered has dimension \( (N \times N) \).

It is defined as follows:

\[
P = \begin{bmatrix}
p_{11} & p_{21} & \cdots & p_{N1} 
p_{12} & p_{22} & \cdots & p_{N2} 
p_{1N} & p_{2N} & \cdots & p_{NN}
\end{bmatrix}.
\]

The above generalization of the MRS process (1) implies that the \( \tau \)-period ahead, future log-return process \( \Delta_t y_{t+\tau} \) and its risk neutral counterpart can be respectively written as follows:

\[
\Delta_t y_{t+\tau} = \left( \mu_i - \frac{1}{2}\sigma_i^2 \right) \tau + \sum_{i=2}^{N} \left[ (\mu_i - \mu_1) - \frac{1}{2} (\sigma_i^2 - \sigma_1^2) \right] O_{\tau}^i
\]

\[
+ \sqrt{\sigma_i^2 \tau + \sum_{i=2}^{N} (\sigma_i^2 - \sigma_1^2) O_{\tau}^i} \omega_{i,\tau}
\]  \( \text{(11)} \)

and

\[
\Delta_t y_{t+\tau} = \left( \mu_i - \frac{1}{2}\sigma_i^2 \right) \tau + \sum_{i=2}^{N} \left[ (\mu_i - \mu_1) - \frac{1}{2} (\sigma_i^2 - \sigma_1^2) \right] O_{\tau}^i Q
\]

\[
+ \sqrt{\sigma_i^2 \tau + \sum_{i=2}^{N} (\sigma_i^2 - \sigma_1^2) O_{\tau}^i Q} \omega_{i,\tau}^Q,
\]  \( \text{(12)} \)

where \( \omega_{i,\tau}^Q \) and \( O_{\tau}^i Q \) constitute the risk neutral counterparts of random variables \( \omega_{i,\tau} \) and \( O_{\tau}^i \), respectively, under measure \( Q \) (see Appendix A3). Note that, under the above generalization of the MRS process (1), the occupation time variable of a representative stock market regime \( i \) now is defined as \( O_{\tau}^i = \sum_{k=1}^{\tau} e_i'S_{t+k} \). This variable will represent the number of periods (times) that the stock market will stay in regime \( i \), for \( i = 2, ..., N \) stock market regimes, over the maturity horizon.
Since $P_{N_i} = 1$, the occupation time variable for regime ‘1’ will be defined as $O_1 = \tau - \sum_{i=2}^{N} O_i$. The risk-neutral counterpart of occupation time variable $O_i$ will be denoted as $O_i^{Q}$. Both variables $O_i$ and $O_i^{Q}$ take values $\zeta_i$ which satisfy the following condition $\sum_{i=2}^{N} \zeta_i \leq \tau$. As in the previous subsection, to calculate the risk neutral probabilities of the values of occupation time variable $O_i^{Q}$, $\zeta_i$, we will assume that the elements of the risk neutral transition matrix $P^Q \equiv [p_{ij}^Q]$ are related to those of its physical counterpart $P \equiv [p_{ij}]$ as follows:

$$p_{ij}^Q = \lambda_{ij} p_{ij},$$

where $\lambda_{ij}$ denotes the price of risk of a shift from regime $i$ to $j$. For analytic convenience, we will assume that, in terms of investors’ preferences, stock market regimes are ordered from regime "1" to "N", where "1" represents the best stock market regime (i.e., a bull regime which is characterized by the highest mean and lowest volatility of the log-return $y_t$, across all different stock market regimes), while regime "N" denotes the worst market regime. For instance, "N" can be taken to represent the crash regime of the market. In this regime, $y_t$ will have the lowest mean and largest volatility.\footnote{Note that the ranking of the alternative market regimes can be thought of as an empirical issue. This will not change the results of our analysis.}

Based on the risk-neutral relationship (12) for log-return $\Delta y_{t+r}$ and its implied underlying risk neutral density, in the next proposition we present an analytic option pricing formula of a European call in the case of $N > 2$ different stock market regimes.

**Proposition 2** Let the log-price of the underlying stock $y_t$ follow a generalization of MRS process (1) allowing for $N > 2$ different stock market regimes. Then, under risk neutral probability measure $Q$, the price of a European call option price $C_t(\tau)$ with strike price $K$ and maturity interval $\tau$ can be analytically derived by the following formula:

$$C_t(\tau) = \sum_{\zeta_N} \ldots \sum_{\zeta_2} C_{t}^{BS} (\sigma^2(\zeta), \nu(\zeta), \tau) \nu(\zeta) \Pr[O_{2}^{Q} = \zeta_2; \ldots; O_{N}^{Q} = \zeta_N | \mathcal{I}_t],$$

where

$$C_{t}^{BS} (\sigma^2(\zeta), \nu(\zeta^{N-1}), \tau) = Y_t N(d_1) - \frac{K}{\nu(\zeta)} e^{-\tau r t} N(d_2),$$

$\zeta = (\zeta_2; \ldots; \zeta_N)$ is a $(N-1)$-dimension vector collecting the values of occupation time variables $O_i^{Q}$ corresponding to regimes $i = 2, 3, \ldots, N$, $\nu(\zeta)$ is a function of...
the elements of vector $\zeta$ defined as

$$
\nu(\zeta) = \frac{\exp \left\{ \sum_{i=2}^{N} (\mu_i - \mu_1) \zeta_i \right\}}{E_Q^2 \left\{ \exp \left( \sum_{i=2}^{N} (\mu_i - \mu_1) \zeta_i \right) \right\}},
$$

and

$$
d_1 = -\ln \frac{K}{\nu(\zeta) + \tau r_{f}^2 + \frac{\sigma^2(\zeta)}{2}} \sigma(\zeta)
$$

$$
d_2 = -\ln \frac{K}{\nu(\zeta) + \tau r_{f}^2 - \frac{\sigma^2(\zeta)}{2}} \sigma(\zeta)
$$

$$
\sigma^2(\zeta) = \sigma^2_{1\tau} + \sum_{i=2}^{N} (\sigma^2_i - \sigma^2_{1\tau}) \zeta_i
$$

The proof of proposition is given in the appendix.

The option pricing formula given by equation (13) has analogous interpretation to that of the case of two market regimes ($N = 2$), given by (8). It calculates European call price $C_t(\tau)$ based on BS call option prices $C_{tBS}(\sigma^2(\zeta), \nu(\zeta), \tau)$, which are conditional on the values of risk-neutral occupation time variables of all possible stock market regimes $i, O_i^Q$, for $i = 2, 3, \ldots, N$. These values now are collected in vector $\zeta = (\zeta_2, \ldots, \zeta_N)$. One difference of formula (13) from (8) is that, in calculating option price $C_t(\tau)$, it allows for the possibility that the stock market will stay in more than one regimes until the expiration date of the option.

### 3 Estimation of the MRS option pricing model

In this Section, we show how to estimate and implement the MRS option pricing model derived in the previous section. Our analysis is focused on examining if regime shifts constitute significant sources of risks which are priced in the options market. We will also evaluate the magnitude of the pricing errors encountered, if we assume that regime shifts are risk neutral for investors in the markets, and we will compare the model to the discrete-time stochastic volatility option pricing model of Nandi and Heston (2000), henceforth HN. The latter is frequently used, in practice, for option pricing (see, e.g., Christofersen and Jacobs (2004), and Moyaert and Petitjean (2011)). This model assumes that the volatility function of stock return distributions follows a GARCH(1,1) process. As noted in the introduction, GARCH effects in the volatility function of stock returns can be
generated by MRS type of shifts in this function. To price call option prices under Markov regime-switching, Duan et al (2002) used GARCH volatility functions to approximate regime type of shifts of this volatility function.

In answering the above all questions, the paper suggests a joint estimation of the option pricing model suggested by the paper and MRS process (1), based on options and stock markets data. This estimation enables us to obtain estimates of the regime-shift price of risk coefficients $\lambda_{ij}$ and, hence, to calculate the risk neutral occupation time probabilities $Pr[O_{t|Q}^i = \zeta_i | \mathcal{I}_t]$, for $i = 2, ..., N$ different regimes. The latter are necessary for the implementation of formula (13), or (8).

The data sets used in our analysis consists of time series observations on the S&P500 stock market index and European call option prices written on this index. These series cover the period from 01/1996 to 10/2010. During this period, the US stock market experienced four financial crises. The first was related to Asian and Russian crises of years 1997 and 1998, respectively. The second and third followed the terrorist attack on September 11, 2001, and the collapses of the Enron and WorldCom companies in year 2002, respectively. The fourth followed the collapse of Lehman Brothers on September 16, 2008. Our option price data set employed in the estimation consists of nonoverlapping at-the-money (ATM), in-the-money (ITM) and out-of-the-money (OTM) European call option prices, with maturity intervals of 5,10,15 and 20 trading days. In total, this set consists of 710 units of option prices and stock returns.

Our empirical analysis proceeds as follows. First, we present estimates of univariate process (1) with the aim of investigating the number of different regimes of the stock market identified by our data. Given this, next we carry out the joint estimation of the MRS process and option pricing formula (13), using stock and options price data. Finally, we assess the consequences of assuming risk-neutral regime shifts on option pricing and we compare the pricing performance of the model to that HN’s model.

### 3.1 Estimates of the univariate MRS process (1)

Table 1 presents estimates of MRS process (1) based on the Maximum Likelihood (ML) method, suggested by Hamilton (1989). The table presents two sets of results. The first assumes that the number of the stock market regimes is two ($N = 2$), while the second that it is three ($N = 3$). To examine which of these two numbers of regime specifications constitutes a better description of the data, the table presents the Akaike Information Criterion ($AIC$) and Ljung-Box

---

A specification of the MRS model with four regimes ($N = 4$) is also estimated. But, this is not found to describe our data satisfactorily.

This is considered as an appropriate criterion in choosing the number of regimes assumed by the MRS process (1). Likelihood ratio (LR) based tests can not be used for this purpose. These
test statistics for serial correlation and conditional heteroscedasticity of the disturbance term of the MRS process $\epsilon_t$. These statistics are denoted as $LB(\cdot)$ and $LB^2(\cdot)$, respectively, where the number of lags are reported in parentheses. They are based on the normally distributed Rosenblatt transformation of the log-return $\Delta y_t$ and they are distributed as chisquared. As shown by Smith (2006), the normal transformation of $\Delta y_t$ improves considerably the finite sample properties of statistics $LB(\cdot)$ and $LB^2(\cdot)$, compared to those based on the standardized generalized disturbance term of the MRS process, defined as $E (\epsilon_t | I_{t-1})$. The latter are nonnormally distributed.

suffer from singularity problems of the information matrix under the null hypothesis that the true number of regimes is $N$ (see, e.g., Psaradakis and Spangolo (2003)).

The Rosenblatt transformation of a random variable implies that, if $\Delta y_t$ has a distribution function $F$, then $Z_t = \Phi^{-1} (F (\Delta y_t))$ will be a standard normal random variable, where $\Phi^{-1} (\cdot)$ denotes the inverse of the standard normal cumulative distribution function. For MRS process (1) allowing for $N$ different market regimes, the normally distributed random variable $Z_t$ is defined as

$$Z_t = \Phi^{-1} \left( \sum_{i=1}^{N} \Pr (S_t = i | I_{t-1}) \int_{-\infty}^{\Delta y_t} f (v | S_t = i, I_{t-1}) dv \right)$$

The Ljung-Box statistics $LB^1(\cdot)$ and $LB^2(\cdot)$ can rely on sample estimates of $Z_t$ and $Z^2_t$ to test for serial correlation and conditional heteroscedasticity of order-$h$ in the disturbance term $\epsilon_t$, respectively. For instance, the Ljung-Box statistic $LB^1(h)$ is calculated based on the following formula:

$$LB(h) = T (T + 2) \sum_{j=1}^{h} \frac{\rho_j^2}{n-j},$$

where $\rho_j$ is the sample autocorrelation of random variable $Z_t$. 

---

14
Table 1: Estimates of MRS process (1)

<table>
<thead>
<tr>
<th>no. reg.</th>
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<th>$N = 2$</th>
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<th>$N = 2$</th>
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<td>0.0026</td>
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<td>$\mu_2$</td>
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<td>(1e-9)</td>
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<td>$p_{32}$</td>
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</tr>
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<td>0.027</td>
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<td>0.0034</td>
<td>$LB^2(1)$</td>
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<td>$p_{21}$</td>
<td>0.006</td>
<td>0.10</td>
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<td>(2e-5)</td>
<td>(4e-4)</td>
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</tbody>
</table>

Notes: The table presents estimates of (1) for $N = 2$ and $N = 3$ number of regimes (no. reg.). Quasi ML (maximum likelihood) estimates of the standard errors of these estimates are in parentheses. $AIC$ denotes the Akaike Information Criterion, and $LB(h)$ and $LB^2(h)$ respectively denote the Ljung-Box test statistics for serial correlation and conditional heteroscedasticity of the disturbance term of MRS process (1) of order $h$. $logL$ is the maximum log-likelihood value of (1).

The results of the table clearly indicate that the specification of MRS (1) allowing for $N = 3$ regimes of the stock market describes better the data. This can be justified both by the values of $LB(\cdot)$ and $LB^2(\cdot)$ statistics and the AIC criterion, reported in the table. The three regimes identified by our data through this specification of the MRS process capture bull, bear and crash conditions of the stock market. The first (bull) regime has the highest mean $\mu_1$ and lowest volatility $\sigma_1$ across the three regimes. The second (bear) regime has a mean $\mu_2$ which is not different than zero. Its volatility $\sigma_2$ is almost twice that of the bull regime. Finally, the third (crash) regime has a negative mean $\mu_3$ and the highest volatility $\sigma_3$, across all three regimes. Estimates of the probabilities that the US stock market lies in these regime at each point of time $t$ are graphically presented in Figures 1, over our sample. These are denoted as $\Pr[S_t = e_i|y_t]$, for $i = 1, 2$ and 3. They are
obtained based on Hamilton’s (1989) filter. The results of this figure clearly show that the crash regime is associated with the financial crises of the US stock market, mentioned before. The bull regime of the market seems to characterize the period from year 2003 to 2007. The bear regime seems to characterize the period from the middle of nineties to year 2003, with the exception of the short periods of the crashes mentioned above.

Another interesting conclusion that can be drawn from the results of the table is that the most likely to happen sequence of regime-switching in the stock market is from bull to bear and from that to crash regime, and the inverse. This can be supported by the values of the transition probabilities $p_{ij}$, reported in the table. The transition probability from the bull to the crash regime is found to be almost zero, which means that it is less likely to be a regime shift from the bull
regime to the crash. Also note that the value of the transition probability from the crash to the bear regime is quite high. This is associated with the fact that the crash regime tend to last for the shortest time-intervals during our sample.

3.2 Joint estimation of the MRS process and option pricing model (13)

The joint estimation of the MRS option pricing model given by formula (13) and MRS process (1) will provide estimates of the risk neutral parameters of this process and regime shift price of risk coefficients $\lambda_{ij}$’s, which are needed for the implementation of the model. Since the stock market regime $S_t$, at time $t$, is not known by the econometrician, in the estimation procedure we will infer this from the data based on Hamilton’s (1989, 1993) filter. This is a standard procedure often followed in the literature to estimate MRS models (see, e.g., Melino and Yang (2003), and Chabi-Yo (2008). The information set $I_{t-1}$ assumed by this estimation procedure consists of past values of $y_t$ and $C_t(\tau)^{obs}$, or their differences, i.e. $I_{t-1} = \{y_{t-1}, C_{t-1}, \Delta y_{t-2}, \ldots\}$. As can be confirmed by our empirical results latter on, exploiting joint information from both the stock and options markets will better help in identifying the current stock market regime $S_t$ by the data, given that options are priced based on market information about this regime.

If we assume that the observed European call price $C_t(\tau)^{obs}$, at time $t$, is measured with an $NIID$ error term $v_t$, i.e. $C_t(\tau)^{obs} = C_t(\tau) + v_t$, where $v_t \sim N(0, \sigma_v^2)$ (see, e.g., ), then the vector of parameters of the MRS process (1) augmented with the risk-neutral counterparts of $p_{ij}$, collected in vector $\theta = (\mu, \sigma, \rho_{12}, \ldots, \rho_{N-1N}, p_{12}^Q, \ldots, p_{N-1N}^Q)$, can be estimated by maximizing the following loglikelihood function:

$$
L = \sum_{t=1}^{T} \log f \left( \Delta y_t, C_t(\tau)^{obs} | I_{t-1}; \theta \right) 
= \sum_{t=1}^{T} \log \left[ \sum_{i=1}^{N} f \left( \Delta y_t, C_t(\tau)^{obs} | S_t; \theta \right) \Pr [S_t = e_i | I_{t-1}] \right],
$$

over a sample of $t = 1, 2, \ldots, T$ observations , where $f \left( \Delta y_t, C_t(\tau)^{obs} | S_t; \theta \right)$ is the joint density function of log-return $\Delta y_t$ and option price $C_t(\tau)^{obs}$, at time $t$. This is the product of two univariate probability density functions, i.e.

---

Joint estimation of option pricing models and the underlying asset price stochastic process is a standard procedure for retrieving risk neutral probabilities from the data (see, e.g., Jackwerth (2000)). It has been also suggested by Chernov and Ghysels (2000) to estimate the price of risk coefficients of the stochastic volatility option pricing model of Heston (1993).
\[
f (\Delta y_t, C_t(\tau)^{obs}|S_t; \theta) = \left(2\pi \sqrt{\sigma^2 (S_t) \sigma_v^2}\right)^{-1} \exp \left\{ \frac{(\Delta y_t - (\mu (S_t) - \frac{1}{2} \sigma^2 (S_t)))^2}{2\sigma^2 (S_t)} \right\} \exp \left\{ \frac{[C_t(\tau)^{obs} - C_t(\tau)]^2}{2\sigma_v^2} \right\},
\]

since error terms \(\epsilon_t\) and \(\nu_t\) are assumed to be independent. Estimates of the regime-shift price of risk coefficients \(\lambda_{ij}\) can be obtained from the estimates of the physical and risk-neutral transition probabilities \(p_{ij}\) and \(p_{ij}^Q\), respectively, based on the following relationship \(p_{ij}^Q = \lambda_{ij}p_{ij}\) (see Sections 2 and 3).

Table 2 presents estimates of vector \(\theta\) based on the above joint ML estimation procedure. This is done for two sets of cross-section data. The first uses ATM European call options, which is the most liquid category of options. The second set includes also ITM and OTM European call options. Both sets of estimates assume that the number of stock market regimes is \(N = 3\) and consider maturity intervals of \(\tau = \{5, 10, 15, 20\}\) trading days. The results of Table 2 are consistent with those of Table 1. They show that the stock and options markets are characterized by three distinct regimes: the bull, bear and crash. The values of the mean and volatility parameters of the MRS process estimated, as well as the transition probabilities across the three regimes are very close to those reported in Table 1. The transition probability of a shift from bull to crash regime \(p_{13}\) and that of its inverse, \(p_{31}\), are very close to zero. This means that the most likely sequence of regime shifts is that from bull to bear and from that to crash, and its inverse. This is also implied by the risk neutral transition probabilities \(p_{ij}^Q\), reported in the table. The above results are robust across the two different sets of moneyness levels considered in our analysis. This result shows that using ATM options, which is the most liquid category of options prices, can precisely estimate the risk neutral parameters of the MRS option pricing model.
Table 2: Joint estimates of the MRS process and option pricing model for $N = 3$

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>ATM</th>
<th>ATM, ITM, OTM</th>
<th>ATM</th>
<th>ATM, ITM, OTM</th>
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</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>0.0042</td>
<td>0.0044 $p_{12}$</td>
<td>0.007</td>
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<tr>
<td>$\mu_2$</td>
<td>0.0020</td>
<td>0.0029 $p_{13}$</td>
<td>9e-9</td>
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</tr>
<tr>
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<td>(0.002)</td>
<td>(7e-7)</td>
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<td>(6e-11)</td>
</tr>
<tr>
<td>$\mu_3$</td>
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<td>-0.028 $p_{21}$</td>
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<tr>
<td>$\sigma_2$</td>
<td>0.027</td>
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Notes: The table presents joint estimates of the parameters of MRS process (1), for $N = 3$ regimes, and those implied by the MRS option pricing model (13). Quasi ML (maximum likelihood) estimates of the standard errors of these estimates are in parentheses. AIC denotes the Akaike Information Criterion, and logL is the maximum log-likelihood value.

Given that the estimates of transition probabilities $p_{13}$ and $p_{31}$ are very close to zero, in Table 3 we present joint estimates of the MRS option pricing model (13) and MRS process (1) under the following restrictions: $p_{13} = 0$ and $p_{31} = 0$. This is done for the two different sets of moneyness levels, considered in Table 2. The likelihood ratio ($LR$) test statistic reported in the table can not reject these
restrictions, implying that this specification of the MRS model can satisfactorily describe the stock and options market. As was expected, the estimates of the MRS option pricing model and its underlying process reported in the table are consistent with those of its more general specification reported in Table 2. In particular, the values of the mean and volatility parameters of the MRS process estimated, as well the physical and risk neutral transition probabilities reported in Table 3 are very close to those of Table 2.

Figure 2 graphically presents estimates of the filtered probabilities of regimes $Pr \left[ S_t = e_i | y_t \right]$, for $i = 1, 2$ and $3$, obtained based on the parameter estimates of Table 3 for the data set which includes ATM, ITM and OTM options.\textsuperscript{9} Inspection of the graphs of the figure indicates that these estimates of $Pr \left[ S_t = e_i | y_t \right]$ are more smoothed than those obtained based on the estimates of univariate MRS process, presented by Figure 1. They can distinguish more clearly the current regime of the stock market $S_t$, perceived by option market investors. This obviously can be attributed to the extra information exploited by the joint estimation of the MRS process and option pricing model, based on options and stock market data.

\textsuperscript{9}Note that analogous graphs of $Pr \left[ S_t = e_i | y_t \right]$ are obtained for the data set which includes only ATM option prices.
Table 3: Joint estimates of the MRS process and option pricing model for $N = 3$ under restrictions $p_{13} = p_{31} = 0$

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<th>Moneyness</th>
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<th>ATM, ITM</th>
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<td>OTM</td>
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</tr>
<tr>
<td></td>
<td>(1e-6)</td>
<td>(4e-7)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: The table presents joint estimates of the parameters of MRS process (1), for $N = 3$ regimes, and those implied by the MRS option pricing model (13). The MRS model is estimated under the following restrictions: $p_{13} = p_{31} = 0$. Quasi ML (maximum likelihood) estimates of the standard errors of these estimates are in parentheses. AIC denotes the Akaike Information Criterion, and logL is the maximum log-likelihood value.
Joint estimates of the filtered probabilities $\Pr[S_t = \epsilon_i | I_t]$ 

The estimates of the physical and risk neutral transition probabilities of Table 3 in the case that the data set includes ATM, ITM and OTM option prices, imply the following price of risk coefficients $\lambda_{ij}$:\textsuperscript{10}

<table>
<thead>
<tr>
<th>$\lambda_{12}$</th>
<th>$\lambda_{21}$</th>
<th>$\lambda_{23}$</th>
<th>$\lambda_{32}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.41</td>
<td>1e-11</td>
<td>8.50</td>
<td>4e-9</td>
</tr>
</tbody>
</table>

These values of $\lambda_{ij}$ are consistent with the risk aversion attitude of investors in the stock and options markets. That is, $\lambda_{ij} > 1$, if $i < j$, and $\lambda_{ij} < 1$, if $i > j$, as is predicted by the theory (see Section 2). They show that the regime shift from the bull to the bear regime constitutes a very important source of risk. The shift from the bear to crash regime constitutes also an important source of risk, which

\textsuperscript{10}Analogous estimates are obtained based on the data set considers only ATM options.
is priced in the options market. But, its price \( \lambda_{23} \) is less than that of the shift from the bear to the bull regime, i.e. \( \lambda_{12} \). The smaller value of \( \lambda_{23} \) than that of \( \lambda_{12} \) means that the options’ market investors consider as more critical a regime shift of the market from the bull to bear regime. As noted before, this can be attributed to the transient nature of this regime to the crash regime. The less than unity values of risk-price coefficients of \( \lambda_{21} \) than \( \lambda_{32} \), capturing the reverse of the above regime shifts (i.e. from the bear regime to the bull and from the crash regime to the bear, respectively) indicate that investors in the options market assign significant risk premia also to these regime shifts, despite the fact that the transition probabilities of these regime shifts are very small. The risk neutral transition probabilities of them are much smaller than those of their physical counterparts, due their risk aversion behaviour.

The reported price of risk coefficients \( \lambda_{ij} \) imply that there is a substantial difference between the physical and risk-neutral probabilities of the occupation time variable \( \text{Pr}[O^i_t = \zeta_i|\mathcal{I}_t] \) and \( \text{Pr}[O^i,Q_t = \zeta_i|\mathcal{I}_t] \), for all regimes \( i = 1, 2 \) and \( 3 \). To see how different in magnitude are the values of \( \text{Pr}[O^i,Q_t = \zeta_i|\mathcal{I}_t] \) from those of \( \text{Pr}[O^i_t = \zeta_i|\mathcal{I}_t] \), in Figure 3 we graphically present estimates of both of them for the number of periods that the market will stay in the crash regime, i.e. for set \( \zeta_3 = \{0, 1, 2, ..., 6\} \). Inspection of this figure reveals the values of \( \text{Pr}[O^3,Q_t = \zeta_3|\mathcal{I}_t] \) are much higher than those of \( \text{Pr}[O^3_t = \zeta_3|\mathcal{I}_t] \) when \( \zeta_3 > 0 \), as is expected by a risk aversion behavior of investors. This is done at the expense of the probability that the market will stay zero times in the crash regime, i.e. \( \zeta_3 = \{0\} \). These results mean that, employing physical values of the occupation time probabilities in option pricing formula (13), or (8), instead of risk neutral, i.e. assuming that regime-sift risks are not price, will lead to serious option mispricing. It will underweight the values conditional on bull or bear BS prices \( C^t_{BS} (\sigma^2 (\zeta), \nu (\zeta^{N-1}), \tau) \), employed in formula (13).

\(^{11}\) Analogous graphs can be obtained for occupation time probabilities of the bear regime.
To further investigate the consequences of assumption that regime-shift risks are not priced in the market on option pricing, implying $\lambda_{ij} = 1$, in this section we present estimates of the implied volatility smile of European call options, across different moneyness levels $\frac{K}{\theta}$, for different values of $\lambda_{ij}$. See Figure 4. These are generated based on the estimates of the MRS option pricing model parameters, reported in Table 3 in the case that our data set include ATM, ITM and OTM options.

The upper set of graphs of the figure presents values of the implied volatility smile function in the case that regime shifts affect both the mean and volatility of
MRS process (1), while the lower presents values of it in the case that regime shifts affect only the volatility function of the MRS process. The comparison of these two different sets of graphs can show which features of the MRS process of return \( y_t \), namely shifts in its mean or volatility function, can explain the pattern of the implied volatility smile documented in practice. For the goals of our analysis, the figure presents values of the implied volatility smile function for prices of regime-shift risk coefficients \( \lambda_{ij} \) which are 2.5 times their estimates, and \( \lambda_{ij} = 1 \), which means that the market regime-shift of risks are not priced in the market.

Inspection of the graphs of Figure 4 (see upper set of graphs) indicates that the pattern of the implied volatility smile generated by the MRS option pricing model is consistent with that observed in practice (see, e.g., Bakshi et al (1997)). The shifts in the mean affect the skew (smirk) of the smile, while those in the volatility affect the curvature of the smile.\(^{12}\) The values of price of risk coefficients \( \lambda_{ij} \) are found to affect positively both the skew and curvature of the smile. In particular, an increase in the value of \( \lambda_{12} \) increases the skew of the smile, while an increase in the value of \( \lambda_{23} \) moves upward the whole volatility smile function. The last result can be attributed to the very high value of volatility \( \sigma_3 \), under the crash regime. This mainly affects the curvature of the volatility smile function. These results imply assuming zero price of regime-shift risks (i.e., setting \( \lambda_{12} = \lambda_{23} = 1 \) and \( \lambda_{32} = \lambda_{21} = 1 \)) will lead to serious mispricing of European calls across all different moneyness levels. Finally, a comparison between the upper and lower sets of graphs of Figure 4 clearly indicates that responsible for the smirk of the implied volatility function is a regime shift in the mean of the MRS process of

\(^{12}\)These results adds to those supporting the view that the MRS option pricing model can consistently explain the shape of the BS implied volatility smile (see, Chourdakis and Tzavalis (2000) and Aingworth et al (2006)).
return $y_t$, while for the volatility is a shift in its volatility.

![Figure: BS implied volatilities generated by the MRS option pricing model](image)

3.4 Option pricing performance

To quantitatively assess the pricing performance of the MRS option pricing model, in Table 4 we present values of the root mean square error ($RMSE$) and mean absolute error ($MAE$) of its pricing errors, across different moneyness categories of our European call option price data set. These are obtained based on the parameter estimates of the model reported in Table 3, for the case of the ATM set of options price data which is the most liquid category of options. Thus, the results of the table for the categories of ITM and OTM options can be thought of as those of an out-of-sample evaluation exercise of the model, using cross-section sets of data.
To examine the performance of the model under different assumptions of interest, the table reports values of the above metrics in the case that the price of risk coefficients $\lambda_{ij}$ are set equal to unity, i.e. $\lambda_{12} = \lambda_{23} = \lambda_{21} = \lambda_{32} = 1$, which means that regime shifts are considered as risk neutral by investors in the options market. Finally, to evaluate the relative pricing performance of the model to another discrete-time option pricing model, the table reports values of the above metrics for the Heston and Nandi (2000) option pricing model, denoted $HN$. This model assumes that the underlying stock follows the risk neutral stochastic process:

$$\Delta y_{t+1} = r_t - \frac{1}{2} \sigma_{t+1}^2 + \sqrt{\sigma_{t+1}^2 z_{t+1}^Q},$$

where $z_{t+1}^Q \sim \text{NIID}(0, 1)$ is the risk-neutral process of the log-return innovation $z_{t+1}$, given as $z_{t+1}^Q = z_{t+1} + (\lambda + \frac{1}{2}) \sqrt{\sigma_{t+1}^2}$, and $\sigma_{t+1}^2$ is the volatility function, given as $\sigma_{t+1}^2 = \varpi + a (z_t^Q - \gamma \sqrt{\sigma_t^2})^2 + \beta \sigma_t^2 + \sqrt{\sigma_{t+1}^4}$. The parameter estimates used to calculate the pricing errors of this option price model are based on two different estimation procedures. The first, often followed in practice, estimates the above univariate process for $\Delta y_{t+1}$ with its volatility function, based only on stock market data. The second relies on a joint estimation of this and $HN$’s option pricing formula, based on stock and options price data (e.g. ATM options). The parameter estimates of these two procedures are given as follows:

$$\varpi = 8.3e-7, \alpha = 4.5e-005, \beta = 0.79, \gamma = 57.12 \text{ and } \lambda = 0.39 \quad \text{and}$$

$$\varpi = 2.7e-21, \alpha = 5.3e-5, \beta = 0.80, \gamma = 48.71, \lambda = 0.37 \text{ and } \sigma_{v, ATM} = 4.34,$$

respectively. These are close to each other, especially those of $\beta, \gamma$ and $\lambda$. They are also very close to those reported by Heston and Nandi (ibid).

<table>
<thead>
<tr>
<th>Table 4: Option pricing performance</th>
</tr>
</thead>
<tbody>
<tr>
<td>MRS (risk averse regime shifts)</td>
</tr>
<tr>
<td>OTM</td>
</tr>
<tr>
<td>RMSE</td>
</tr>
<tr>
<td>MAE</td>
</tr>
<tr>
<td>MRS (risk neutral regime shifts)</td>
</tr>
<tr>
<td>OTM</td>
</tr>
<tr>
<td>RMSE</td>
</tr>
<tr>
<td>MAE</td>
</tr>
<tr>
<td>$HN$ (joint estimates)</td>
</tr>
<tr>
<td>RMSE</td>
</tr>
<tr>
<td>MAE</td>
</tr>
<tr>
<td>$HN$ (univariate estimates)</td>
</tr>
<tr>
<td>RMSE</td>
</tr>
<tr>
<td>MAE</td>
</tr>
</tbody>
</table>
A number of interesting conclusions can be drawn from the results of Table 4. First, they clearly indicate that ignoring the price of regime shift of sources of risk leads to higher pricing errors. These are higher for the ATM and ITM European call options. Second, the pricing performance of the MRS model is better than that of the \( HN \) model, which can be thought of as its competent given that GARCH effects can mimic (or approximate) MRS type of shifts in volatility as mentioned before. This result holds across all different moneyness levels examined. The above results can be supported by the values of both the RMSE and MAE reported in the table.

4 Conclusions

This paper has introduced a discrete time option pricing model for European calls which assumes that the underlying asset (stock) follows a stochastic process which allows for discontinuous regime shifts in the mean or volatility of stock market returns. These shifts reflect switches among the alternative regimes of the stock market such as between the bull and bear regimes, or between the bear and crash regimes. We employ the model to assess if the risk of regime shifts is priced in the options market, as many studies in the literature make the assumption that investors are risk-neutral with regard to regime shifts and do price them explicitly.

We apply the derived option pricing formula to a set of European call option price data, and we show that regime shifts indeed constitute separate sources of risks which are priced in the market. We find that shifts from the bull to bear and from bear to crash regimes carry substantial prices of risk, a finding which is consistent with investors exhibiting significant risk aversion to adverse regime changes.

A Appendix

In this Appendix, we derive the main theoretical results of the paper. We also present an algorithm calculating the occupation time probabilities.

A.1 Derivation of the mean of the risk neutral MRS process (7)

First, assume that the random variables of (4) are defined under measure \( Q \). Then, add and subtract in this process interest rate \( r_{f}^{t} \) and the yet undetermined compensating quantity \( A \). This will yield
\[
\Delta_t y_{t+\tau} = \tau r_t + \left[ (\mu_2 - \mu_1) - \frac{1}{2} \left( \sigma_2^2 - \sigma_1^2 \right) \right] O_{t+\tau}^{2,Q} \\
- A + \sqrt{\sigma_1^2 \tau + (\sigma_2^2 - \sigma_1^2) O_{t+\tau}^{2,Q}} \omega_{t+\tau}^{Q},
\]

(16)

where
\[
\omega_{t+\tau}^{Q} = \omega_{t,\tau} + \frac{\left( \mu_1 - \frac{1}{2} \sigma_1^2 \right) \tau + \frac{1}{2} \tau \sigma_1^2 - \tau r_t}{\sqrt{\sigma_1^2 \tau + (\sigma_2^2 - \sigma_1^2) O_{t+\tau}^{2,Q}}}
\]

and \(O_{t+\tau}^{2,Q}\) denotes the occupation time variable \(O_t\) under risk neutral measure \(Q\). To derive a closed-form solution for \(A\), solve forwards equation (16) for the price \(Y_{t+\tau}\), i.e.

\[
Y_{t+\tau} = Y_t \exp \left\{ \tau r_t + \left[ (\mu_2 - \mu_1) - \frac{1}{2} \left( \sigma_2^2 - \sigma_1^2 \right) \right] O_{t+\tau}^{2,Q} - A \\
+ \sqrt{\sigma_1^2 \tau + (\sigma_2^2 - \sigma_1^2) O_{t+\tau}^{2,Q}} \omega_{t+\tau}^{Q} \right\}.
\]

Taking the conditional on \(\mathcal{I}_t\) expectation of the last relationship of \(Y_{t+\tau}\) under measure \(Q\) yields

\[
E_t^Q (Y_{t+\tau}) = Y_t e^{\tau r_t} \sum_{\zeta_2=0}^{\tau} \exp \left\{ \left[ (\mu_2 - \mu_1) - \frac{1}{2} \left( \sigma_2^2 - \sigma_1^2 \right) \right] \zeta_2 - A \\
+ \frac{1}{2} \sigma_1^2 \tau + \frac{1}{2} \left( \sigma_2^2 - \sigma_1^2 \right) \zeta_2 \right\} P \left[ O_{t+\tau}^{2,Q} = \zeta_2 | \mathcal{I}_t \right].
\]

(17)

Substituting (17) into (6) gives the following relationship for \(A\):

\[
A = \ln E_t^Q \left[ \exp \left\{ (\mu_2 - \mu_1) O_{t+\tau}^{2,Q} \right\} \right] + \frac{1}{2} \sigma_1^2 \tau.
\]

(18)

Substituting the last relationship into (16) yields the risk neutral representation of \(\Delta_t y_{t+\tau}\) given by (7). The conditional on \(\mathcal{I}_t\) mean of this process is given as

\[
E_t^Q (\Delta_t y_{t+\tau}) = \left( r_t - \frac{1}{\tau} \ln E_t^Q \left[ \exp \left\{ (\mu_2 - \mu_1) O_{t+\tau}^{2,Q} \right\} \right] - \frac{1}{2} \sigma_1^2 \right) \\
+ \left[ (\mu_2 - \mu_1) - \frac{1}{2} \left( \sigma_2^2 - \sigma_1^2 \right) \right] O_{t+\tau}^{2,Q}.
\]
Following analogous steps to the above, we can derive the risk neutral MRS representation of $\Delta_t y_{t+\tau}$ given by (12) and its mean, for the multi-regime case.

### A.2 Proof of Proposition 1

To prove this proposition, first we will derive an analytic formula of the conditional on $I_t$ probability density of log-price $y_{t+\tau}$ (or its implied log-return $\Delta_t y_{t+\tau}$) under physical and risk neutral measures, denoted as $f_y (y_{t+\tau}|I_t)$ and $f'^y (y_{t+\tau}|I_t)$, respectively. The physical density $f_y (y_{t+\tau}|I_t)$ can be easily derived from process (3) by applying Bayes’ rule. This will give

$$f_y (y_{t+\tau}|I_t) = \sum_{\zeta_2=0}^\tau f_y (y_{t+\tau}|\mathcal{H}_{t,\tau} (\zeta_2, I_t)) \Pr \left[ O^i_{\tau} = \zeta_2 | I_t \right],$$

where the information set $\mathcal{H}_{t,\tau} (\zeta_2, I_t)$, defined as

$$\mathcal{H}_{t,\tau} (\zeta_2, I_t) = I_t \cup \{ O^i_{\tau} = \zeta_2 \},$$

constitutes an extension of $I_t$ with the occupation time values $\zeta_2$, while the conditional on $\mathcal{H}_{t,\tau} (\zeta_2, I_t)$ density function $f_y (y_{t+\tau}|\mathcal{H}_{t,\tau} (\zeta_2, I_t))$ is normal given as

$$f_y (y_{t+\tau}|\mathcal{H}_{t,\tau} (\zeta_2, I_t)) = \left[ 2\pi \sigma^2 (\zeta_2) \right]^{-\frac{1}{2}} \exp \left\{ -\frac{(y_{t+\tau} - y_t - \mu (\zeta_2))^2}{2\sigma^2 (\zeta_2)} \right\}.$$ 

The mean and variance of this distribution are given as

$$\mu (\zeta_2) = \left( \mu_1 - \frac{1}{2} \sigma^2_1 \right) \tau + \left( \mu_2 - \mu_1 - \frac{1}{2} (\sigma^2_2 - \sigma^2_1) \right) \zeta_2$$

and

$$\sigma^2 (\zeta_2) = \sigma^2_1 \tau + (\sigma^2_2 - \sigma^2_1) \zeta_2,$$

respectively.\(^\text{13}\)

The risk neutral density $f'^y (y_{t+\tau}|I_t)$ has an analogous formula to its physical

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\(^{13}\)These moments can be derived immediately by noticing that the conditional on the information set $\mathcal{H}_{t,\tau} (\zeta, I_t)$ distribution of the sum of random variables $\sum_{i=1}^\tau \sigma (S_{t+i}) \epsilon_{t+i}$ is given as

$$\sum_{i=1}^\tau \sigma (S_{t+i}) \epsilon_{t+i} | \mathcal{H}_{t,\tau} (\zeta, I_t) \sim N \left( 0, \tau \sigma^2_1 + (\sigma^2_2 - \sigma^2_1) \zeta \right),$$

by $\epsilon_t \sim NIID (0, 1)$ and the assumption that $\epsilon_t$ and the vector os state variables $S_t$ are independent, for all $t$. 

30
counterpart, given by (19). It is defined as

\[
f^Q_y (y_t) = \sum_{\zeta_2=0}^{\tau} f^Q_{y_t} (y_t | \mathcal{H}_{t,\tau} (\zeta_2, \mathcal{I}_t)) \Pr [O_{\tau}^{2} Q = \zeta_2 | \mathcal{I}_t],
\]

(22)

where

\[
f^Q_{y_t} (y_t | \mathcal{H}_{t,\tau} (\zeta_2, \mathcal{I}_t)) = \frac{1}{2\pi \sigma^2 (\zeta_2)^{\frac{1}{2}}} \exp \left\{ -\frac{[y_t - y_t - \mu^Q (\zeta_2)]^2}{2\sigma^2 (\zeta_2)} \right\},
\]

\[
\mu^Q (\zeta_2) = \left( r^f_t - \frac{1}{\tau} \ln E^Q_t [\exp \{ (\mu_2 - \mu_1) \zeta_2 \}] - \frac{1}{2} \sigma^2 (\zeta_2)^2 \right) \tau + \left( (\mu_2 - \mu_1) - \frac{1}{2} (\sigma_2^2 - \sigma_1^2) \right) \zeta_2
\]

and

\[
\sigma^2 (\zeta_2) = \sigma_1^2 \tau + (\sigma_1^2 - \sigma_2^2) \zeta_2.
\]

Having derived the analytic formula of the risk neutral density \( f^Q_y (y_t | \mathcal{I}_t) \), the option pricing formula (8), given by Proposition 1, can be derived using Cox, Ross and Rubinstein’s (1979) formula. According to this, a European call option price with \( \tau \)-periods to maturity at time \( t \) and strike price \( K \) satisfies the following relationship under measure \( Q \):

\[
C_t (\tau) = e^{-r^f_t \tau} E^Q_t [Y_{t+\tau} - K]^+
= e^{-r^f_t + y_t} E^Q_t \left[ \exp \{ \Delta_t y_{t+\tau} \} - \frac{K}{Y_t} \right]^+,
\]

(23)

where \( \Delta_t y_{t+\tau} = y_{t+\tau} - y_t \) and \( [Y_{t+\tau} - K]^+ \) constitutes the payoff function of the option. Using the analytic formula of the risk neutral density function \( f^Q_y (y_t | \mathcal{I}_t) \), given by (22), the last relationship can be written as follows:

\[
C_t (\tau) = e^{-r^f_t + y_t} \int_{-\infty}^{+\infty} \left[ \exp \{ \Delta_t y_{t+\tau} \} - \frac{K}{Y_t} \right]^+ f^Q (y_{t+\tau} | \mathcal{I}_t) dy_{t+\tau}
= \sum_{\zeta_2=0}^{\tau} Y_t e^{-r^f_t} E^Q_t \left[ \exp \{ \Delta_t y_{t+\tau} \} - k_t | \mathcal{H}_{t,\tau} (\zeta_2, \mathcal{I}_t) \right]^+ \Pr [O_{\tau}^{2} Q = \zeta_2 | \mathcal{I}_t],
\]

where \( k_t = K/Y_t \) and \( E^Q_t \left[ \exp \{ \Delta_t y_{t+\tau} \} - k_t | \mathcal{H}_{t,\tau} (\zeta_2, \mathcal{I}_t) \right] \) is the expectation of con-
ditional density (21), i.e. \( \Delta_t y_{t+\tau} | \mathcal{H}_{t,\tau}(\zeta_2, \mathcal{I}_t) \sim N \left( \mu^Q(\zeta_2), \sigma^2(\zeta_2) \right) \).

Next, define the random variable \( w_t(\tau) \) as follows: \( \Delta_t y_{t+\tau} = w_t(\tau) + \mu^Q(\zeta_2) - \tau \eta^2_t + \frac{\sigma^2(\zeta_2)}{2} \), where \( w_t(\tau) \) \( | \mathcal{H}_{t,\tau}(\zeta_2, \mathcal{I}_t) \sim N \left( \tau \eta^2_t - \frac{\sigma^2(\zeta_2)}{2}, \sigma^2(\zeta_2) \right) \). Given this definition, the last relationship of the call option price \( C_t(\tau) \) can be rewritten as

\[
C_t(\tau) = \sum_{\zeta_2=0}^{\tau} \nu (\zeta_2) Y_t e^{-\tau \eta^2_t} E^Q_t \left[ \exp \{ w_t(\tau) \} \right. \left. - \frac{k_t}{\nu(\zeta_2)} \mathcal{H}_{t,\tau}(\zeta_2, \mathcal{I}_t) \right] \right] + \Pr [ O^{2,\zeta}_r = \zeta_2 | \mathcal{I}_t ],
\]

where

\[
\nu (\zeta_2) = \exp \left\{ \mu^Q(\zeta_2) - \tau \eta^2_t + \frac{\sigma^2(\zeta_2)}{2} \right\} = \frac{\exp \left\{ \left( \mu_2 - \mu_1 \right) \zeta_2 \right\}}{E^Q_t \left[ \exp \left\{ \left( \mu_2 - \mu_1 \right) \zeta_2 \right\} \right].
\]

The last relationship proves Proposition 1. Under the assumptions of MRS process (1) made in Section 2, \( Y_t e^{-\tau \eta^2_t} E^Q_t \left[ \exp \{ w_t(\tau) \} \right. \left. - \frac{k_t}{\nu(\zeta_2)} \mathcal{H}_{t,\tau}(\zeta_2, \mathcal{I}_t) \right] \right] \) constitutes the conditional on \( \zeta_2 \) Black-Scholes option pricing formula which is denoted in the proposition as as \( C_t^{B,S}(\sigma^2(\zeta_2), \nu(\zeta_2), \tau) \).

A.3 Proof of Proposition 2

To prove this proposition we will follow analogous steps to those of the proof of Proposition 1. In so doing, first note that the innovation error terms process \( \omega^Q_t \) and \( \omega_{t,\tau} \) are linked through the following relationship:

\[
\omega^Q_t = \omega_{t,\tau} + \left( \mu_1 - \frac{1}{2} \sigma_1^2 \right) \tau + \ln E^Q \left[ \exp \left\{ \sum_{i=2}^N (\mu_i - \mu_1) O^{i,Q}_t \right\} \right] + \frac{1}{2} \tau \sigma_1^2 - \tau \eta^2_t,
\]

while the risk neutral density \( f^Q_y \left( y_{t+\tau} | \mathcal{I}_t \right) \) is given as

\[
f^Q_y \left( y_{t+\tau} | \mathcal{I}_t \right) = \sum_{\zeta_N} \sum_{\zeta_2} f^Q_y \left( y_{t+\tau} | \mathcal{H}_{t,\tau}(\zeta_{N-1}, \mathcal{I}_t) \right) \Pr [ O^{2,\zeta}_r = \zeta_2, ..., O^{N,\zeta}_r = \zeta_N | \mathcal{I}_t ], \quad (24)
\]
where \( \zeta = (\zeta_2, \zeta_3, \ldots, \zeta_N) \), \( \mathcal{H}_{t,r}(\zeta, I_t) = \{ O^2_r = \zeta_2, \ldots, O^N_r = \zeta_N \} \cup I_t \) constitutes an extension of the information set \( I_t \) with the vector of occupation time values \( \zeta \) and

\[
f_y(y_{t+r}|\mathcal{H}_{t,r}(\zeta, I_t)) = (2\pi\sigma^2(\zeta))^{-\frac{1}{2}} \exp \left\{ -\frac{(\Delta(t_{t+r}) - \mu^Q(\zeta))^2}{2 \sigma^2(\zeta)} \right\},
\]

with

\[
\mu^Q(\zeta) = \left( \mathcal{F}_{1-',1'} \ln E^Q \left[ \exp \left\{ \sum_{i=2}^{N} (\mu_i - \mu_1)\zeta_i \right\} \right] - \frac{1}{2} \sigma^2(\zeta) \right) \tau + \sum_{i=2}^{N} \left( \mu_i - \mu_1 \right) - \frac{1}{2} \left( \sigma^2 - \sigma^2_1 \right) \zeta_i \]

and

\[\sigma^2(\zeta) = \sigma^2 + \sum_{i=2}^{N} \left( \sigma^2 - \sigma^2_1 \right) \zeta_i.\]

Using the analytic formula of the \( f_y^Q(y_{t+r}|I_t) \) given by (24) and the fundamental option formula (23), we can derive the option pricing formula given by Proposition 2.

### A.4 An algorithm calculating the occupation time probabilities

We will describe this algorithm for the calculation of the physical values of the occupation time probabilities \( \Pr(O^i_r) \). An analogous procedure can be followed to calculate the risk neutral values of them, denoted as \( \Pr(O^i_r^Q) \).

First, note that the joint probability of the occupation time values conditioned on a value of the initial state \( S_0 = e_j \) can be written as

\[
\Pr \left( O^1_r = \zeta_1, O^2_r = \zeta_2, \ldots, O^{N-1}_r = \zeta_{N-1}|S_0 = e_j \right) = \\
\Pr \left( O^1_r = \zeta_1|S_0 = e_j \right) \cdot \Pr \left( O^2_r = \zeta_2|O^1_r = \zeta_1, S_0 = e_j \right) \ldots \\
\ldots \Pr \left( O^{N-1}_r = \zeta_{N-1}|O^1_r = \zeta_1, O^2_r = \zeta_2, \ldots, O^{N-2}_r = \zeta_{N-2}, S_0 = e_j \right),
\]

where

\[
\Pr \left( O^2_r = \zeta_2|O^1_r = \zeta_1, S_0 = e_j \right) = \begin{cases} 
\Pr \left( O^2_{r-\zeta_1} = \zeta_2|S_0 = e_j \right) & \text{if } 0 \leq \zeta_2 \leq \tau - \zeta_1 \\
0 & \text{otherwise}
\end{cases}
\]
\[
O_{\tau}^{N-1} = \zeta_{N-1}, O_{\tau}^{1} = \zeta_1, O_{\tau}^{2} = \zeta_2, \ldots, O_{\tau}^{N-2} = \zeta_{N-2}, S_0 = e_j
\]

\[
= \begin{cases} 
\Pr \left( O_{\tau}^{N-1} = \zeta_{N-1} \mid O_{\tau}^{1} = \zeta_1, \ldots, O_{\tau}^{N-2} = \zeta_{N-2}, S_0 = e_j \right) & \text{if } 0 \leq \zeta_{N-1} \leq \tau - \sum_{i=1}^{N-2} \zeta_i \\
0 & \text{otherwise.}
\end{cases}
\]

Given the above relationships, the occupation time probabilities \( \Pr (O_i^t) \) can be computed recursively as follows.

Consider the transition probability of moving from state \( i \) to state \( j \) in the next period \( p_{ij} = \Pr (S_{t+1} = j \mid S_t = i) \) and the transition probability matrix \( P \equiv [p_{ij}]_{1 \leq i, j \leq N} \), where the \((j, i)\) element of matrix \( P \) equals the \( p_{ij} \) transition probability. Using the Chapman-Kolmogorov identity, we can compute the \( u \)-th power of the transition probability matrix \( P^u \equiv P^u \), \( (u = 1, \ldots, \tau) \), where the \((j, i)\) element of \( P^u \) is defined as \( p_{ij}(u) = \Pr (S_{t+u} = e_j \mid S_t = e_i) \).

Next, define the first passage probability as

\[
f_{ij}(t) = \Pr (Y_t = e_j, Y_{t-1} \neq e_j, \ldots, Y_1 \neq e_j \mid Y_0 = e_i), \quad i, j = 1, 2, \ldots, N; t = 1, \ldots, \tau,
\]

where the first passage probability can be computed recursively using the following equations:

\[
f_{ij} = p_{ij}(t) - \sum_{k=1}^{t-1} p_{jj}(t-k) f_{ij}(k), \quad (t = 2, \ldots, \tau)
f_{ij}(1) = \Pr (S_1 = e_j \mid S_0 = e_i) = p_{ij}.
\]

Based on the above the definition of the first passage probability, we can obtain recursive equations calculating the occupation time probability of \( i \) \((i = \{1, \ldots, N\})\) given an initial state \( S_0 = e_j \) as follows:

\[
\Pr (O_i^t = 1 \mid S_0 = e_j) = p_{ji}
\]

\[
\Pr (O_i^t = 0 \mid S_0 = e_j) = \Pr (S_t \neq e_i, S_1 \neq e_i \mid S_0 = e_j) = 1 - \sum_{u=1}^{t} f_{ji}(u), \quad (t = 1, \ldots, \tau)
\]
\[
\Pr(O^i_t = 1|S_0 = e_j) = \sum_{u=1}^{t-1} \Pr(S_u = e_i, S_{u-1} \neq e_i, ..., S_1 \neq e_i|S_0 = e_j)
\times \Pr\left(\sum_{l=u+1}^{t} e'_l S_l = 0 \bigg| S_0 = e_j\right)
+ \Pr(S_t = e_i, S_{t-1} \neq e_i, ..., S_1 \neq e_i|S_0 = e_j)
\]
\[
= \sum_{u=1}^{t-1} f_{ji}(u) \Pr(O^i_{t-u}|S_0 = e_j) + f_{ji}(t)
\]

\[
\Pr(O^i_t = \zeta_i|S_0 = e_j) = \sum_{u=1}^{t-\zeta_i+1} \Pr(S_u = e_i, S_{u-1} \neq e_i, ..., S_1 \neq e_i|S_0 = e_j)
\times \Pr\left(\sum_{l=u+1}^{t} e'_l S_l = \zeta_i - 1 \bigg| S_0 = e_j\right)
\]
\[
= \sum_{u=1}^{t-\zeta_i+1} f_{ji}(u) \Pr(O^i_{t-u} = \zeta_i - 1|S_0)
, (t = 2, ..., \tau; \ \zeta_i = 2, ..., t)
\]
References


Pliska, S.R. Introduction to mathematical finance, Blackwell.


