Pricing and hedging contingent claims using variance and higher-order moment futures

by

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Abstract

This paper suggests perfect hedging strategies of contingent claims under stochastic volatility and/or random jumps of the underlying asset price. This is done by enlarging the market with appropriate futures contracts whose payoffs depend on higher-order sample moments of the underlying asset price process. It also derives a model-free relation between these higher-order moment futures contracts and the value of a composite portfolio of European options, which can be employed to perfectly hedge variance futures contracts. Based on the theoretical results of the paper and on options and variance futures contracts price data written on the S&P 500 index, it is shown that, first, random jumps are priced in the market and, second, hedging strategies for European options employing variance and higher-order moment futures considerably improves upon the performance of traditional delta hedging strategies. This happens because these strategies account for volatility and jump risks.

Keywords: Variance futures contracts, higher-order moments, volatility and jump risks, hedging strategies.

JEL: C14, G11, G13

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1 Introduction

Since the seminal papers of Black and Scholes (1973), and Merton (1973) on pricing and hedging contingent claims, there has been a vast amount of studies trying to develop parametric option pricing models based on more general assumptions of the underlying asset price stochastic process. Prominent examples of these studies include the stochastic volatility (SV) option pricing model of Heston (1993) and its extension allowing for random jumps in the underlying asset price (denoted as SVJ), suggested by Bates (1996), as well as its various extensions (see, e.g., Eraker, Johannes and Polson (2003)). Despite the plethora of studies extending the Black-Scholes (BS) model for option pricing, there are a few studies focused on hedging derivatives under more general assumptions about the stochastic process of the underlying asset price (see, e.g., Pham (2000) for a theoretical survey and Bakshi, Cao and Chen (1997) for an empirical application).\(^1\) This happens because hedging derivatives under more general model specification assumptions is not so straightforward, compared to the BS framework. Indeed, as the market is incomplete perfect hedging against all sources of risks and, in particular, volatility and jump risks associated with a position in derivatives is not feasible.

In this paper, we extend the BS framework to provide perfect hedging strategies of contingent claims under more general assumptions of the underlying asset price allowing for stochastic volatility and/or random jumps. This is done by enlarging the market with appropriate futures contracts, whose payoffs depend on higher-order sample moments of the asset price process. These contracts can efficiently hedge against volatility and jump risks. The paper contributes into the literature on many fronts. It derives new hedging ratio formulas of the above futures contracts, the underlying asset and the zero-coupon bond of the self-financing hedging portfolio for both SV and SVJ models. For the SV model, we demonstrate that enlarging the market with a variance futures (or swap) contract, which is nowadays traded in the market, can perfectly hedge positions in contingent claims written on the underlying asset and/or its volatility. This makes the market complete. For this model, it is also shown that the price of a variance futures contract can be perfectly hedged by the value of a composite portfolio of out-of-the-money (OTM) European call and put

\(^1\)These studies develop the so-called quadratic hedging strategies which are based on risk minimization and mean-variance criteria.
options (see, e.g., Gatheral (2006a)). The paper shows how to calculate the value of this portfolio, numerically, based on a discrete set of data and it derives approximation error bounds of its value. The latter can be proved very useful in practice to assess the magnitude of the numerical errors encountered when calculating composite portfolios of OTM options.

For the SVJ model, the paper shows that, in addition to variance futures, higher than second-order moment futures contracts should be included in the self-financing portfolio in order to hedge the exposure of a contingent claim price against random jumps. If this claim is a variance futures contract, this can be done together with the composite portfolio of OTM options. Since the size of jumps is random, a sufficiently large number of higher-order moment futures contracts is required for perfect hedging. As this number goes to infinity, the paper shows that the value of the self-financing hedging portfolio converges to the price of the contingent claim, thus making the market approximately (or quasi) complete in the sense of Björk, Kabanov and Runggaldier (1997) and Jarrow and Madan (1999). In practice, someone can approximate the value of this hedging portfolio with a finite number of higher-order moment futures contracts. The above results of the paper can find applications to different markets. These include the stock market (see Bakshi and Kapadia (2003), and Bollen and Whaley (2004)), the fixed-income security market (see Li and Zhao (2006) and Jarrow, Li and Zhao (2007)), the mortgage-backed security market (see Boudoukh, Whitelaw, Richardson and Stanton (1997)) and the credit default swap market (see Brigo and El-Bachir (2010)).

The results of the paper are used to empirically address two questions, which have important portfolio management implications. The first is if random jumps are priced in the market. According to theory, the price of jump risk must be reflected in the difference between the price of a variance futures contract and the value of the composite options portfolio. This will be examined without relying on any parametric model of the market. The second question is if a position in a European call, or put, can be efficiently hedged based on a two-instruments hedging strategy, which considers a variance futures contract as a hedging vehicle, or it requires hedging strategies which rely on higher-order moment futures. This can shed light on the ability of variance futures contracts to hedge against volatility risk and the number of higher-order moment futures contracts needed for approximately completing the market in the presence of random jumps. To answer the above questions, we rely on options and variance futures price data written on the S&P 500 index.
The empirical results of the paper lead to the following main conclusions. First, random jumps are indeed priced in the variance futures market. The prices of these contracts are found to be significantly bigger in magnitude than those of the composite options portfolio, which spans variance futures contracts under the assumption of no jumps. Second, a two-instruments hedging strategy, which also includes a variance futures contracts in the self-financing portfolio, is found to considerably improve upon the performance of the traditional delta hedging strategy often used in practice, which only includes the underlying asset and the zero-coupon bond in this portfolio. The improving performance of this strategy comes from the fact that variance futures contracts can efficiently hedge the exposure of a call, or a put, option to volatility risk. The inclusion of a finite (up to the 4th-order) number of higher-order moment futures contracts into the self-financing portfolio is found to further improve the performance of the above two-instruments hedging strategy, especially for short-term at-the-money (ATM) calls and OTM puts which are more sensitive to jump risk.

The paper is organized as follows. Section 2 introduces the variance and higher-order moment futures contracts and gives the relation between them, and the value of the composite portfolio of OTM options. Section 3 shows how these derivatives can be used to perfectly replicate the price of a contingent claim under the SV and SVJ models. Section 4 addresses the empirical questions of the paper. Section 5 concludes the paper. All derivations are given in a technical Appendix.

2 Variance and higher-order moment futures contracts

2.1 Variance futures contracts

Variance futures (or swaps) are derivative contracts in which one counterparty agrees to pay the other a notional amount times the difference between a fixed level and a realized level of variance. The fixed level is the variance futures price. Realized variance is determined by non-central second-order sample moment of the underlying asset over the life of the contract. More precisely, let $T - \tilde{\tau} = t_0 < t_1 < ... < t_n = T$ be a partition of time interval $[T - \tilde{\tau}, T]$ into $n$ equal segments of length $\Delta t$, where $\tilde{\tau}$ is accrual period of the contract, given as $n/252$. Then, the payoff of this variance futures contract is given as

$$V_{T-\tilde{\tau},T}^{(n)} = \frac{252}{n} \sum_{i=1}^{n} \ln \left( \frac{S_i}{S_{i-1}} \right)^2,$$
where \( S_t \) is the price of the underlying asset at time \( t_i \). Since this contract worths zero on the inception date, no arbitrage dictates that the time-\( t \) price of the above variance futures contract is given by the risk-neutral expected value of the contract payoff, i.e.,

\[
FV_t^{(n)} = E_t^Q \left[ V_{T-\tilde{T},T}^{(n)} \right],
\]

where \( Q \) denotes the risk-neutral measure. If time is continuous, i.e. \( n \to \infty \), payoff \( V_{T-\tilde{T},T}^{(n)} \) converges in probability to the annualized quadratic variation of the log-price process of the underlying asset, denoted as \( \frac{1}{T} \langle X, X \rangle_{T-\tilde{T},T} \) (see Protter (1990)), where \( X = \ln S \). Then, the variance futures price \( FV_t^{(n)} \) converges to its continuous-time limit, defined as \( FV_t \), given by the risk-neutral expected value of \( \frac{1}{T} \langle X, X \rangle_{T-\tilde{T},T} \), i.e.,

\[
\lim_{n \to \infty} FV_t^{(n)} = E_t^Q \left[ \frac{1}{T} \langle X, X \rangle_{T-\tilde{T},T} \right] \equiv FV_t,
\]

(1)

see Jarrow et al (2011).\(^2\)

Under the assumption that log-price \( X \) has no discontinuous component, variance futures contracts can be priced and perfectly hedged based on a portfolio of European call and put contracts (see, e.g. Carr and Lee (2009), Friz and Gatheral (2005) and Gatheral (2006)). However, this is not true if \( X \) contains a discontinuous component, which is found to characterize the price dynamics of many financial assets (see, e.g., Barndorff-Nielsen and Shepard (2006b)). To construct such a portfolio, next proposition gives the price of variance futures contracts \( FV_t \) under more general assumptions of the underlying asset price allowing for a discontinuous component in log-price \( X \). The proposition gives exact formulas of price \( FV_t \) for the following two cases: (i) \( T - \tilde{T} \leq t \leq T \), and (ii) \( t < T - \tilde{T} \). The second case is not considered in the literature. It assumes that time-\( t \) is prior to the starting date of the accrual period.

**Proposition 1** Let the stochastic process of the underlying asset log-price \( (X_u)_{u \in [0,T]} \) be a

\(^2\)As aptly shown by Broadie and Jain (2008), the effects of assuming continuous time on pricing variance futures contracts is negligible. This allows us to rely on continuous-time models in pricing and hedging contingent claims.
semimartingale. Then, the price of variance futures contract \( FV_t \) is given as

\[
(i) \quad FV_t = \frac{1}{\tau} \langle X, X \rangle_{T-\tau, t} + \frac{2e^{\tau T}}{\tau} \left[ \int_{S_t}^{+\infty} \frac{1}{K^2} C_t(\tau, K) dK + \int_0^{S_t} \frac{1}{K^2} P_t(\tau, K) dK \right] + \frac{2}{\tau} E_t^Q \left[ \int T dS_u \frac{1}{S_u-} - \frac{S_T - S_t}{S_t} \right] - \frac{2}{\tau} \sum_{j=3}^{\infty} \frac{1}{j!} E_t^Q \left[ \sum_{t<u\leq T} (\Delta X_u)^j \right]
\]

for \( T - \tau \leq t \leq T \) and

\[
(ii) \quad FV_t = \frac{2e^{\tau T}}{\tau} \left[ \int_{S_t}^{+\infty} \frac{1}{K^2} C_t(\tau_1, K) dK + \int_0^{S_t} \frac{1}{K^2} P_t(\tau_1, K) dK \right] - \frac{2e^{\tau T}}{\tau} \left[ \int_{S_t}^{+\infty} \frac{1}{K^2} C_t(\tau, K) dK + \int_0^{S_t} \frac{1}{K^2} P_t(\tau, K) dK \right] + \frac{2}{\tau} E_t^Q \left[ \int_{T-\tau}^T dS_u \frac{1}{S_u-} - \frac{S_T - S_T - \tau}{S_T} \right] - \frac{2}{\tau} \sum_{j=3}^{\infty} \frac{1}{j!} E_t^Q \left[ \sum_{T-\tau<u\leq T} (\Delta X_u)^j \right]
\]

for \( t < T - \tau \), where \( \tau = T - t \) and \( \tau_1 = T - \tau - t \), and \( C_t(\tau, K) \) and \( P_t(\tau, K) \) respectively denote the prices of European call and put options at time \( t \), with strike price \( K \) and maturity interval \( \tau \).

Both formulas (2) and (3), given by Proposition 1, indicate that variance futures price \( FV_t \) is determined by similar terms. Their interpretation is as follows. Consider, for instance, formula (2). The integral terms, defined as

\[
V_{o,t} = \frac{2e^{\tau T}}{\tau} \left[ \int_{S_t}^{+\infty} \frac{1}{K^2} C_t(\tau, K) dK + \int_0^{S_t} \frac{1}{K^2} P_t(\tau, K) dK \right], \quad \text{(4)}
\]

give the value of a composite portfolio of options, \( V_{o,t} \), which depends on the values of two subportfolios of OTM European call and put options with expiration date \( T \). Term \( E_t^Q \left[ \int_{t+}^T dS_u \frac{1}{S_u-} - \frac{S_T - S_t}{S_t} \right] \) of formula (2) depends on the underlying asset of the variance futures contract. If this asset is a stock (or a stock market index) that pays a constant dividend yield \( \delta \), then this term becomes

\[
E_t^Q \left[ \int_{t+}^T dS_u \frac{1}{S_u-} - \frac{S_T - S_t}{S_t} \right] = 1 + (r - \delta)\tau - e^{(r-\delta)\tau}.
\]

If the underlying asset is a futures or swap contract, this term becomes zero. Finally, term \( \sum_{j=3}^{\infty} \frac{1}{j!} E_t^Q \left[ \sum_{t<u\leq T} (\Delta X_u)^j \right] \) of formula (2) is due to the discontinuous component of process \( X \).
The decomposition of price \( FV_t \) into the three terms discussed above provides the correct framework to replicate and, thus, hedge variance futures contracts. For instance, under the assumption that the log-price process \( X \) does not contain a discontinuous component, formula (2) implies that a long position in a variance futures contract taken at time \( t \), where \( T - \tilde{T} \leq t \leq T \), can be perfectly replicated by a portfolio constructed by holding: (a) a static long position in \( \frac{2}{\pi K \tau} \) calls at strikes \( K > S_t \) and \( \frac{2}{\pi K \tau} \) puts at strikes \( K < S_t \) with time to maturity \( \tau \), (b) a dynamic position in \( \frac{2}{\pi K} \left( \frac{1}{S_u} - \frac{1}{S_t} \right) \) shares at any time \( u \in [t, T] \), and (c) \( e^{-rt} \frac{1}{\tau} \langle X, X \rangle_{T-\tilde{T}, \tau} \) in cash (see, e.g., Carr and Lee (2009)). In case (ii) of Proposition 1, where \( t < T - \tilde{T} \), the replicating portfolio should include: (a) a static long position in \( \frac{2}{\pi K \tau} \) calls at strikes \( K > S_t \) and \( \frac{2}{\pi K \tau} \) puts at strikes \( K < S_t \) with time to maturity \( \tau \), (b) a static short position in \( \frac{2}{\pi K \tau} \) calls at strikes \( K > S_t \) and \( \frac{2}{\pi K \tau} \) puts at strikes \( K < S_t \) with time to maturity \( \tau_1 \), and (c) a dynamic position in \( \frac{2}{\pi \tau} \left( \frac{1}{S_u} - \frac{1}{S_t} \right) \) shares at any time \( u \in [T - \tilde{T}, T] \).

Proposition 1 clearly indicates that the above portfolios can not perfectly replicate a long position in variance futures if process \( X \) contains a discontinuous component, due to the sum of higher-order moment terms \( \sum_{j=3}^{\infty} \frac{1}{j!} E_t^Q \left[ \sum_{t<u<T} (\Delta X_u)^j \right] \) for case (i), or \( \sum_{j=3}^{\infty} \frac{1}{j!} E_t^Q \left[ \sum_{T-\tilde{T}<u<T} (\Delta X_u)^j \right] \) for case (ii). Ignoring these terms will lead to an imperfect hedge of variance futures contract by means of traded option contracts. The magnitude of the hedging errors encountered will be investigated in Section 4, based on market data. To replicate these terms, in the next section we will introduce futures contracts whose payoffs are defined by higher than second-order sample moments of \( X \).

2.2 Higher-order moment futures contracts

Let define the \( j \)th-order moment futures contract, for \( j \geq 3 \), as a contract in which one counterparty agrees to pay the other a notional amount times the difference between a fixed level (contract’s price) and a realized level of the \( j \)th-order non-central sample moment of process \( X \). The payoff of this contract is given as

\[
V^{(n)}_{(j), (T-\tilde{T}, T)} = \frac{252}{n} \sum_{i=1}^{n} \ln \left( \frac{S_i}{S_{i-1}} \right)^j, \text{ for } j \geq 3,
\]

where \( n \) are segments of time interval \([T - \tilde{T}, T]\) of length \( \Delta t \). The price of this contract under the no arbitrage principle is given by the risk-neutral expected value of \( V^{(n)}_{(j), (T-\tilde{T}, T)} \).
i.e. 

\[ F^{(n)}_{(j),t} = E_t^Q \left[ V^{(n)}_{(j), (T-\tau, T)} \right]. \]

In continuous time (i.e. \( n \to \infty \)), it can be shown that payoff \( V^{(n)}_{(j), (T-\tau, T)} \) converges in probability to \( \frac{1}{\tau} \sum_{T-\tau < u \leq T} (\Delta X_u)^j \) (see Lepingle (1976), for a proof). Thus, the continuous-time value of the above contract, denoted as \( F_{(j),t} \), can be defined as

\[ \lim_{n \to \infty} F^{(n)}_{(j),t} = E_t^Q \left[ \frac{1}{\tau} \sum_{T-\tau < u \leq T} (\Delta X_u)^j \right] \equiv F_{(j),t}. \]

(5)

This definition of \( F_{(j),t} \) implies that the prices of odd-order moment futures will be negative, if large drops in \( X \) occurring between \( T - \tau \) and \( T \) dominate its movements. In this case, positions in these futures contracts should be reversed. That is, a long position in a higher-order moment futures will be taken as a short position, and vice versa.

If we assume that a finite number of \( j = 3, 4, \ldots, N \) higher-order moment futures are traded in the market, then formulas (2) and (3) of Proposition 1 imply that long positions in variance futures contracts can be hedged against discontinuous movements in \( X \), captured by term \( \sum_{j=3}^{\infty} \frac{1}{\tau} E_t^Q \left[ \sum_{T-\tau < u \leq T} (\Delta X_u)^j \right] \). This will happen, if the replicating portfolios, defined in the previous section for cases (i) and (ii) of Proposition 1, include also a short position in \( \frac{2}{\tau/j} j \)th-order moment futures contracts, for \( j = 3, 4, \ldots, N \). For sufficiently large \( N \), it can be easily proved that these replicating portfolios can approximately hedge long positions in variance futures contracts, even when the underlying asset price contains discontinuous components. These results mean that existence of higher-order moment futures can make the market approximately (or quasi) complete (see Björk, Kabanov and Runggaldier (1997), or Jarrow and Madan (1999)).

Note, at this point, that the above higher-order moment futures contracts can not capture the variation of the discontinuous component of \( X \), defined as \( \frac{1}{\tau} \sum_{T-\tau < u \leq T} (\Delta X_u)^2 \). This variation is part of the annualized quadratic variation of \( X \), \( \frac{1}{\tau} \langle X, X \rangle_{T-\tau, T} \), since we have

\[ \frac{1}{\tau} \langle X, X \rangle_{T-\tau, T} = \frac{1}{\tau} \langle X, X \rangle^{c}_{T-\tau, T} + \frac{1}{\tau} \sum_{T-\tau < u \leq T} (\Delta X_u)^2, \]

(6)

Note also that Proposition 1 implies that the \( j \)th-order moment futures contract can be hedged by means of European calls and puts, the variance futures contract and a finite number of \( s \)th-order moment futures contracts, for \( s \neq j \). This can be done by solving (2) or (3) for \( F_{(j),t} \), using the definition of \( F_{(j),t} \) given by (5).
where \( \langle X, X \rangle^c \) denotes the continuous part of \( \langle X, X \rangle \). Since

\[
\frac{1}{\sqrt{2/\pi}} \sum_{i=1}^{n-1} \left| \ln \left( \frac{S_i}{S_{i-1}} \right) \right| \ln \left( \frac{S_{i+1}}{S_i} \right) \xrightarrow{p} \frac{1}{\tau} \langle X, X \rangle^c_{T-\tau, T},
\]

where "\( \xrightarrow{p} \)" signifies convergence in probability (see Barndorff-Nielsen and Shephard (2003, 2004)), and, hence,

\[
\frac{252}{n} \left( \sum_{i=1}^{n} \ln \left( \frac{S_i}{S_{i-1}} \right)^2 - \frac{1}{\sqrt{2/\pi}} \sum_{i=1}^{n-1} \left| \ln \left( \frac{S_i}{S_{i-1}} \right) \right| \ln \left( \frac{S_{i+1}}{S_i} \right) \right) \xrightarrow{p} \frac{1}{\tau} \sum_{T-\tau < u \leq T} (\Delta X_u)^2,
\]

one can introduce a new futures contract to price the discontinuous component of \( \langle X, X \rangle \). This contract will be henceforth referred to as bipower variation futures contract and its discrete-time payoff at time \( T \) is given as

\[
\frac{252}{n} \left( \sum_{i=1}^{n} \ln \left( \frac{S_i}{S_{i-1}} \right)^2 - \frac{1}{\sqrt{2/\pi}} \sum_{i=1}^{n-1} \left| \ln \left( \frac{S_i}{S_{i-1}} \right) \right| \ln \left( \frac{S_{i+1}}{S_i} \right) \right).
\]

As can be seen in Section 3, this contract can help us to separate the volatility from the jump risk both of them encountered when investing in variance futures contracts. Furthermore, it is necessary to perfectly hedge contingent claims, like European options and options written on volatility swaps, against the above two sources of risk. Under the risk-neutral measure \( Q \), the time-\( t \) price of a bipower variations futures contract is given as\(^4\)

\[
F_{(2), t} = E_t^Q \left[ \frac{1}{\tau} \sum_{T-\tau < u \leq T} (\Delta X_u)^2 \right]. \quad (7)
\]

2.3 Numerical errors in calculating the value of the composite options portfolio

Proposition 1 indicates that, in pricing and hedging variance futures contracts, we need to calculate value \( V_{o,t} \) of the composite portfolio of OTM European calls and puts. This is

\[^4\text{Note that subtracting } F_{(2), t} \text{ from } F_{V_t} \text{ yields}
\]

\[
F_{V_t} - F_{(2), t} = E_t^Q \left[ \frac{1}{\tau} \langle X, X \rangle^c_{T-\tau, T} \right],
\]

which shows that a long position in a variance futures contract and a short position in a bipower variation futures contract forms a portfolio whose payoff depends only on the continuous part of quadratic variation \( \langle X, X \rangle^c \). The price of this portfolio can be thus used to determine the market price of volatility risk.
given by formula (4), for the case (i) of Proposition 1.\footnote{Note that the analysis of this section applies also to case (ii) of Proposition 1.} To this end, we need to rely on an efficient numerical method which fits the integral functions of (4) into finite, discrete cross-sectional sets of OTM European call and put option prices. Following recent literature, a method which can be used to this end employs an interpolation-extrapolation scheme of the implied by the BS volatilities of the above option prices.\footnote{See Dennis and Mayhew (2002) and Jiang and Tian (2005), inter alia. Note that interpolation and extrapolation of implied volatilities, instead of option prices, has been suggested in the literature in order to avoid numerical difficulties in fitting smooth functions into option prices. To convey the observed option prices into implied volatilities and vice versa, we use the BS formula. This methodology does not require the BS model to be the true option pricing model.} This scheme interpolates these volatilities over the observed closed interval of strike prices $[K_{\min}, K_{\max}]$, where $K_{\min}$ and $K_{\max}$ denote the minimum and maximum strike prices available in the options market, respectively, and it extrapolates them over intervals $(0, K_{\min}]$ and $[K_{\max}, +\infty)$, where option prices are not available. The interpolation of the above implied volatilities can be based on cubic splines. These constitute smooth functions which can provide accurate estimates of implied volatilities over interval $[K_{\min}, K_{\max}]$.

On the other hand, the extrapolation of implied volatilities over intervals $(0, K_{\min}]$ and $[K_{\max}, +\infty)$ can be done either based on a linear, or constant, function. Both of these functions are truncated at strike prices $K_0$, which is considered as an approximation of a zero strike price, and $K_\infty$, which is considered as an approximation of a strike price which tends to infinity. The strike prices $K_0$ and $K_\infty$ are calculated to correspond to OTM put and call option prices, respectively, which are very close to zero, e.g., smaller than $10^{-3}$. An alternative to the above extrapolation schemes is to set $K_0=K_{\min}$ and $K_\infty=K_{\max}$, which means to rely only on available option data. That is, to choose not to extrapolate. This can be justified economically by the fact that investing in the composite portfolio of OTM options in order to hedge variance futures does not imply trading on artificial options, but only market available options defined over strike price interval $[K_{\min}, K_{\max}]$. According to Proposition 1, the value $V_{o,t}$ of such a portfolio should be also reflected in variance futures prices.

The numerical interpolation-extrapolation procedures suggested above can lead to an approximation error related to the curve-fitting scheme of implied volatilities over interval $[K_0, K_\infty]$. This error depends on the number of option prices available, the length of interval $[K_{\min}, K_{\max}]$ and the extrapolation scheme chosen. Given that this error can not
be measured because the true theoretical implied volatility function is not known, the next proposition enables us to appraise its size effects on $V_{o,t}$, by deriving upper and lower bounds for it. This can shed some light on the importance of this numerical error on possible biases of $V_{o,t}$ in replicating market prices of variance futures contracts, especially relative to that due to the discontinuous component of the log-price process $X$.

**Proposition 2** Let $\tilde{C}_t(\tau, y)$ and $\tilde{P}_t(\tau, y)$ denote the approximated values of the true call and put prices $C_t(\tau, y)$ and $P_t(\tau, y)$, with respect to variable $y = \ln(K/S_t)$, respectively, based on an interpolation-extrapolation scheme. If $\tilde{V}_{o,t}$ denotes the approximated value of the composite portfolio of options $V_{o,t}$, given by formula (4), based on $\tilde{C}_t(\tau, y)$ and $\tilde{P}_t(\tau, y)$, then we can derive the following approximation error bound of $\tilde{V}_{o,t}$:

$$
\left| \tilde{V}_{o,t} - V_{o,t} \right| \leq \frac{2e^{\tau y}}{S_t} \left[ C_{\text{error}} \left( 1 - e^{-y\infty} \right) + P_{\text{error}} \left( e^{-y_0} - 1 \right) + \varepsilon \right],
$$

(8)

where $C_{\text{error}} = \max_{y \in (0, y_\infty)} \left| C_t(\tau, y) - \tilde{C}_t(\tau, y) \right|$ and $P_{\text{error}} = \max_{y \in (y_0, 0)} \left| P_t(\tau, y) - \tilde{P}_t(\tau, y) \right|$ constitute upper bounds of the approximation errors of the European call and put option pricing function, respectively, $y_0 = \ln(K_0/S_t)$, $y_\infty = \ln(K_\infty/S_t)$, and $\varepsilon$ is a truncation error of strike prices $K$ at endpoints $K_\infty$ and $K_0$, respectively, defined as

$$
\varepsilon = \left| \int_{K_\infty}^{+\infty} \frac{1}{K^2} C_t(\tau, K)dK + \int_{0}^{K_0} \frac{1}{K^2} P_t(\tau, K)dK \right|.
$$

The approximation error bound formula (8), given by Proposition 2, can be used as a metric of choosing between alternative interpolation-extrapolation schemes to estimate $V_{o,t}$ (see our empirical Section 4). As shown by (8), the upper and lower bounds of approximation error $\tilde{V}_{o,t} - V_{o,t}$ depends critically on the call and put option pricing function approximation errors $C_{\text{error}}$ and $P_{\text{error}}$. These are multiplied by functions $(1 - e^{-y})$ and $(e^{-y} - 1)$, respectively. To investigate more rigorously the effects of these two functions on error bound $\left| \tilde{V}_{o,t} - V_{o,t} \right|$, in the next corollary we analyze some of their properties, while in Figure 1 we plot their values over $y$. Note that this figure gives the values of function

$$
f(y) = \begin{cases} 
e^{-y} - 1 & \text{if } y < 0 \\ 1 - e^{-y} & \text{if } y \geq 0, \end{cases}
$$

(9)

which nests the two functions $(1 - e^{-y})$ and $(e^{-y} - 1)$. 

11
Corollary 1  Taking the limit $y \to +\infty$ or $y \to -\infty$, we have

$$\lim_{y \to +\infty} (1 - e^{-y}) = 1 \quad \text{and} \quad \lim_{y \to -\infty} (e^{-y} - 1) = +\infty,$$

(10) respectively. We can also prove that

$$(1 - e^{-y}) \leq 1 \quad \forall \ y \in \mathbb{R}_+.$$  

(11)

The results of the above corollary indicate that function $1 - e^{-y}$, which multiplies $C_{error}$, is bounded by 1. This means the size effects of this term on approximation error $|\hat{V}_{o,t} - V_{o,t}|$ is bounded. This can be confirmed by the graph of Figure 1. On the other hand, Corollary 1 indicates that function $e^{-y} - 1$, which multiplies $P_{error}$, is not bounded. This function tends to infinity as $y \to -\infty$, which means that the size effects of approximation error $P_{error}$ on $|\hat{V}_{o,t} - V_{o,t}|$ can be unbounded. This result implies that, in implementing formula (4) to a discrete set of option prices, we should avoid extrapolating implied volatilities up to extreme values of $K_0$, or we should not extrapolate implied volatility functions at all. Otherwise, approximation error $P_{error}$ may be considerably magnified.

3  Pricing and hedging contingent claims using variance and higher-order moment futures contracts

In the previous section, we show how to price and perfectly hedge a variance future contracts in the presence of a discontinuous component in the underlying log-price process $X$. In this section, we generalize these results to the case of any contingent claim written on the price of a stock and/or its volatility. These contingent claims include plain vanilla European options, volatility swaps or options written on these swaps. The latter constitute more complex derivatives, which have non-linear payoffs with respect to volatility (see Carr and Lee (2009) and Friz and Gatheral (2005)). Most of these contingent claims are traded in the market. To price and hedge them, we will initially assume that stock prices follow the stochastic volatility (SV) model (see, e.g., Heston (1993)). Then, our analysis will be extended to the SV model allowing for jumps (SVJ) (see, e.g., Bates (1996)). This model has been found to better describe the dynamics of stock prices or their volatility and, thus, is considered as a common specification in the literature. Our analysis starts with the SV
model, as this model enables us to more clearly see the role of variance future contracts to perfectly hedge positions in contingent claims. Next, we demonstrate that in order to complete the market for the SVJ model we need to enlarge it with the bipower variation and higher-order moment futures contracts.

3.1 The SV model

Assume that the underlying asset is a stock which pays dividends at rate \( \delta \). Then, the SV model assumes that its price \( S \) and volatility \( V \) obey the following processes:

\[
\frac{dS_t}{S_t} = \mu_t^S dt + \sqrt{V_t} dW_t^{(1)} \\
\frac{dV_t}{V_t} = \kappa (\theta - V_t) dt + \sigma \sqrt{V_t} dW_t^{(2)},
\]

where \( \mu_t^S = (r_t - \delta_t + \gamma_S V_t) \) is the expected return of the underlying asset, where \( r_t \) is the rate of return of a zero-coupon bond and \( \gamma_S V_t \) is the market price of risk, and processes \( W^{(1)} \) and \( W^{(2)} \) are two correlated Brownian motions, with correlation coefficient \( \rho \).\(^7\) The last assumption implies that there exists a Brownian motion process \( W^{(3)} \) which is independent of \( W^{(1)} \) and satisfies the following relationship: \( dW_t^{(2)} = \rho dW_t^{(1)} + \sqrt{1 - \rho^2} dW_t^{(3)} \). Under the SV model, the stochastic discount factor (SDF) process \( \Lambda_t \), used for asset pricing, is given as

\[
\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \gamma_S \sqrt{V_t} dW_t^{(1)} - \frac{\gamma_V \sqrt{V_t}}{\sigma \sqrt{1 - \rho^2}} dW_t^{(3)},
\]

where parameter \( \gamma_V \) accounts for the market price of volatility risk (see Chernov and Ghysels (2000)).

This market is incomplete in the sense that a position in a contingent claim cannot be perfectly replicated by a self-financing portfolio, referred to as hedging portfolio, consisting of a position in the underlying stock and the zero-coupon bond. In this section, we will show that this market can be completed if the hedging portfolio contains, in addition to the above two assets, a position in a variance futures contract. To this end, our analysis starts with pricing variance futures contracts.

\(^7\) Note that our results do not depend on the specification of the volatility process \( V \). They hold under the following more general specification of \( V \):

\[
dV_t = \alpha(t, V_t) dt + \beta(t, V_t) dW_t^{(2)},
\]

where \( \alpha \) and \( \beta \) are appropriately defined functions.
3.1.1 Pricing variance futures contracts

Consider that, apart from the stock and zero-coupon bond, a variance futures contract is traded in the market. This is written on stock price \( S \) and is defined over time interval \([0, T]\). The payoff of this contract at time \( T \) is given as \( \frac{1}{T} \int_0^T V_u du \). The following proposition gives analytic formulas of the price of this contract and its expected return, \( E_t \left[ \frac{dFV_t}{FV_t} \right] \), under the SV model.

**Proposition 3** Assume that no-arbitrage opportunities exist in the market. Then, under the SV model, the price of a variance futures contract at time \( t \in [0, T] \) is given as

\[
FV_t = \frac{1}{T} \left( \int_0^t V_u du + F(t, V_t) \right),
\]

where \( F(t, V_t) = \tau \left( \psi_t V_t + (1 - \psi_t) \theta^Q \right) \) with \( \psi_t = (1 - e^{-\kappa^Q \tau}) / (\kappa^Q \tau) \) and \( \tau = T - t \), while its expected return, denoted \( \mu^FV_t \), is given as

\[
\mu^FV_t dt \equiv E_t \left[ \frac{dFV_t}{FV_t} \right] = \frac{\partial \ln FV}{\partial \ln V} \gamma_V dt,
\]

where \( \theta^Q \) and \( \kappa^Q \) constitute risk-neutral counterparts of \( \theta \) and \( \kappa \), respectively, and \( \gamma_V = \sigma \rho \gamma_S + \gamma_V \).

Proposition 3 shows that the expected return of a variance futures contract \( \mu^FV_t \) depends on the market price of risk \( \gamma_S \), due to the correlation between \( S \) and \( V \), and the price of volatility risk \( \gamma_V \). If \( \gamma_V < 0 \), as shown in many empirical studies (see, e.g., Carr and Wu (2010)),\(^8\) then \( \mu^FV_t \) will be negative, given that \( \gamma_S > 0, \rho < 0 \) (due to the leverage effect) and \( \partial \ln FV / \partial \ln V > 0 \). The negative expected return of this contract is something to expect, since it pays when volatility unexpectedly increases. Due to aversion to this event, investors wish to pay more to acquire variance futures contracts. Another interesting result of Proposition 3 is that, under the SV model, the price of volatility risk \( \gamma_V \) can be uniquely determined by expected returns \( \mu^FV_t \) and \( \mu^S_t \), since \( \gamma_V = \gamma_S - \sigma \rho \gamma_S \). This result implies

\(^8\)The negative sign of \( \gamma_V \) can be also explained theoretically based on Lucas’ consumption asset pricing model in which the representative investor has a coefficient of risk aversion \( \gamma \). Under these assumptions, discount factor \( \Lambda_t \) becomes \( \Lambda_t = C_t^{-\gamma} \), where \( C_t \) denotes consumption at time \( t \). Based on this model, it can be shown that \( -\gamma E_t \left[ \frac{dC_t}{C_t} dV_t \right] = E_t \left[ \frac{d\Lambda_t}{\Lambda_t} dV_t \right] = -\gamma_V V_t dt \). The last relationship implies that, if \( E_t \left[ \frac{d\Lambda_t}{\Lambda_t} dV_t \right] > 0 \), meaning that the representative investor substitutes future with current consumption at a greater rate when volatility tend to increase, then \( \gamma_V < 0 \).
that a position in any contingent claim can be perfectly replicated by a portfolio consisting of a number of stocks, bonds and a position in variance futures contract. This will be shown more rigorously in the next section.

3.1.2 Pricing and hedging contingent claims

Consider now a contingent claim $C$. Its price at time $t$, denoted $C_t$, depends on both the stock price and the spot variance, i.e. $C_t = C(t, S_t, V_t)$, where function $C(\cdot)$ has continuous second order partial derivatives. As noted before, this claim can be a plain vanilla European option, a volatility swap or an option written on this swap. The self-financing portfolio needed to hedge (replicate) a position in this claim consists of a position in the underlying stock, the zero-coupon bond and a variance futures contract. The numbers of these hedging instruments at time $t \in [0, T]$, known as "deltas", will be defined by the following vector of real-value adapted processes $\phi_t = (\phi^S_t, \phi^B_t, \phi^{FV}_t)'$. At time $t$, the value of this portfolio will be equal to the price of the contingent claim $C_t$, i.e.

$$C_t = \phi^S_t S_t + \phi^B_t B_t,$$  \hfill (17)

where $B$ denotes the price of zero-coupon bond. Note that the number of variance futures contracts $\phi^{FV}_t$ is omitted from equation (17), since it costs nothing to take a position in them.

Next proposition derives the values of the vector of deltas $\phi_t$. These can be employed to perfectly hedge changes in contingent claim price $C_t$ driven by stochastic movements in stock price $S$ and volatility $V$. The proposition also derives the instantaneous expected return of contingent claim $C$, defined as $E_t \left[ \frac{dC_t}{C_t} \right]$.

**Proposition 4** Consider a contingent claim $C$ written on the underlying stock and/or its volatility, with price function $C_t = C(t, S_t, V_t)$, where $C(\cdot)$ is a continuous function which has second order partial derivatives. Then, under the SV model and the no-arbitrage principle, the deltas of the self-financing portfolio replicating contingent claim price $C_t$, at $t \in [0, T]$, are given as follows:

$$\phi^S_t = \partial C / \partial S \quad \text{and} \quad \phi^B_t = B_t^{-1} \left( C_t - \phi^S_t S_t \right)$$
for the underlying stock and zero-coupon bond, respectively, and

$$\phi_t^{FV} = \partial C / \partial FV$$

for the variance futures contract. The instantaneous expected return of $C$ is given as

$$E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln S} (\mu_t^S + \delta_t - r_t) dt + \frac{\partial \ln C}{\partial \ln FV} \mu_t^{FV} dt,$$  \hspace{1cm} (18)

where $\mu_t^S$ and $\mu_t^{FV}$ are the expected returns of the underlying stock and variance futures contract, respectively, defined in Proposition 3.

Proposition 4 proves that, under the SV model, a contingent claim $C$ written on a stock and/or its volatility can be perfectly hedged by a self-financing portfolio consisting of a zero-coupon bond, the underlying stock and a variance futures contract. The price of this contingent claim, $C_t$, is uniquely determined by the prices (or expected returns) of these three financial instruments. As shown in the appendix, this price can be obtained by solving Heston’s (1993) partial differential equation (PDE). In contrast to Heston’s approach, which obtains this PDE relying on equilibrium approach, we have derived it explicitly by eliminating all stochastic terms determining the contingent claim price. These results mean that, under the SV model, the market enlarged with a variance futures contract becomes complete. As mentioned before, contingent claim $C$ can also include volatility swaps and options written on them. The results of Proposition 4 imply that these assets can be perfectly hedged by variance futures contracts, whose price equals the value of the composite portfolio of OTM options $V_{\alpha,t}$, under the SV model (see Proposition 1). Thus, these more complex derivatives, with non-linear payoffs, can be perfectly hedged with respect to volatility by a portfolio of OTM options.

Formula (18) of Proposition 4 indicates that the expected return of contingent claim $C$, $E_t \left[ \frac{dC_t}{C_t} \right]$, in addition to the risk-free rate $r$ and expected return of the stock (implied by the BS model), also depends on the volatility risk premium. The latter is reflected in the expected return of the variance futures contract $\mu_t^{FV}$ (see equation (16)). This result means that, if the hedging portfolio of contingent claim $C$ does not include the variance futures contract, then this will lead to non-zero delta-hedged gains, as it will not perfectly replicate claim $C$. As shown by Bakshi and Kapadia (2003), these delta-hedged gains are
expected to be negative under the physical measure, when $\mu^F_t < 0$ and $\frac{\partial \ln C}{\partial \ln FV} > 0$. These results are consistent with the empirical findings of Coval and Shumway (2001), which show that BS model overestimate at-the-money (ATM) and long-term European options returns. This happens because $\frac{\partial \ln C}{\partial \ln FV}$ is positive and larger in magnitude for these two categories of options.

The results of Proposition 4 have also an interesting empirical implication. They can facilitate estimation of the parameters of the SV model under risk-neutral measure $Q$. This can be done by exploiting information from options, variance futures and stock price data, jointly. In particular, the values of latent variable $V_t$, for all $t$, can be substituted by a linear function of variance futures prices $FV_t$, which are observable in the market, using formula (15). This will reduce the number of parameters required in the estimation of the model and will increase the efficiency of their estimates, given that it exploits additional market information.

3.2 The SVJ model

Under the SVJ model, the processes driving stock price and volatility changes are given as follows:

\[
\frac{dS_t}{S_{t-}} = \mu_t^S dt + \sqrt{V_t} dW^{(1)}_t + J_t dN_t - \lambda \mu dt
\]

\[
dV_t = \kappa (\theta - V_t) dt + \sigma \sqrt{V_t} dW^{(2)}_t,
\]

where $\mu_t^S$ now is defined as $\mu_t^S = (r_t - \delta_t + \gamma_S V_t + \lambda (\bar{\mu} - \bar{\mu}^Q))$, $J_t$ is the random percentage jump conditional on a jump occurring distributed as $\ln(1 + J_t) \sim N (\mu_J, \sigma_J^2)$, $N_t$ is a Poisson process with intensity parameter $\lambda$, i.e., $\text{prob}(dN_t = 1) = \lambda dt$, $\bar{\mu} = \exp (\mu_J + \frac{1}{2} \sigma_J^2) - 1$ and $\bar{\mu}^Q$ is the risk neutral counterpart of $\bar{\mu}$ (see, e.g., Bates (1996)). As with the SV model, $W^{(1)}$ and $W^{(2)}$ are two Brownian motion processes which are correlated with each other.

For the SVJ model, the SDF process $\Lambda$ is given as

\[
\frac{d\Lambda_t}{\Lambda_{t-}} = - (r_t + \lambda \bar{\mu}_\Lambda) dt - \gamma_S \sqrt{V_t} dW^{(1)}_t - \frac{\gamma \sqrt{V_t}}{\sigma \sqrt{1 - \rho^2}} dW^{(3)}_t + J_{\Lambda,t} dN_t,
\]

where $\ln(1 + J_{\Lambda,t}) \sim N (\mu_{\Lambda_J}, \sigma_{\Lambda_J}^2)$ and $\bar{\mu}_\Lambda$ denotes the mean jump size $\bar{\mu}_\Lambda = \exp (\mu_{\Lambda_J} + \frac{1}{2} \sigma_{\Lambda_J}^2) - 1$. Following Pan (2002) and Broadie, Chernov and Johannes (2007), we set $\bar{\mu}_\Lambda = 0$. 

17
Under the SVJ model, a contingent claim $C$ written on a stock and/or the volatility can not be perfectly replicated by a portfolio consisting of the underlying stock and the zero-coupon bond. To perfectly hedge $C$, in the next subsections we show that the self-financing portfolio must contain, in addition to the above two instruments, a variance futures contract, as happens with the SV model, and the higher-order moment futures contracts defined in Section 2. Before proving this result, we need to derive the price and expected return of these futures contracts, under the SVJ model.

### 3.2.1 Pricing variance and higher-order moment futures contracts

Under the SVJ model, the annualized quadratic variation of log-price process $X$ is given as

$$
\frac{1}{T} \langle X, X \rangle_{0,T} = \frac{1}{T} \left( \int_0^T V_u du + \int_0^T \tilde{J}_u^2 dN_u \right)
$$

where $\tilde{J} = \ln (1 + J)$. The following proposition gives the price and instantaneous expected return of a variance futures contract.

**Proposition 5** Under the SVJ model and the no-arbitrage principle, the price of a variance futures contract at time $t \in [0, T]$ is given as

$$
FV_t = \frac{1}{T} \left( \int_0^t V_u du + \int_0^t \tilde{J}_u^2 dN_u + F(t, V_t) + G(t) \right),
$$

where $F(t, V_t)$ is defined in Proposition 3, $G(t) = \lambda \left( \mu_{(2)}^Q \right) \tau$, where $\mu_{(2)}^Q$ is the second-order non-central moment of $\tilde{J}$ under the risk-neutral measure $Q$. Its expected return is given as

$$
\mu_t^{FV} dt = E_t \left[ \frac{dFV_t}{FV_t} \right] = \frac{\partial \ln FV}{\partial \ln V} \tilde{\gamma}_V dt + \frac{\lambda}{FV_t} \left( \mu_{(2)} - \mu_{(2)}^Q \right) dt,
$$

where $\tilde{\gamma}_V = \sigma \rho \gamma_S + \gamma_V$ and $\mu_{(2)}$ is the second-order non-central moment of $\tilde{J}$ under the physical probability measure $P$.

Proposition 5 indicates that the expected return of a variance futures contract $\mu_t^{FV}$ depends, in addition to the market and volatility risk premium, on a risk premium related to the jump component of the underlying stock price $S$. Given recent evidence that $\mu_{(2)}^J < \mu_J < 0$ and $\left( \sigma_{(2)}^J \right)^2 > \sigma_J^2$, where $\mu_{(2)}^J$ and $\sigma_{(2)}^J$ are respectively the risk-neutral counterparts of $\mu_J$ and $\sigma_J$ (see Broadie, Chernov and Johannes (2007)), this premium must be negative,
implying \( \mu^{Q}_{(2)} - \mu^{Q}_{(2)} < 0 \). This means that its presence will reduce the expected return of a variance futures further than the volatility premium due to investors’ aversion towards jumps.

The prices and their expected changes of the bipower variation and higher-order moment futures contracts are given in the next proposition. Instead of the expected returns, the proposition gives the expected price changes of the above contracts, since their expected returns depend on the sign of the current prices \( F_{(j),t} \), which can be negative for odd \( j \). The sign of the expected price changes of an investment in a higher-order moment futures contract is only determined by that of jump risk premium. Note that, under the SVJ model, the payoffs of these financial instruments are respectively defined as

\[
1_T R_0 e^{J_u dN_u} \quad \text{and} \quad 1_T R_0 e^{J_{(j)} u dN_u}, \quad \text{for} \quad j \geq 3.
\]

**Proposition 6** Under the SVJ model and the no-arbitrage principle, the price of the bipower variation and higher-order moment futures contracts at time \( t \in [0,T] \) are given as

\[
F_{(j),t} = \frac{1}{T} \left( \int_0^t \tilde{J}_u dN_u + \lambda \mu^{Q}_{(j)} \right)
\]

for \( j = 2 \) and \( j \geq 3 \), respectively. The expected changes of the prices of these derivatives are given as

\[
\mu^F_{(j)} dt = E_t \left[ dF_{(j),t} \right] = \lambda \left( \mu_{(j)} - \mu^{Q}_{(j)} \right) dt,
\]

where \( \mu_{(j)} \) and \( \mu^{Q}_{(j)} \) are the \( j \)-th-order non-central moment of \( \tilde{J} \) under the physical and risk-neutral measure, respectively. These moments can be calculated respectively as \( \mu_{(j)} = \frac{\partial^j M(x)}{\partial x^j} \bigg|_{x=0} \) and \( \mu^{Q}_{(j)} = \frac{\partial^j M^Q(x)}{\partial x^j} \bigg|_{x=0} \), where \( M(x) = \exp \left( \mu_j x + (1/2) (\sigma_j)^2 x^2 \right) \) and \( M^Q(x) = \exp \left( \mu^{Q}_{(j)} x + (1/2) \left( \sigma^{Q}_{(j)} \right)^2 x^2 \right) \) are the moment-generating functions of random variable \( \tilde{J} \) under measures \( P \) and \( Q \), respectively.

Proposition 6 implies that the expected change of the price of the bipower variation futures \( \mu^F_{(2)} \) is negative and lower in magnitude than that of the variance futures. This can be obviously attributed to the fact that, by definition, the bipower variation futures pays only when jumps occur in the market during time period \([0,T]\), whereas the variance futures contract accounts also for an increase in volatility. Using formulas (22) and (24), for \( j = 2 \), it can be easily seen that \( \tilde{\gamma}_V = \sigma \rho \gamma_S + \gamma_V \), which depends on the price of market
and volatility risk, can be calculated as

\[ \tilde{\gamma}_V = \frac{FV_t}{\partial FV / \partial \ln V} \mu_t^{FV} - \frac{1}{\partial FV / \partial \ln V} \mu_t^{F(2)}. \]  

This relationship implies that the expected gains of a portfolio constructed by holding

\[ \frac{1}{\partial FV / \partial \ln V} \]  

of the notional in a long position of a variance futures and \[ \frac{1}{\partial FV / \partial \ln V} \]  

of the notional in a short position of bipower variation futures, respectively, will be related to the volatility risk premium. This comes from the fact that the return of this portfolio does not depend on the jump risk premium, as noted in Section 2 (see fn 4).

The results of Proposition 6 indicate that the sign of the expected changes of the prices of higher-order moment futures \( \mu_t^{F(j)} \), for \( j \geq 3 \), depends on the sign of moments’ difference \( \mu_{(j)} - \mu_{(j)}^Q \). Note at this point that moments \( \mu_{(j)} \) and \( \mu_{(j)}^Q \), for all \( j \), exist due to the normality assumption of the log-jump size \( \tilde{J} \). For \( j = 3 \), the sign of \( \mu_{(3)} - \mu_{(3)}^Q \) is expected to be positive, which implies that the expected price change of the 3rd-order moment futures contract \( \mu_t^{F(3)} \) will be positive. This compensates investors taking a long position in this futures contract for bearing the risk of possible negative jumps occurring during time period \([0, T]\). Following analogous arguments, we can generalize the above results to the case of higher-order odd or even moments as follows:

\[ \mu_t^{F(j)} = \begin{cases} > 0, & \text{if } j \text{ is odd} \\ < 0, & \text{if } j \text{ is even} \end{cases} \]  

3.2.2 Pricing and hedging contingent claims

To price and hedge contingent claim \( C \) under the SVJ model, we will assume that the self-financing portfolio, which replicates the price of \( C \), \( C_t = C(t, S_t, V_t) \) for \( t \in [0, T] \), consists, in addition to the financial instruments of the corresponding portfolio for the SV model, of higher-order moment futures contracts. That is, the vector of deltas \( \phi_t \) now is defined as \( \phi_t = (\phi_t^S, \phi_t^B, \phi_t^{FV}, \phi_t^{F(2)}, \ldots, \phi_t^{F(N)})' \), where its elements denote the number of the

\[ \mu_{(3)} - \mu_{(3)}^Q > 0. \]
underlying stock, zero-coupon bond, variance futures, bipower variation futures (for \( j = 2 \)) and higher-order moment futures contracts (for \( j \geq 3 \)), at time \( t \), respectively. Note that this replicating portfolio is assumed that consists of a finite, but sufficiently large number \( N \) of higher-order moment futures contracts, which is adequate to approximately replicate \( C \) in the sense that

\[
\lim_{N \to \infty} dV_t(\phi_t) = dC_t \tag{27}
\]

for all \( t \in [0, T] \), where \( V_t(\cdot) \) denotes the value of the self-financing portfolio. In the next proposition, we derive analytic formulas of the elements of vector of deltas \( \phi_t \) and instantaneous expected return of contingent claim \( C \).

**Proposition 7** Consider a contingent claim \( C \) written on the underlying stock and/or its volatility, with price function \( C_t = C(t, S_t, V_t) \), where \( C(\cdot) \) is a continuous function which has partial derivatives of any order. Then, under the SVJ model and the no-arbitrage principle, the deltas of the self-financing portfolio replicating contingent claim price \( C_t \), at \( t \in [0, T] \), are given as follows:

\[
\begin{align*}
\phi_t^S &= \frac{\partial C}{\partial S}, \quad \phi_t^B = B_t^{-1} (C_t - \phi_t^S S_t), \quad \phi_t^{FV} = \frac{\partial C}{\partial FV} \\
\phi_t^{(2)} &= \frac{1}{2!} \left( \frac{\partial (2) C}{\partial \ln S^{(2)}} - \frac{\partial C}{\partial \ln S} \right) - \frac{\partial C}{\partial FV} \quad \text{and} \quad \phi_t^{(j)} = \frac{1}{j!} \left( \frac{\partial (j) C}{\partial \ln S^{(j)}} - \frac{\partial C}{\partial \ln S} \right), \quad \text{for} \ j \geq 3,
\end{align*}
\tag{28}
\]

while the instantaneous expected return of \( C \) is given as

\[
E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln S} (\mu_t^S + \delta_t - r_t) dt + \frac{\partial \ln C}{\partial \ln FV} \mu_t^{FV} dt + \left( \frac{1}{2} \left( \frac{\partial (2) C}{\partial \ln S^{(2)}} - \frac{\partial C}{\partial \ln S} \right) - \frac{\partial \ln C}{\partial FV} \right) \mu_t^{(2)} dt \\
+ \lim_{N \to \infty} \sum_{j=3}^{N} \frac{1}{j!} \left( \frac{\partial (j) C}{\partial \ln S^{(j)}} - \frac{\partial \ln C}{\partial \ln S} \right) \mu_t^{(j)} dt. \tag{29}
\]

The results of Proposition 7 imply that enlarging the market with variance, bipower variation and higher-order moment futures contracts enables us to approximately replicate contingent claim \( C \). For sufficiently large \( N \), this market becomes approximately (or quasi) complete, thus implying that the delta-hedged gains of the self-financing portfolio given by the proposition converge to zero. In such a market, there is a unique risk-neutral measure \( Q \) (see Björk, Kabanov and Runggaldier (1997), or Jarrow and Madan (1999)). Under
this, we can derive the price of contingent claim $C_t$ by solving Bates’s (1996) PDE (see Appendix). As with the SV model, we have derived this PDE by eliminating all stochastic terms determining stochastic movements in $C_t$, and not based on equilibrium approach.

The formula of the expected return of the contingent claim $E_t \left[ \frac{dC_t}{C_t} \right]$, given by Proposition 7, indicates that, in addition to the expected returns of the underlying asset and the variance futures contract, it also depends on those of the bipower variation and higher-order moment futures contracts. The bipower variation futures contract has two effects on $E_t \left[ \frac{dC_t}{C_t} \right]$ which are complementary to each other. The first adjusts $E_t \left[ \frac{dC_t}{C_t} \right]$ for its bias due to estimating the volatility premium based on the expected return of a variance futures contract, which also depends on the jump risk (see equation (22)). The second accounts for the exposure of contingent claim price $C_t$ to the jump risk, especially, its component related to $\tilde{J}^2$ (see (24), for $j = 2$). The total of the above two effects on $E_t \left[ \frac{dC_t}{C_t} \right]$ depend on the sign and magnitude of delta coefficients $\phi^{F(j)}$, for $j > 2$. For European option contracts, which is the focus of our empirical analysis in next section (Section 4), these coefficients depend on the maturity and moneyness of the option contract. Since the above definitions of $\phi^{F(j)}$ are new in the literature, below we give a more detailed discussion about their sign and magnitude. This is done across different categories of moneyness of European calls and puts, and over short and long maturity intervals. To help this discussion, in Figure 2 we graphically present estimates of $\phi^{F(j)}$, for $j = 2, 3, 4$, across different strike prices $K$ and maturity intervals of one and six months. These rely of parameter estimates of the SVJ model, based on our empirical results of Section 4. Note that the put-call parity relation implies that $\phi^{F(j)}$ will be the same for calls and puts of the same maturity interval and strike price, for all $j$. This discussion is also very helpful in understanding sources of delta-hedged gains which can be found in practice (see Section 4).

For deep-OTM puts and calls, delta coefficients $\phi^{F(j)}$ will be almost zero, for all $j = 2, 3, 4$. In particular, $\phi^{F(2)}$ will be close to zero, because both vega and gamma are very close to zero given that strike price $K$ takes very small or large values. $\phi^{F(3)}$ and $\phi^{F(4)}$ will be also close to zero, because deep-OTM puts and calls are very little influenced by the expectation

\[ \phi^{F(2)} = \frac{1}{2} S_t^2 \frac{\partial^{(2)} C}{\partial S^{(2)}} - \frac{\partial C}{\partial FV}. \]
of a jump occurrence, which is priced by the 3rd and 4th-order moment futures. The practical implication of these results is that a two-instruments hedging strategy, involving a short position in the underlying stock and a variance futures contract, will be sufficient to hedge the exposure of deep-OTM options to all sources of risk, as jump risk is negligible for this category of options. This is also true for long-term options of any moneyness. As long as volatility risk is hedged through variance futures, the exposure of these options to jump risk becomes also negligible.

In contrast to long-term, the sign of $\phi^{F(j)}$ for short-term options changes significantly across moneyness. In particular, for short-term OTM puts $\phi^{F(2)}$ will be negative. This can be attributed to the fact that a position held in variance futures contracts is in excess of that required to hedge the exposure of the option to a jump, independently of the sign of the jump. OTM puts benefit more by an increase in volatility, which can cause the price of the underlying stock to decrease due to leverage effect, compared to an equally possible positive or negative jump effect on the stock price, measured by $\tilde{J}^2$. This means that the option’s vega should be larger than the option’s gamma, implying $\phi^{F(2)} < 0$. For short-term OTM calls, the sign of $\phi^{F(2)}$ is expected to be positive. In this case, gamma should be larger than vega, given that a positive or negative jump is preferable than an increase in volatility.

Regarding the sign of higher-order deltas $\phi^{F(3)}$ and $\phi^{F(4)}$, this will be respectively negative and positive for short-term OTM puts. For $\phi^{F(3)}$, this happens because this category of options is considered by investors as a security instrument against severe adverse movements of the stock market. Thus, a long position on them embody a short position in a 3rd-order moment futures contract. The positive sign of $\phi^{F(4)}$ can be explained by the fact that OTM puts also embody a long position in the 4th-order moment futures contract, which pays when extreme negative or positive jumps occur. For short-term OTM calls, $\phi^{F(3)}$ and $\phi^{F(4)}$ are both positive. In this case, an investor who takes a long position in this category of options is averse to jumps, and thus embodies a long position in the 3rd and 4th-order moment futures contracts. The long position in the 3rd-order moment futures contract sells protection for a possible negative jump in the market. The long position in the 4th-order moment futures contract compensates the investor for an extreme positive or negative jump.

Finally, for short-term ATM options, the sign of $\phi^{F(j)}$ is expected to be negative, for all $j = 2, 3, 4$. More specifically, $\phi^{F(2)}$ will be negative, since a positive or negative jump is
preferable than an increase in volatility causing leverage effects, as it happens with OTM puts. The negative value of $\phi^{F(3)}$ can be attributed to the fact that taking a long position in ATM options either leads to immediate gains in case of a put or to limited loses in case of a call. On the other hand, the negative sign of $\phi^{F(4)}$ can be explained by the fact that an extreme jump can render ATM options unprofitable.

4 Can volatility and jump risks be efficiently hedged by variance and higher-order moment futures?

Using data on variance futures contracts and European call and put prices, in this section we answer the following two questions based on the theoretical relationships derived in the previous section. The first is if random jumps are priced in the variance futures market. As shown by Proposition 2, this must be reflected in the difference between the price of a variance futures contract and the value of the composite options portfolio, i.e. $FV_t - V_{o,t}$. This question will be examined without relying on any parametric model of the market. The second question is if a long position in a European call or put can be efficiently hedged based on the two-instruments hedging strategy which considers a variance futures contract as a hedging vehicle, or it requires hedging strategies which rely on higher-order moment futures. To address this question, we will rely on estimates of deltas of the above hedging strategies based on the formulas of Proposition 7, for the SVJ model. Answering the above questions has important portfolio management implications. They can indicate if it is beneficial, first, to include variance futures contracts in the traditional delta hedging strategy and, second, to consider higher-order moment futures for hedging options and variance futures positions against random jumps.

4.1 The data

The options price data used in our empirical analysis are taken from the OptionMetrics Ivy data base. These are daily market closing prices of European ATM and OTM calls, with $K \geq S_t$, and OTM puts, with $K < S_t$, written on the S&P 500 index. We use ATM and OTM calls and puts because, as is well known in the literature, these options are more actively traded compared to in-the-money (ITM). We have excluded option prices violating the boundary conditions and options with maturity intervals less than 2 weeks. Finally,
following Driessen, Maenhout and Vilkov (2009), we have removed options with zero open interest for liquidity reasons. Apart from option prices, we have also used the reported implied volatility surfaces (IVS) of the above data base, with maturity intervals 1, 2, 3, 6 and 9 months. This is done in order to estimate the parameters of the SVJ model needed in our analysis and calculate the values of the composite options portfolio. This data set contains implied volatilities on both call and puts on a grid of 13 strike prices.

The variance futures price data used in our analysis are written on the S&P 500 index. These contracts are traded over-the-counter. This data set consists of closing prices of variance futures net of the accrued realized variance with maturity intervals of 1, 2, 3, 6 and 9 months. Thus, if we have a traded variance futures contract during period \([T - \tau, T]\), then the recorded quote of \(FV_t\) is adjusted as follows: 

\[
FV_t \equiv E^Q_t \left[ \frac{1}{\tau} (X, X)_{T-\tau, T} \right] = FV_t - \frac{1}{\tau} (X, X)_{T-\tau, T}.
\]

Finally, the series of the risk-free interest rate and dividend yield are also taken from the OptionMetrics data base. The interest rate is derived by British Banker’s Association LIBOR rates and settlement prices of Chicago Mercantile Exchange Eurodollar futures. The dividend yield is estimated by the put-call parity relation of ATM option contracts.

Our data set covers the period from March 30, 2007 to October 29, 2010. This sample interval consists of 900 trading days and it includes the period of the recent financial crisis, the most serious at least since the 1930’s. Our analysis presents results for subsamples before and after the date that this crisis seems to be intensified. This date is found to be closely related to that of the collapse of Lehman Brothers.

4.2 Fitting the SVJ model into the data

To estimate the parameters of the SVJ model, we rely on daily values of IVS of put options written on the S&P 500 index, with constant maturity intervals 1, 2, 3 and 6 months.\(^{11}\) Fitting the SVJ model into constant-maturity IVS provides estimates of its parameters which are less sensitive to available maturity intervals every day. Since the following parameters of the SVJ model: \(\sigma, \rho, \lambda\) and \(\kappa\theta\) are the same under the physical and risk neutral measures, in the estimation procedure we will set them to certain values provided in the literature, following Broadie, Chernov and Johannes (2007). In particular, these are set

\(^{11}\)Note that the choice between European calls and puts in estimating the SVJ model does not affect the results, due to put-call parity relation.
to $\kappa \theta = 5.04 \times 0.04 = 0.2016$, $\sigma = 0.52$, $\rho = -0.66$ and $\lambda = 1.8$, obtained by Kaeck and Alexander (2011) using a sample of data covering our sample interval interval 2007-2010. Given these values, we then estimate the vector of the risk-neutral parameters of the SVJ model, collected in vector $\Theta = (\kappa^Q, \mu^Q, \sigma^Q_j)'$. This is done by solving the following least squares problem:

$$
\min_{\Theta} \sum_{i=1}^{M_t} \sum_{j=1}^{N_i} \left( \ln \tilde{P}_t(\tau_i, K_j, S_t, FV_t; \Theta) - \ln P_t(\tau_i, S_t, K_j) \right)^2,
$$

for all $t$, where $\tilde{P}_t(\tau_i, K_j, S_t, FV_t; \Theta)$ denotes estimates of put option prices implied by the SVJ model, with maturity intervals $\tau_i$ and strike prices $K_j$, and $P_t(\tau_i, K_j)$ are their corresponding market prices.\footnote{See, e.g., Bakshi, Cao and Chen (1997). Note that, in this estimation procedure, we use logarithms of put prices to avoid the problem of assigning more weight to relatively expensive options (e.g., ITM long-term options) and less weight to cheaper options (e.g., OTM options). See also Lin and Chang (2010).} $M_t$ denotes the number of different maturity intervals $\tau$, considered at any point in time $t$ (i.e. $M_t = 4$), and $N_i$ denotes the number of different put prices employed in the estimation procedure, for all $t$, i.e. $N_i = 13$. These values $M_t$ and $N_i$ imply that, for each time point (day) of our sample, a quite large cross-section set of 52 put option prices are used to fit the SVJ model into the data. Note that the estimates of put option prices $\tilde{P}_t(\tau_i, K_j, S_t, FV_t; \Theta)$, derived by the above estimation procedure, rely on variance futures contract prices $FV_t$. As mentioned in the previous section, in this estimation procedure these values of $FV_t$ can substitute out those of volatility $V_t$, which is a latent variable. As noted before, this helps in obtaining more precise estimates of vector $\Theta$ from the data, as it exploits all available market information.

Table 1 presents the average values of the estimates of the elements of vector $\Theta$ over all time-points of the sample. Standard errors of these values are given in parentheses. Note that table presents values of the elements of $\Theta$ for the whole sample and two subsamples before and after September, 3 2008. This date is almost two weeks before the date of the collapse of Lehman Brothers on September, 16 2008. It is chosen to split the sample into two subsamples based on the daily estimates of $\Theta$, which indicate a clear cut shift in jump parameters $\mu^Q, \sigma^Q_j$ after September 3, 2008 (see Figure 3). It reveals that the market was expecting a significant negative jump two weeks before the collapse of this bank. To see how well the SVJ model fits into the data, the table also presents values of the RMSE
(root mean square error) and the RMSPE (RMS pricing error) between the estimated and actual put option prices \( \hat{P}_t \left( \tau_i, K_j, S_t, \bar{FV}_t; \Theta \right) \) and \( P_t(\tau_i, K_j) \), respectively. These values are found to be very small for the whole sample and the two subsamples considered, which indicates that the SVJ fits satisfactorily into the data.

Regarding the elements of \( \Theta \), the results of the table clearly indicate that the estimates of parameter \( \kappa Q \) is significantly smaller than estimates of it under the physical measure reported in the literature (i.e. \( \kappa = 5.04 \), see, e.g., Kaek and Alexander (2011)). This is true for the whole sample and the two subsamples considered. For instance, for the whole sample \( \kappa Q \) is found to be 0.72. This result was expected and it is consistent with recent evidence provided in the literature (see, e.g., Egloff, Leippold and Wu (2010)). It implies that the price of volatility risk \( \gamma_V \) is negative. The estimates of risk-neutral parameters \( \bar{\mu} Q \) and \( \sigma_{\nu} Q \), reported in the table, are also consistent with the literature (see, e.g., Eraker (2004)). As was expected, these estimates are found to be bigger in absolute terms for the subsample after the date of the Lehman Brothers collapse. This can be also confirmed by the inspection of Figure 3, which graphically presents the estimates of \( \bar{\mu} Q \) and \( \sigma_{\nu} Q \), over the whole sample. The plots of this figure reveal that the estimates of \( \bar{\mu} Q \) and \( \sigma_{\nu} Q \) are virtually zero until September, 3 2008. After that date, they vary substantially due to the subsequent turmoils in the market triggered by the recent financial crisis sequence of events.\(^{14}\)

### 4.3 Are random jumps priced in the variance futures market?

To empirically investigate if random jumps are priced in the market of variance futures, we will rely on the predictions of Proposition 1. According to this proposition, the difference between the price of a variance futures contract \( FV_t \) and value of the composite portfolio of European OTM calls and puts \( V_{o,t} \), given by formula (4), must be significantly different than zero. This difference, adjusted for the accrued realized variance effects \(-\frac{1}{\tau} \left< X, X \right>_{T-t} \) and \( \frac{2}{\tau} (1 + (r - \delta)T - e^{(r-\delta)T}) \), will be henceforth written as \( \tilde{FV}_t - \tilde{V}_{o,t} \), where \( \tilde{FV}_t = FV_t - \frac{1}{\tau} \left< X, X \right>_{T-t} \) and \( \tilde{V}_{o,t} = V_{o,t} + \frac{2}{\tau} (1 + (r - \delta)T - e^{(r-\delta)T}) \). It should reflect the sum of higher-order moment effects \( \sum_{j=3}^{\infty} \frac{1}{j!} E_t^Q \left[ \sum_{t<u<T} (\Delta X_u)^j \right] \), due to the discontinuous component of log-price \( X \). Note that, in our above definitions, we have set \( \bar{\tau} = \tau \), since in our data set

\(^{13}\)Note that Egloff, Leippold and Wu (2010) estimate volatility premium using variance futures quotes.

\(^{14}\)Substantial time-variation in the values of \( \bar{\mu} Q \) and \( \sigma_{\nu} Q \) during periods of market stress and turmoil are also reported in the literature by Santa-Clara and Yan (2010).
the accrued period of the variance futures contract coincides with the maturity interval of the European options, \( \tau \).

To address the above question, we first need to retrieve values of \( V_{o,t} \), based on OTM call and put prices written on the S&P 500 index. To this end, we use weekly values of the IVS data of maturity intervals \( \tau = 1, 2, 3, 6, 9 \) months taken every Wednesday of the sample, and we rely on a numerical interpolation-extrapolation scheme from those described in Section 2.3. In particular, we present results of two of these schemes, which differ with respect to the extrapolation procedure employed. The first sets endpoints \( K_0 \) and \( K_\infty \) to their sample minimum and maximum values \( K_{\min} \) and \( K_{\max} \), respectively. That is, it chooses not to extrapolate the implied volatility functions, while the second assumes constant extrapolation.\(^{15}\) In Table 2, we present average values of the estimates of \( V_{o,t} \) over the whole sample and their corresponding approximation error bounds, obtained based on the error bound formula (8). Since the true values of \( V_{o,t} \) are unknown, these error bounds are calculated based on estimates of the whole set of parameters of the SVJ model. These are obtained by fitting the model into weekly option prices of our data. They are found to provide very small values of the root mean square errors ranging from 0.12 to 0.27, for all maturity intervals \( \tau \) considered.

The results of Table 2 indicate that the values of \( V_{o,t} \) obtained by the interpolation-extrapolation scheme of the implied volatility function which chooses not to extrapolate this function provide much smaller approximation error bounds, for all \( \tau \).\(^{16}\) We will thus choose this scheme to calculate values of difference \( \tilde{FV}_t - \tilde{V}_{o,t} \), employed in our empirical analysis. The very small in size numerical approximation errors of this numerical scheme guarantee that the values of \( \tilde{FV}_t - \tilde{V}_{o,t} \) will not be serially affected by numerical errors. In Table 3, we present average values of \( \tilde{FV}_t - \tilde{V}_{o,t} \), based on this scheme of calculating \( V_{o,t} \). This is done for the whole sample and the two subsamples considered in our analysis, and for maturity intervals \( \tau = 1, 2, 3, 6, 9 \) months. In parentheses, the table presents values of the t-statistic (t-stat) testing if the reported average values of of \( \tilde{FV}_t - \tilde{V}_{o,t} \) are different

\(^{15}\)We do not present results for the interpolation-extrapolation scheme which assumes linear extrapolation, since it is found to perform much more worse than the two other schemes.

\(^{16}\)The worse performance of the numerical approximation scheme which assumes constant extrapolation of implied volatility function can be attributed to the following three effects. First, it decreases the truncation error \( \epsilon \). Second, it increases \( C_{error} \) and \( P_{error} \) since the intervals \((y_0, 0)\) and \((0, y_\infty)\) increases, respectively. Finally, it increases the value of the function \( e^{-y} - 1 \) that controls for the effect of \( P_{error} \) on the bound. If the negative effect on \( P_{error} \) dominates the positive effect on \( \epsilon \), then the extrapolation of the implied volatility function would increase the approximation error bounds, which explains the numbers of Table 2.
than zero, for all $\tau$.

A number of interesting conclusions emerge from the results of Table 3. First, the reported values of difference $\tilde{F}V_t - \tilde{V}_{o,t}$ are positive and significantly different than zero, for all $\tau$. This indicates that random jumps are priced in the market of variance futures contracts, according to the predictions of Proposition 1. Second, the values of $\tilde{F}V_t - \tilde{V}_{o,t}$ increase both in terms of magnitude and volatility during the subsample which covers the recent financial crisis period (see also Figure 4). This result should be expected, since this period is one of market stress and fears of financial crisis accelerate. Finally, the results of Table 3 indicate that the mean values of $\tilde{F}V_t - \tilde{V}_{o,t}$ tend to increase with maturity interval $\tau$. According to the results of Proposition 1, this can be attributed to the increase of the probability of a jump occurrence or its size with maturity interval $\tau$.

Having established that difference $\tilde{F}V_t - \tilde{V}_{o,t}$ is significant, next we examine if its variation can be explained by term $-\frac{2}{3} \sum_{j=3}^{\infty} \frac{1}{3 j!} E_t^Q \left[ \sum_{t<u\leq T} (\Delta X_u)^j \right]$, as the theory suggests. To this end, we regress logarithm values of $\tilde{F}V_t - \tilde{V}_{o,t}$ on estimates of this term, taken also in logarithms. In our analysis, we consider two different sets of estimates of this term. The first, denoted as $JT_t$, is obtained based on the estimates of the SVJ model, presented in Section 4.2, and it is calculated as

$$JT_t = 2\lambda \left( \mu_j^Q + \frac{1}{2} \left( \sigma_j^Q \right)^2 + \frac{1}{2} \left( \sigma_j^Q \right)^2 - e^{\sigma^Q + (\sigma^Q)^2 / 2} \right),$$

for all $t$ (see Appendix, for a proof). Figure 4 graphically presents estimates of $JT_t$ against difference $\tilde{F}V_t - \tilde{V}_{o,t}$. The second method of estimating term $-\frac{2}{3} \sum_{j=3}^{\infty} \frac{1}{3 j!} E_t^Q \left[ \sum_{t<u\leq T} (\Delta X_u)^j \right]$ is model-free. It is based on estimates of the third-order risk-neutral moment of the underlying stock log-return implied from OTM option prices, denoted as $\mu_{3,t}$ (see Lemma 1 of Appendix).

$^{17}$Note that equation (30) holds also under a more general version of the SVJ model, which allows for jumps in volatility which are independent of those in the price level of the underlying asset often assumed in the literature (see, e.g., Broadie, Chernov and Johannes (2007)).

$^{18}$ $\mu_{3,t}$ can be obtained by setting $m = 3$ to formula (39), given in the Appendix. It is an approximation of $-\frac{2}{3} \sum_{j=3}^{\infty} \frac{1}{3 j!} E_t^Q \left[ \sum_{t<u\leq T} (\Delta X_u)^j \right]$, which can be seen by writing

$$E_t^Q \left[ \sum_{t<u\leq T} (\Delta X_u)^3 \right] = \mu_{3,t} - 3E_t^Q \left[ \int_{t+}^{T} (X_u - X_t)^2 dX_u \right] - 3E_t^Q \left[ \int_{t+}^{T} (X_u - X_t) d(X,X)_u \right].$$

This is derived by applying Ito’s lemma to function $f(x) = x^3$. To estimate $\mu_{3,t}$, we adopt the numerical procedure used to obtain values of $V_{o,t}$.
Least squares (LS) estimates of the coefficients of the above regressions are reported in Table 4. The variables (log-transformations) involved in these regressions are denoted as follows: \( LNFVO_t \) stands for the logarithm of \( \tilde{FV}_t - \tilde{V}_{o,t} \), \( LNJT_t \) for the logarithm of \( JT_t \), and \( LN\mu_{3,t} \) for the logarithm of \( -\mu_{3,t} \), which is positive for all \( t \). Panel A of the table presents estimation results of the regression of \( LNFVO_t \) on \( LNJT_t \), while panel B presents those of the regression of \( LNFVO_t \) on \( LN\mu_{3,t} \). Note that, since the values of \( JT_t \) are almost zero for the pre-crisis subsample (see Figure 4), the first of the above regressions was estimated only for the second subsample, covering the post-crisis period. In addition to LS estimates, the table presents fully-modified LS (FMLS) estimates of the regression coefficients, since all variables involved in these regressions are found to be integrated series of order one, I(1). FMLS estimates correct the standard LS coefficient estimates for small-sample bias due to the omission of short-run dynamics of regression variables. To test if the estimated regressions are cointegrated, the table presents Phillips’ and Ouliaris (1990) \( Z_t \) and \( Z_a \) test statistics for cointegration. Rejection of cointegration between variables \( LNFVO_t \) and \( LNJT_t \), or \( LNFVO_t \) and \( LN\mu_{3,t} \), implies that variables \( JT_t \) and \( -\mu_{3,t} \) can not explain long-run shifts in \( \tilde{FV}_t - \tilde{V}_{o,t} \). This will constitute strong evidence against the theory.

The results of Table 4 indicate that variables \( JT_t \) and \( -\mu_{3,t} \), capturing random jumps, can explain long-run movements of difference \( \tilde{FV}_t - \tilde{V}_{o,t} \), as the theory suggests. This is true for all maturity intervals \( \tau \) considered. The \( p \)-values of test statistics \( Z_t \) and \( Z_a \), reported in the table, clearly indicate that the regressions of \( LNFVO_t \) on \( LNJT_t \), or \( LNFVO_t \) and \( LN\mu_{3,t} \), constitute cointegrating regressions. These results can be also confirmed pictorially by the inspection of the graph of Figure 4, which shows that \( JT_t \) follows closely persistent shifts of \( \tilde{FV}_t - \tilde{V}_{o,t} \), over the whole sample. The above results highlight the need to introduce higher-order moment futures contracts to hedge a position in variance futures contracts, as suggested in Section 2.

Regarding the regression coefficient estimates, the results of Table 4 (see Panel A) indicate that both the LS and FMLS estimates of \( \beta_1 \) are in the right direction, but they are less than unity which is their predicted value by the theory. These deviations of the estimates of \( \beta_1 \) from unity can be attributed to numerical approximation errors in the estimates of \( V_{o,t} \), the existence of transaction costs and/or the inadequacy of the SVJ to fully capture the discontinuous component of stock price index dynamics. The last may be
though of as the most plausible explanation of them (see, e.g., Bates (2003)), given that the error bounds of $V_{o,t}$, reported in Table 2, and the effects of transaction costs on our results are found to very small.\footnote{To investigate if transaction costs constitute a significant source of the deviations of the estimates of $\beta_1$ from unity, we estimated $V_{o,t}$ based on ask option prices, instead of the mid bid-ask prices. These estimation results are not reported for reasons of space. They are very close to those reported in Table 4. This can be attributed to the fact that the average cost that should be paid to take a long position in the composite portfolio of options with $\tau = 1$ month maturity interval is $0.003$, which is very small. This value becomes smaller, as $\tau$ increases.}

4.4 Evaluation of dynamic hedging strategies accounting for volatility and jump risks

The findings of the previous section that random jumps are priced in the market of variance futures contracts implies that hedging strategies accounting for this source of risk are required in practice. Since this risk together with that of volatility are reflected in variance futures contracts, it will be interesting in examining if these assets can be employed to improve upon the performance of the traditional delta hedging strategy. The latter, denoted as $HP_1$, hedge a long position in a European option by holding a self-financing portfolio consisting of $\phi_t^S = \partial C/\partial S$ number of shares of the underlying stock and invest the residuals, defined as $U_t \equiv C_t - \phi_t^S S_t$, in a zero-coupon bond. In addition to this, it will be also interesting in investigating if there exist potential gains by adding higher-order moment futures in the hedging portfolio.

To empirically investigate the above questions, we will compare the following two hedging strategies with strategy $HP_1$: The first is a two-instruments hedging strategy, denoted as $HP_2$, which assumes that $HP_1$ includes also a short position in $\phi_t^{FV} = \partial C/\partial FV$ number of a variance futures contracts. The second, denoted as $HP_3$, is a strategy which extends $HP_2$ to also include the bipower, 3rd and 4th-order moment futures contracts in the self-financing portfolio at numbers $\phi_t^{FV(2)} = \frac{1}{2!} \left( \frac{\partial^2 C}{\partial \ln S^2} - \frac{\partial C}{\partial \ln S} \right) - \frac{\partial C}{\partial FV}$, for $j = 2$, and $\phi_t^{FV(j)} = \frac{1}{j!} \left( \frac{\partial^j C}{\partial \ln S^j} - \frac{\partial C}{\partial \ln S} \right)$, for $j = 3, 4$, respectively, as is implied by Proposition 7.\footnote{We have chosen to extend up to the 4th-order moment futures contracts strategy $HP_2$ for two main reasons. First, due to the interest of the literature for the implications of skeweness and excess kurtosis on asset pricing (see, e.g., Bakshi, Kapadia and Madan (2003)) and, second, for practical reasons, i.e., for trading a limited number of higher-order moments for hedging purposes. Furthermore, the results of a simulation study, whose results are available upon request, has shown that the inclusion of higher than 4th-order moment futures contracts do not critically change the hedging performance $HP_3$, with the exception of short-term OTM calls. For this category of options we have found that we need to include up to the 15th-order moment futures contract so as the performance of $HP_3$ to be critically improved.} The
hedging errors of the above all strategies over discrete-time interval $t + \Delta t$ are defined as follows:

$$HP_1 : H(t + \Delta t)$$
$$= [C_{t+\Delta t} - C_t] - [\phi^S_t (S_{t+\Delta t} - S_t) + r_t U_t \Delta t + \delta_t \phi^S_t S_t \Delta t]$$

$$HP_2 : H(t + \Delta t)$$
$$= [C_{t+\Delta t} - C_t] - [\phi^S_t (S_{t+\Delta t} - S_t)$$
$$+ \phi^{FV}_t (FV_{t+\Delta t} - FV_t) + r_t U_t \Delta t + \delta_t \phi^S_t S_t \Delta t]$$,

and

$$HP_3 : H(t + \Delta t)$$
$$= [C_{t+\Delta t} - C_t] - [\phi^S_t (S_{t+\Delta t} - S_t) + \phi^{FV}_t (FV_{t+\Delta t} - FV_t) +$$
$$+ \sum_{j=2}^{4} \phi^{F(j)}_t (F(j)_{t+\Delta t} - F(j)_t) + r_t U_t \Delta t + \delta_t \phi^S_t S_t \Delta t].$$

All these strategies are evaluated for OTM puts, with moneyness levels $K/S_t$ of 0.8 (deep-OTM) and 0.9, an ATM call, with $K/S_t = 1$, and OTM calls, with $K/S_t$ of 1.1 and 1.2 (deep-OTM). The expiration dates of these options lie within the maturity intervals of 15 to 35, 40 to 60 and 100 to 160 days. These intervals corresponds to options with time-to-maturity periods around one, two and a half and six months, respectively. In our analysis, we divide our sample in weeks and, for each week, we consider a daily rebalancing of the self-financing portfolio, implying that $\Delta t = 1/252$. The delta coefficients $\phi^S_t$, $\phi^{FV}_t$ and $\phi^{F(j)}_t$, for $j = 2, 3, 4$, are calculated based on daily estimates of the parameters of the SVJ model, reported in Section 4.2. Since the bipower and higher-order moment futures contracts are not traded, to evaluate strategy $HP_3$ we rely on estimates of their prices under the SVJ model (see equation (23)). The drift term of these prices are estimated based on the average of their daily estimates of parameters $\overline{p}^Q$ and $\sigma^Q_j$, for all weeks of the sample. This guarantees that the prices of these contracts are consistent with the predictions of the SVJ model, for all weeks.\textsuperscript{21}

\textsuperscript{21}To see if our results are sensitive to the estimator of the drift component of higher-order moment futures
Tables 5A, 5B and 5C present values of the following evaluation metrics of the hedging errors $H(t + \Delta t)$ of the above strategies across different moneyness levels and maturity intervals:

$$\text{RMSE} = \sqrt{\frac{1}{D} \sum_{s=1}^{D} H(t + s\Delta t)^2},$$

(34)

where RMSE stands for root mean square error and $D$ is the number of days of our sample,

$$\text{MAE} = \frac{1}{D} \sum_{s=1}^{D} |H(t + s\Delta t)|,$$

(35)

where MAE stands for the mean absolute error, and

$$\Sigma = \frac{1}{W} \sum_{j=1}^{W} \sum_{s=1}^{W_j} H(t + s\Delta t)$$

(36)

is the average cumulative delta-hedged gains across all weeks of the sample, where $W$ denotes the number of weeks of the sample and $W_j$ denotes the number of the trading days per week. Table 5A presents results for the whole sample, while Tables 5B and 5C for the two subsamples before and after the recent financial crisis, respectively. To test if there are significant hedging gains across the three strategies $HP_1$, $HP_2$ and $HP_3$, the table also reports $p$-values of an overidentified restrictions GMM based test statistic, denoted as $J$-stat (see Chernov and Ghysels (2000)). This is distributed as $\chi^2$ with one degree of freedom. This statistic can test if the following $(2 \times 1)$-dimension vector of second-order moment conditions are satisfied by the data:

$$\begin{pmatrix}
\frac{1}{D} \sum_{s=1}^{D} H_k(t + s\Delta t)^2 - \mu \\
\frac{1}{D} \sum_{s=1}^{D} H_i(t + s\Delta t)^2 - \mu
\end{pmatrix} = 0,$$

for any pair of $k \neq l$ hedging strategies, where $H_i(t + s\Delta t)$, for $i = \{k, l\}$, is the hedging error of hedging strategy $HP_i$, for $i = 1, 2, 3$, and $\mu$ is a constant. This constant is assumed to be the same across any pair of strategies $(k, l)$ examined. If this hypothesis is not true, prices, we have also conducted our hedging evaluation exercise assuming that $\hat{\pi}^Q$ and $\sigma^Q_J$ are given as the mean of the daily estimates of these parameters across the sample period. The results, which can be provided upon request, are very close to those reported in the current section. This is due to the fact that the performance of a hedging strategy mainly depends on the stochastic term of the price, which is approximated by the accrued realized non-central moment rather than the drift term, which is affected by the parameters of the jump component.
then $J$-stat will reject the above moment conditions, which means that hedging strategy $HP_k$ differs from $HP_l$, for $k \neq l$.

The results of Tables 5A-5C lead to the following conclusions. First, strategy $HP_2$, which, in addition to the underlying asset, includes also in the hedging portfolio short positions in a variance futures contract clearly outperforms traditional strategy $HP_1$, which considers only a short position in the underlying stock. This is true for all different maturity intervals and moneyness levels examined, as well as the two subsamples considered. It can be justified by the values of metrics RMSE and MAE, and the $p$-values of $J$-stat reported in the tables. The latter always reject the overidentified restrictions for the pair of strategies $HP_1$ and $HP_2$. The higher benefits of $HP_2$ than those of $HP_1$ are observed for ATM and long-term options. This can be attributed to the fact that these categories of options are more affected by a change in volatility, as is pointed out by the analysis of Section 3.1.2.

Second, the values of the RMSE and MAE metrics reported in the table indicate that $HP_3$ outperforms $HP_2$ for all options categories, with the exception of short-term OTM calls. The highest hedging benefits of $HP_3$ compared to $HP_2$ are observed in cases of the ATM call and the short-term OTM put with $K/S_t = 0.9$. This can be also justified by the $p$-values of the $J$-stat reported in the tables. The outperformance of $HP_3$ over $HP_2$ in these cases can be attributed to the fact that the above categories of options are exposed more to jump risk. These results are in accordance with our theoretical predictions of Section 3.2.2, which claim that ATM calls and short-term OTM puts are more sensitive to random jumps compared to deep-OTM calls and puts, as well as long-term OTM puts. As noted before (see fn 21), the worse performance of $HP_3$ than $HP_2$ for short-term OTM calls can be attributed to the fact that higher than 4th-order moment futures are needed to approximate the discontinuous component of option prices. Finally, the results of the tables indicate that, for the post-crisis period, the benefits of $HP_3$ strategy compared to $HP_2$ become much higher for the deep-OTM put. This is consistent with the results of Section 4.3. It can be attributed to the significance of the jump risk premium for this period.

Third, the estimates of the average weekly delta-hedged gains $\Sigma$, reported in the tables, are negative for strategy $HP_1$, as they reflect risk premium effects due to volatility and random jumps. This is consistent with evidence provided in the literature (see Bakshi and Kapadia (2003)). These gains tend to increase with the maturity interval, which indicates that jump risk premium, which affects more short-term options rather than long-term,
dominates volatility risk premium (see also Branger and Schlag (2008)). With respect to
moneyness, the values of $\Sigma$ become closer to zero for OTM calls and puts rather than ATM
calls, which means that the impact of volatility and jump premia is smaller for away-from-the-
money options. For strategy $HP_2$, the values of $\Sigma$ tend to zero for most options categories
examined, especially for deep-OTM and long-term options. This result means that the
expected gains of strategy $HP_2$ are no longer affected by the volatility risk premium. The
jump risk premium does affect the gains of strategy $HP_2$ for these two categories of options,
since its effect is negligible (see our theoretical discussion in Section 3.2.2). This is not true
for short-term OTM puts and ATM calls. For these categories of options, the value of $\Sigma$
become negative due to the significant influence of the jump risk premium effects. This sign
of $\Sigma$ can be analytically explained by that of delta coefficients $\phi^{F(j)}$, for $j = 2, 3, 4$, discussed
in Section 3.2.2 and the sign of the expected price changes of higher-order moment futures
contracts $\mu^{F(j)}_t$, for $j = 2, 3, 4$, discussed in Section 3.2.1.

In particular, the negative sign of $\Sigma$ for short-term OTM puts means that the total
effect of $\phi^{F(3)}$ and $\phi^{F(4)}$ on $E_t \left[ \frac{dC_t}{C_t} \right]$ (see formula (29)), which is negative, dominates that
of $\phi^{F(2)}$, which is positive. The negative sign of $\Sigma$ for short-term ATM calls means that the
total effect of $\phi^{F(3)}$ on $E_t \left[ \frac{dC_t}{C_t} \right]$, which is negative, dominates that of $\phi^{F(2)}$ and $\phi^{F(4)}$, which
is positive. This negative sign of strategy $HP_2$ for short-term OTM puts and ATM calls
can be explained by noticing that a long position in these contracts and a short position in
the hedging portfolio of strategy $HP_2$ compensates investors for a negative jump, against
which they are averse. For short-term OTM calls, the positive sign of $\Sigma$ can be attributed
to the fact that these options are heavily exposed to a possible negative jump, which will
tend to make them deep-OTM. When adopting strategy $HP_3$, the values $\Sigma$ tend to zero for
OTM puts, for the whole sample.

The robustness of the above results has been investigated by carrying out two exercises.
In the first, a monthly interval is chosen to construct the alternative hedging portfolios using
daily rebalancing (see also Bollen and Whaley (2004)), while, in the second, the monthly
interval of the hedging portfolios is combined with weekly rebalancing. The results of these
two exercises do not change the above conclusions about the performance of the hedging
strategies examined.
4.5 Delta-hedged gains, volatility and jump risk premia

To verify that the reported in the previous section delta-hedged gains depend on volatility and/or jump risk premia, we run the following regressions for ATM options:

\[
\Delta \hat{\Sigma}_t^{HP_j}/S_t = \beta_0 + \beta_1 \Delta VOL_t + \beta_2 \Delta FVO_t + \varepsilon_{jt}, \quad j = 1, 2, 3, \tag{37}
\]

where \(\Delta \hat{\Sigma}_t^{HP_j}/S_t\) are the first differences of the cumulative weekly delta-hedged gains of strategies \(HP_j\), for \(j = 1, 2, 3\), scaled by the current stock price \(S_t\), \(\Delta VOL_t\) is the first difference of historical volatility computed over the week prior to \(t\), and \(\Delta FVO_t\) is the first difference of variable \(\bar{V}_t - \tilde{V}_{o,t}\), which captures the sum of higher-order moment effects \(\sum_{j=3}^{\infty} \frac{1}{j!} \mathcal{E}^Q_{t} \left[ \sum_{t<\tau\leq T} (\Delta X_u)^j \right]\) depending on jump premium. Regression (37) has been suggested by Bakshi and Kapadia (2003) to investigate if volatility risk is reflected in the expected delta-hedged gains of ATM options of strategy \(HP_1\). We have extended this regression to two directions. First, we also consider hedging strategies \(HP_2\) and \(HP_3\), and, second, we employ a market based, direct measure of jump risk premium in its right hand side, given by \(\Delta FVO_t\). Testing whether volatility or jump risk are not priced are equivalent to testing if \(\beta_1 = 0\) or \(\beta_2 = 0\) can not be rejected by the data.

Since regressions (37), involve contemporaneous terms, to avoid any endogeneity bias estimation problems of slope coefficients \(\beta_1\) and \(\beta_2\) we have estimated them based on the GMM procedure, using the following instruments: the constant, and lagged values of the dependent variable \(\Delta \hat{\Sigma}_t^{HP_j}/S_t\) and the two independent variables \(\Delta VOL_t\) and \(\Delta FVO_t\) two periods back. The estimation results are presented in Table 6. This is done for the three maturity intervals considered in Tables 5A-5C. Newey-West standard errors corrected for heteroscedasticity and serial correlation are reported in parentheses. The order of serial correlation is chosen based on the Schwarz information criterion. The results of the table reveal the following.

First, the delta-hedged gains of strategy \(HP_1\) reflect volatility and jump risk premia, given that the estimates of coefficients \(\beta_1\) and \(\beta_2\) are clearly different than zero at 5% significance level. The negative estimates of \(\beta_1\) and \(\beta_2\) indicate that the volatility and jump risk premia are negative, which is consistent with the findings of Bakshi and Kapadia (2003).

\(^{22}\)This weekly interval of the estimation of historical volatility ensures that our data series are not overlapping.
Second, the delta-hedged gains of strategy $HP_2$ does not reflect volatility risk premium, since the estimates of $\beta_1$ are not significant. This result means that $HP_2$ can efficiently hedge against volatility risk. As was expected, the only source of risk that is reflected in the delta-hedged gains of strategy $HP_2$ is due to random jumps, as the estimates of $\beta_2$ are different than zero. The estimates of $\beta_2$ become insignificant for strategy $HP_3$, which also includes bipower and higher-order moment futures contracts in the hedging portfolio. This indicates that the delta-hedged gains of strategy $HP_3$ can also account for jump risk. The only exception is for long-term options. This may be attributed to the fact that delta-hedged gains of strategies $HP_2$ and $HP_3$ are very close to each other for this category of options (see the results of Table 5A). This can obviously attributed to the fact that the SVJ model can not fully capture the dynamics of the data, as argued in Section 4.3. As can be seen from Figure 2, $\phi^{\hat{R}(j)}$ are very small for long-term options. Thus, a small difference in the estimates of $\phi^{\hat{F}(j)}$, due to a slight mispecification of the SVJ model, can lead to significant biases in the delta-hedged gains.

5 Conclusions

This paper suggests perfect hedging strategies of contingent claims, including variance futures contracts, volatility swaps and plain vanilla European options, under the assumption that the underlying asset price follows the stochastic volatility (SV) and the stochastic volatility with jumps (SVJ) models. This is done by enlarging the market with appropriate futures contracts whose payoffs depend on higher-order sample moments of the underlying asset price. For the SV model, we demonstrate that enlarging the market with a variance futures (or swap) contract, which is nowadays traded in the market, makes it complete and, thus, perfectly hedge positions in contingent claims written on the underlying asset and/or its volatility. Under this model, it is also shown that the price of a variance futures contract can be perfectly hedged by the value of a composite portfolio of out-of-the money (OTM) European call and put options. The paper shows how to numerically calculate the value of this portfolio based on a discrete set of data and it derives approximation error bounds of its value.

For the SVJ model, the paper shows that, in addition to variance futures contracts, higher than second-order moment futures contracts should be included in the self-financing portfolio to replicate the price of a contingent claim. These futures contracts can hedge the
exposure of a contingent claim price to random jumps. If this claim is a variance futures contract, this can be done together with the composite portfolio of OTM options. Since the size of jumps is random, the value of the self-financing hedging portfolio converges to the price of the contingent claim, as the number of higher-order moment futures contracts goes to infinity, thus making the market approximately (or quasi) complete. In practice, someone can satisfactorily approximate the value of this hedging portfolio with a finite number of higher-order moment futures contracts which can be traded in the market.

The results of the paper are used to empirically address two questions on pricing and hedging options. The first is if random jumps are priced in the market. This price of risk is reflected in the difference between the price of a variance futures contract and the value of the composite options portfolio. It can be estimated from market data without relying on any parametric model. The second question is if a position in a European option can be efficiently hedged based on a two-instruments hedging strategy, which considers a variance futures contract as a hedging vehicle, or it requires hedging strategies which rely on higher-order moment futures. To answer these questions, we rely on options and variance futures price data written on the S&P 500 index. The empirical results of the paper clearly indicate that, indeed, random jumps are priced in the variance futures market. The paper clearly shows that the prices of these contracts significantly exceed those of the composite options portfolio, thus implying that random jumps are priced in the variance futures market.

Regarding the second question posed by the paper, the empirical results indicate that a two-instruments hedging strategy, which also includes a variance futures contracts in the self-financing portfolio, is found to considerably improves upon the performance of the traditional delta hedging strategy often used in practice, which only includes the underlying asset and the zero-coupon bond in this portfolio. The improving performance of this strategy comes from the fact that variance futures contracts can efficiently hedge the exposure of a call, or a put, to volatility risk. This provides evidence that the inclusion of a finite (up to the 4th-order) number of higher-order moment futures contracts into the self-financing portfolio can further improve the performance of the above two-instruments hedging strategy, especially for short-term at-the-money (ATM) calls and OTM puts which are more sensitive to jump risk. These results have important portfolio management implications. They indicate the need, first, to introduce variance futures contracts in the traditional delta hedging strategy in order to account for volatility risk, and second, to consider higher-order
moment futures for hedging options and variance futures positions against their exposure to random jumps.

6 Appendix

In this appendix, we provide proofs of the propositions presented in the main text. For some of these proofs, we rely on a number of Lemmas which are also proved in this appendix.

6.1 Proof of Proposition 1

To prove Proposition 1, we first need to show how the conditional risk-neutral moments of future period log-return distribution can be given as the value of a composite portfolio of out-of-the money (OTM) call and put options. This is done in the next lemma.

**Lemma 1** Let $(\Omega, \mathcal{F}, Q)$ be the probability space restricted to time interval $[0, T]$, with its filtration $\mathcal{F}_t = \{\mathcal{F}_t; t \in [0, T]\}$. Let stochastic process $(S_t)_{t \in [0, T]} \in \mathbb{R}$ represent the price of a traded asset. Then, the conditional on filtration $\mathcal{F}_t$ mth-order non-central risk-neutral moment of asset log-return $\ln \left( \frac{S_T}{S_t} \right)$ is given as follows

\[
\mu_{m,t} = e^{rt} \left\{ \int_{S_t}^{S_T} G(K; m) C_t(\tau, K) dK + \int_0^{S_t} G(K; m) P_t(\tau, K) dK \right\}, 
\]

for $m = 1$, and

\[
\mu_{m,t} = e^{rt} \left\{ \int_{S_t}^{S_T} G(K; m) C_t(\tau, K) dK + \int_0^{S_t} G(K; m) P_t(\tau, K) dK \right\}, 
\]

for $m \geq 2$, where $G(K; m) = m - 1 - \ln \left( \frac{K}{S_t} \right)$, $r$ is the risk-free rate, and $C_t(\tau, K)$ and $P_t(\tau, K)$ respectively denote the prices of a European call and put options written on the asset $S$, with maturity interval $\tau = T - t$ and strike price $K$.

**Proof.** To prove Lemma 1, we first retrieve the conditional characteristic function of $\ln \left( \frac{S_T}{S_t} \right)$ from European option prices. To this end, we are based on the following result. If
$f$ is a twice continuously differentiable function, then we have

$$
\begin{align*}
f(S_T) &= f(S_t) + f'(S_t)(S_T - S_t) + \\
&\quad + \int_{S_t}^{+\infty} f''(K)(S_T - K)_+dK + \int_0^{S_t} f''(K)(K - S_T)_+dK.
\end{align*}
$$

(40) (see Bakshi and Madan (2000)). Suppose that $f : \mathbb{R}_+ \to \mathbb{C}$ and $x \mapsto e^{iu\ln(\frac{S_T}{S_t})}$, $\forall u \in \mathbb{R}$, then equation (40) implies

$$
e^{iu\ln(\frac{S_T}{S_t})} = 1 + \frac{iu}{S_t}(S_T - S_t) + \\
&\quad + \int_{S_t}^{+\infty} iu(iu - 1) \frac{1}{K^2} e^{iu\ln(\frac{K}{S_t})}(S_T - K)_+dK + \\
&\quad + \int_0^{S_t} iu(iu - 1) \frac{1}{K^2} e^{iu\ln(\frac{K}{S_t})}(K - S_T)_+dK.
$$

Multiplying both sides of the last relationship with $e^{-\tau\gamma}$ and taking the conditional expectation with respect to measure $Q$ yields

$$
\psi_t(u) = 1 + iu \left( E_t^Q \left[ \frac{S_T}{S_t} \right] - 1 \right) + \\
&\quad + e^{\tau\gamma} iu(iu - 1) \int_{S_t}^{+\infty} \frac{1}{K^2} e^{iu\ln(\frac{K}{S_t})} C_t(S_t, K)dK + \\
&\quad + e^{\tau\gamma} iu(iu - 1) \int_0^{S_t} \frac{1}{K^2} e^{iu\ln(\frac{K}{S_t})} P_t(S_t, K)dK,
$$

where $\psi_t(.)$ is the conditional characteristic function of $\ln \left( \frac{S_T}{S_t} \right)$ under the risk-neutral measure $Q$. To derive the moment functions given by Lemma 1, we first present function $\psi_t(u)$ as

$$
\psi_t(u) = 1 + iu \left( E_t^Q \left[ \frac{S_T}{S_t} \right] - 1 \right) + \\
&\quad + e^{\tau\gamma} iu(iu - 1) \int_{S_t}^{+\infty} \frac{1}{K^2} \sum_{m=0}^{\infty} \frac{(iu)^m}{m!} \ln \left( \frac{K}{S_t} \right)^m C_t(S_t, K)dK + \\
&\quad + e^{\tau\gamma} iu(iu - 1) \int_0^{S_t} \frac{1}{K^2} \sum_{m=0}^{\infty} \frac{(iu)^m}{m!} \ln \left( \frac{K}{S_t} \right)^m P_t(S_t, K)dK,
$$

40
by expanding $e^u$ as $e^u = \sum_{m=0}^{\infty} \frac{u^m}{m!}$. After tedious algebraic manipulations, the last relationship yields

\[
\psi_t(u) = 1 + iu \left( \left( E_t^Q \left[ \frac{S_T}{S_t} \right] - 1 \right) - e^{\kappa_T} \left[ \int_{S_t}^{+\infty} \frac{1}{K^2} C_t(\tau, K) dK + \int_0^{S_t} \frac{1}{K^2} P_t(\tau, K) dK \right] \right) + \sum_{m=2}^{\infty} e^{\kappa_T} \left\{ \int_{S_t}^{+\infty} G(K; m) C_t(\tau, K) dK + \int_0^{S_t} G(K; m) P_t(\tau, K) dK \right\} \frac{(iu)^m}{m!},
\]

where $G(K; m) = m! \left( \ln \left( \frac{K}{S_t} \right) \right)^{m-2} \left[ m - 1 - \ln \left( \frac{K}{S_t} \right) \right]$. Based on the last relationship, we can derive formulas (38) and (39). This can be done based on the moment representation of a characteristic function, i.e. $\psi_t(u) = \sum_{m=0}^{\infty} \mu_{m,t} \frac{(iu)^m}{m!}$, where $\mu_{m,t}$ is the $m$th-order non-central moment of the asset log-return at time $t$. ●

Given Lemma 1, next we prove Proposition 1.

**Proof.** First note that, by applying Ito’s lemma for semimartingales (see Protter (1990)) to function $S = e^X$, we have

\[
dS_u = S_u dX_u + \frac{1}{2} S_u d\langle X, X \rangle_u + \left( S_u - S_{u^-} - S_u^- \Delta X_u - \frac{1}{2} S_{u^-} (\Delta X_u)^2 \right),
\]

which implies

\[
\frac{1}{2} d\langle X, X \rangle_u = \frac{dS_u}{S_{u^-}} - dX_u + \left( \Delta X_u + \frac{1}{2} (\Delta X_u)^2 - \frac{S_u - S_{u^-}}{S_{u^-}} \right).
\]

By Taylor’s series expansion, the discontinuous component of the last relationship can be written as

\[
\Delta X_u + \frac{1}{2} (\Delta X_u)^2 - \frac{S_u - S_{u^-}}{S_{u^-}} = \Delta X_u + \frac{1}{2} (\Delta X_u)^2 + 1 - e^{\Delta X_u} = - \sum_{j=3}^{\infty} \frac{(\Delta X_u)^j}{j!},
\]

which implies that equation (41) can be written as

\[
\frac{1}{2} d\langle X, X \rangle_u = \frac{dS_u}{S_{u^-}} - dX_u - \sum_{j=3}^{\infty} \frac{(\Delta X_u)^j}{j!}.
\]
Integrating the last formula yields

\[
\frac{1}{2} \langle X, X \rangle_{T - \bar{\tau}, T} = \int_{T - \bar{\tau}}^{T} \frac{dS_u}{S_{u-}} - \left( \ln \left( \frac{S_T}{S_t} \right) - \ln \left( \frac{S_{T - \bar{\tau}}}{S_t} \right) \right) - \sum_{j=3}^{\infty} \frac{1}{j!} \left( \sum_{T - \bar{\tau} < u < T} (\Delta X_u)^j \right).
\]

(42)

Assume first that \( T - \bar{\tau} \leq t \leq T \) (see case (i) of Proposition 1). Taking risk-neutral expectations of equation (42) conditional on filtration \( \mathcal{F}_t \) and, then, multiplying both sides of the resulting equation with \( 2/\bar{\tau} \) yields

\[
F_V = \frac{1}{\bar{\tau}} \langle X, X \rangle_{T - \bar{\tau}, T} + \frac{2}{\bar{\tau}} E_t^Q \left[ \int_{t}^{T} \frac{dS_u}{S_{u-}} \right] - \frac{2}{\bar{\tau}} E_t^Q \left[ \ln \left( \frac{S_T}{S_t} \right) \right] - \frac{2}{\bar{\tau}} \sum_{j=3}^{\infty} \frac{1}{j!} E_t^Q \left[ \sum_{T - \bar{\tau} < u < T} (\Delta X_u)^j \right].
\]

Substituting the first-order risk-neutral moment, given by formula (38), into the last formula and rearranging terms yields

\[
F_V = \frac{1}{\bar{\tau}} \langle X, X \rangle_{T - \bar{\tau}, t} + \frac{2}{\bar{\tau}} E_t^Q \left[ \int_{t}^{T} \frac{dS_u}{S_{u-}} - \frac{S_T - S_t}{S_t} \right] - \frac{2}{\bar{\tau}} \sum_{j=3}^{\infty} \frac{1}{j!} E_t^Q \left[ \sum_{T - \bar{\tau} < u < T} (\Delta X_u)^j \right],
\]

where \( \tau = T - t \), which proves formula (2) of Proposition 1. Assume that \( t < T - \bar{\tau} \) (see case (ii) of Proposition 1). Again taking risk-neutral expectations of equation (42) and, then, multiplying both sides with \( 2/\bar{\tau} \) yields

\[
F_V = \frac{2}{\bar{\tau}} E_t^Q \left[ \int_{T - \bar{\tau}}^{T} \frac{dS_u}{S_{u-}} \right] - \frac{2}{\bar{\tau}} E_t^Q \left[ \ln \left( \frac{S_T}{S_t} \right) \right] + \frac{2}{\bar{\tau}} E_t^Q \left[ \ln \left( \frac{S_{T - \bar{\tau}}}{S_t} \right) \right] - \frac{2}{\bar{\tau}} \sum_{j=3}^{\infty} \frac{1}{j!} E_t^Q \left[ \sum_{T - \bar{\tau} < u < T} (\Delta X_u)^j \right].
\]
Substituting the first-order risk-neutral moment given by formula (38) into the last equation and rearranging terms yields

\[
FV_t = \frac{2}{\bar{T}} E_t^Q \left[ \int_{T-\bar{T}}^T \frac{dS_u}{S_{u-}} - \frac{S_T - S_{T-\bar{T}}}{S_t} \right] \cdot \frac{2e^{rT}}{\bar{T}} \left[ \int_{S_t}^{+\infty} \frac{1}{K^2} C_t(\tau, K) dK + \int_0^{S_t} \frac{1}{K^2} P_t(\tau, K) dK \right] - \frac{2e^{r\bar{T}}}{\bar{T}} \left[ \int_{S_t}^{+\infty} \frac{1}{K^2} C_t(\bar{T}, K) dK + \int_0^{S_t} \frac{1}{K^2} P_t(\bar{T}, K) dK \right] - \frac{2}{\bar{T}} \sum_{j=3}^{\infty} \frac{1}{j!} E_t^Q \left[ \sum_{T-\bar{T} < u < T} (\Delta X_u)^j \right],
\]

where \( \tau = T - t \) and \( \tau_1 = T - \bar{T} - t \), which proves formula (3) of Proposition 1. ■

### 6.2 Proof of Proposition 2

Given the definitions of \( \tilde{V}_{t, \tau} \) and \( V_{t, \tau} \), the approximation error \( |\tilde{V}_{t, \tau} - V_{t, \tau}| \) can be analytically written as

\[
|\tilde{V}_{t, \tau} - V_{t, \tau}| = \frac{2e^{r\bar{T}}}{\bar{T}} \left[ \int_{S_t}^{K_\infty} \frac{1}{K^2} \left( \tilde{C}_t(\tau, K) - C_t(\tau, K) \right) dK + \int_{K_0}^{S_t} \frac{1}{K^2} \left( \tilde{P}_t(\tau, K) - P_t(\tau, K) \right) dK \right] - \int_{K_\infty}^{+\infty} \frac{1}{K^2} C_t(\tau, K) dK - \int_0^{K_0} \frac{1}{K^2} P_t(\tau, K) dK \right] \leq \frac{2e^{r\bar{T}}}{\bar{T}} \left[ \int_{S_t}^{K_\infty} \frac{1}{K^2} \left| \tilde{C}_t(\tau, K) - C_t(\tau, K) \right| dK + \int_{K_0}^{S_t} \frac{1}{K^2} \left| \tilde{P}_t(\tau, K) - P_t(\tau, K) \right| dK + \varepsilon \right],
\]

where

\[
\varepsilon = \left[ \int_{K_\infty}^{+\infty} \frac{1}{K^2} C_t(\tau, K) dK + \int_0^{K_0} \frac{1}{K^2} P_t(\tau, K) dK \right].
\]

The above relationship can be rewritten as follows

\[
|\tilde{V}_{t, \tau} - V_{t, \tau}| \leq \frac{2e^{r\bar{T}}}{\bar{T}} \left[ C_{error} \int_{S_t}^{K_\infty} \frac{1}{K^2} dK + P_{error} \int_{K_0}^{S_t} \frac{1}{K^2} dK + \varepsilon \right] = \frac{2e^{r\bar{T}}}{\bar{T}} \left[ C_{error} \int_{S_t}^{y_\infty} e^{-y} dy + P_{error} \int_{y_0}^{S_t} e^{-y} dy + \varepsilon \right] = \frac{2e^{r\bar{T}}}{\bar{T}} \left[ C_{error} \left( 1 - e^{-y_\infty} \right) + P_{error} \left( e^{-y_0} - 1 \right) + \varepsilon \right],
\]

which proves equation (8) of Proposition 2.
6.3 Proof of Proposition 3

For the proof of formula (15), see Gatheral (2006b). To derive formula (16), note that (15) implies that

\[ dFV_t = V_t dt + dF(t, V_t) = \left( V_t + \frac{\partial FV}{\partial t} \right) dt + \frac{\partial FV}{\partial V} dV_t, \tag{43} \]

since \( \frac{\partial^2 FV}{\partial V^2} = 0 \). Substituting in the last relationship volatility process (13), we obtain

\[ dFV_t = \left( V_t + \frac{\partial FV}{\partial t} + \frac{\partial FV}{\partial V} \kappa (\theta - V_t) \right) dt + \frac{\partial FV}{\partial V} \sigma \sqrt{V_t} dW_t^{(2)}. \]

Since the variance futures price must be a martingale under measure \( Q \), implying \( E_t^Q [dFV_t] = 0 \), the last relationship implies that

\[ V_t + \frac{\partial FV}{\partial t} + \frac{\partial FV}{\partial V} \kappa (\theta - V_t) = \frac{\partial FV}{\partial V} \tilde{\gamma}_V V_t \tag{44} \]

or

\[ dFV_t = \frac{\partial FV}{\partial V} \tilde{\gamma}_V V_t dt + \frac{\partial FV}{\partial V} \sigma \sqrt{V_t} dW_t^{(2)}. \]

The last equation implies that the instantaneous expected return of the variance futures price under the physical measure \( P \) is given as

\[ \mu_t^{FV} dt = E_t \left[ \frac{dFV_t}{FV_t} \right] = \frac{\partial \ln FV}{\partial V} \tilde{\gamma}_V V_t dt = \frac{\partial \ln FV}{\partial V} \tilde{\gamma}_V dt, \]

which proves formula (16) of Proposition 3.

6.4 Proof of Proposition 4

The portfolio is self-financing implying that

\[ dC_t = \phi_t^{S} dS_t + \phi_t^{B} dB_t + \phi_t^{FV} dFV_t + \delta_t \phi_t^{S} S_t dt. \]

By applying Ito’s lemma to contingent claim price process \( C_t \) we obtain

\[ dC_t = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial V} dV_t + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 V_t dt + \frac{1}{2} \frac{\partial^2 C}{\partial V^2} \sigma^2 V_t dt + \frac{\partial C}{\partial S} \rho \sigma S_t V_t dt \]
Equating the last two equations and using equation (43) yields

\[ \phi_t S_t + \phi_t^B B_t r_t dt + \phi_t^FV \left( \frac{\partial FV}{\partial t} dt + V_t dt + \frac{\partial FV}{\partial V} dV_t \right) + \delta_t \phi_t S_t dt \]

\[ = \left( \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 V_t + \frac{1}{2} \frac{\partial^2 C}{\partial V^2} \sigma^2 V_t + \frac{\partial C}{\partial S} \rho \sigma S_t V_t \right) dt + \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial V} dV_t + \delta_t \phi_t S_t dt \]

Set \( \phi_t^S = \partial C/\partial S \) and \( \phi_t^FV = \frac{\partial C/\partial V}{\partial V/\partial V} = \partial C/\partial FV \), and substitute equations (17) and (44) into the last formula. This yields

\[ C_t r_t - \frac{\partial C}{\partial S} S_t (r_t - \delta_t) + \frac{\partial C}{\partial V} (V_t \gamma_V - \kappa (\theta - V_t)) \]

\[ = \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 V_t + \frac{1}{2} \frac{\partial^2 C}{\partial V^2} \sigma^2 V_t + \frac{\partial C}{\partial S} \rho \sigma S_t V_t. \]

Replacing in the last relationship the parameters of volatility process \( V_t \) with their risk-neutral counterparts and rearranging terms yields

\[ \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 V_t + \frac{\partial C}{\partial S} S_t (r_t - \delta_t) - C_t r_t + \frac{\partial C}{\partial S} \rho \sigma S_t V_t + \frac{1}{2} \frac{\partial^2 C}{\partial V^2} \sigma^2 V_t + \frac{\partial C}{\partial S} \rho \sigma S_t V_t = 0, \]

which is the partial differential equation (PDE) of contingent claim price \( C_t \), derived by Heston (1993) under equilibrium approach. This PDE implies that the drift term of \( dC_t \) is equal to

\[ r_t C_t - \frac{\partial C}{\partial S} S_t (r_t - \delta_t) + \frac{\partial C}{\partial V} (\gamma_V V_t - \kappa (\theta - V_t)). \]

Given this, the stochastic process driving \( C_t \) can be written as

\[ dC_t = \left( r_t C_t - \frac{\partial C}{\partial S} S_t (r_t - \delta_t) + \frac{\partial C}{\partial V} (\gamma_V V_t - \kappa (\theta - V_t)) \right) dt + \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial V} dV_t + \frac{\partial C}{\partial \gamma_S} S_t \gamma_S dt + \frac{\partial C}{\partial \gamma_V} V_t \gamma_V dt + \left( \frac{\partial C}{\partial S} S_t \sqrt{V_t} + \frac{\partial C}{\partial V} \sigma \sqrt{V_t} \right) dW_t^{(1)} + \frac{\partial C}{\partial \gamma \sigma} \sqrt{1 - \rho^2 \gamma \sqrt{V_t} dW_t^{(3)}}, \]

which implies

\[ E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln S} \gamma_S V_t dt + \frac{\partial \ln C}{\partial \ln V} \gamma_V V_t dt. \]
Using $\gamma_t V_t = \delta \mu_t^S + \delta_t - r_t$ and formula (16), the last equation yields

$$E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln S} (\mu_t^S + \delta_t - r_t) dt + \frac{\partial \ln C}{\partial \ln FV} \mu_t^FV dt,$$

which proves formula (18) of Proposition 4.

### 6.5 Proof of Proposition 5

Under the no-arbitrage principle, we have

$$FV_t = \frac{1}{T} E_t^Q \left[ (X, X)_{0,T} \right] = \frac{1}{T} \left( \int_0^T V_u du + \int_0^T \tilde{J}_u^2 dN_u + E_t^Q \left[ \int_0^T V_u du + \int_0^T \tilde{J}_u^2 dN_u \right] \right)$$

$$= \frac{1}{T} \left( \int_0^T V_u du + \int_0^T \tilde{J}_u^2 dN_u + F(t, V_t) + G(t) \right),$$

where $F(t, V_t) = \tau (\psi_t V_t + (1 - \psi_t) \theta^Q)$ and $\psi_t = (1 - e^{-\kappa Q \tau}) / (\kappa Q \tau)$ (see Proposition 3), and $G(t) = \lambda \left( \mu^Q \right) \tau$, where $\mu^Q$ is the second-order non-central moment of $\tilde{J}$ under measure $Q$. The last relationship proves formula (21) of Proposition 5. This relationship implies that

$$dFV_t = V_t dt + \tilde{J}_t^2 dN_t + dF(t, V_t) + dG(t)$$

$$= \left( V_t + \frac{\partial FV}{\partial t} + \frac{\partial G}{\partial t} \right) dt + \frac{\partial FV}{\partial V} dV_t + \tilde{J}_t^2 dN_t. \quad (46)$$

Substituting into the last relationship volatility process $V_t$ and equation (44), which also holds under the SVJ model, we obtain

$$dFV_t = \left( \frac{\partial FV}{\partial V} \tilde{\gamma}_t V_t + \frac{\partial G}{\partial t} \right) dt + \frac{\partial FV}{\partial V} \sigma \sqrt{V_t} dW_t^{(2)} + \tilde{J}_t^2 dN_t.$$ 

The last relationship implies that the instantaneous expected return of the variance futures contract is given as

$$\mu_t^FV dt \equiv E_t \left[ \frac{dFV_t}{FV_t} \right] = \frac{\partial \ln FV}{\partial \ln V} \tilde{\gamma}_t dt + \frac{\lambda}{FV_t} \left( \mu^{(2)}_t - \mu^Q \right) dt,$$

where $\mu^{(2)}_t$ is the second-order non-central moment of $\tilde{J}$ under $P$. This proves formula (22) of Proposition 5.
6.6 Proof of Proposition 6

Under the no-arbitrage principle, we have

\[
F_{(j),t} = \frac{1}{T} \mathbb{E}^Q \left[ \int_0^T \tilde{J}_u^j dN_u \right] = \frac{1}{T} \left( \int_0^t \tilde{J}_u^j dN_u + \mathbb{E}^Q \left[ \int_{t}^{T} \tilde{J}_u^j dN_u \right] \right)
\]

for \( j \geq 2 \), where \( \mu^Q_{(j)} \) is the \( j \)th-order non-central moment of \( \tilde{J} \) under \( Q \). The last relationship proves formula (23). To prove formula (24), note that futures contracts price \( F_{(j),t} \) follows the data-generating process:

\[
dF_{(j),t} = \tilde{J}_t^j dN_t - \lambda \mu^Q_{(j)} dt. \tag{47}
\]

Taking the conditional expectation of the last relationship yields

\[
\mu_t F_{(j)} dt \equiv \mathbb{E}_t \left[ dF_{(j),t} \right] = \lambda \left( \mu_{(j)} - \mu^Q_{(j)} \right) dt,
\]

where \( \mu_{(j)} \) is the \( j \)th-order non-central moment of \( \tilde{J} \) under the physical measure. The last relationship proves formula (24) of Proposition 6.

6.7 Proof of Proposition 7

Before proving Proposition 7, we need to prove the following lemma.

**Lemma 2** Consider contingent claim price function \( C_t = C(t, S_t, V_t) \) which is infinitely differentiable with respect to \( S \) and assume that

\[
\sup_{t^*, S^*, V^*} \sum_{n=2}^{\infty} \left| \frac{\partial^{(n)} C}{\partial \ln S^{(n)}} - \frac{\partial C}{\partial \ln S} \right| R^n < \infty \tag{48}
\]

for all \( t^*, S^*, V^* > 0 \) and \( R > 0 \). Then, the following result holds:

\[
\Delta C_t - \frac{\partial C}{\partial S} \Delta S_t = \lim_{N \to \infty} \sum_{n=2}^{N} \left( \frac{\partial^{(n)} C}{\partial \ln S^{(n)}} - \frac{\partial C}{\partial \ln S} \right) \tilde{J}_t^n \frac{n!}{n!}, \tag{49}
\]

where \( \tilde{J} = \ln(1 + J) = \ln \left( 1 + \frac{\Delta S}{S} \right) \).
Proof. By taking a Taylor’s series expansion of the discontinuous component of the contingent claim price with respect to the stock price, we have that

$$
\Delta C_t - \frac{\partial C}{\partial S} \Delta S_t = \Delta C_t - \frac{\partial C}{\partial \ln S} \Delta S_t = \Delta C_t - \frac{\partial C}{\partial \ln S} J_t = \\
= \Delta C_t - \frac{\partial C}{\partial \ln S} \lim_{N \to \infty} \left( \sum_{n=1}^{N} \frac{J^n_t}{n!} \right). \tag{50}
$$

The last equation comes from the fact that

$$
J_t = e^{\ln(1+J_t)} - 1 = \lim_{N \to \infty} \left( \sum_{n=1}^{N} \frac{J^n_t}{n!} \right).
$$

Note that formula (50) is equivalent to

$$
\Delta C_t - \frac{\partial C}{\partial \ln S} \tilde{J}_t - \frac{\partial C}{\partial \ln S} \lim_{N \to \infty} \left( \sum_{n=2}^{N} \frac{\tilde{J}^n_t}{n!} \right) = \lim_{N \to \infty} \sum_{n=2}^{N} \frac{\partial^{(n)} C}{\partial \ln S^{(n)}} \frac{\tilde{J}^n_t}{n!} - \frac{\partial C}{\partial \ln S} \lim_{N \to \infty} \left( \sum_{n=2}^{N} \frac{J^n_t}{n!} \right) = \\
= \lim_{N \to \infty} \sum_{n=2}^{N} \left( \frac{\partial^{(n)} C}{\partial \ln S^{(n)}} - \frac{\partial C}{\partial \ln S} \right) \frac{\tilde{J}^n_t}{n!},
$$

where the sum converges because of (48), which proves equation (49).

Given Lemma 2, next we prove Proposition 7.

Proof. The portfolio is self-financing portfolio which implies that

$$
dV_t(\phi) = \phi_t^S dS_t + \phi_t^B dB_t + \phi_t^{FV} dFV_t + \sum_{n=2}^{N} \phi_t^{F^{(n)}} dF_{(n),t} + \delta_t \phi_t^S S_t dt.
$$

This portfolio replicates the contingent claim price in the sense that, as $N \to \infty$, we have

$$
\lim_{N \to \infty} dV_t(\phi) = dC_t,
$$

which implies that

$$
dC_t = \phi_t^S dS_t + \phi_t^B dB_t + \phi_t^{FV} dFV_t + \lim_{N \to \infty} \sum_{n=2}^{N} \phi_t^{F^{(n)}} dF_{(n),t} + \delta_t \phi_t^S S_t dt.
$$
Applying Ito’s lemma to contingent price function \( C_t = C(t, S_t, V_t) \) and substituting equations (46) and (47) into the last relationship yields

\[
\begin{align*}
\frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial V} dV_t + 1 \frac{\partial^2 C}{2 \partial S^2} S_t^2 V_t dt + 1 \frac{\partial^2 C}{2 \partial V^2} \sigma^2 V_t dt + \frac{\partial C}{\partial S} \rho \sigma S_t V_t dt & + \left( \Delta C_t - \frac{\partial C}{\partial S} \Delta S_t \right) dN_t \\
= \phi_t^S dS_t + \phi_t^B dB_t + \phi_t^{FV} \left( V_t + \frac{\partial FV}{\partial t} + \frac{\partial G}{\partial t} \right) dt + \phi_t^{FV} \frac{\partial FV}{\partial V} dV_t + \phi_t^{FV} \tilde{\tau}_t^2 dN_t + & \\
+ \lim_{N \to \infty} \sum_{n=2}^N \phi_t^{F(n)} \left( \tilde{J}_n^t dN_t - \lambda \mu_Q^{(n)} dt \right) & + \delta_t \phi_t^S S_t dt
\end{align*}
\]

Assume now that price function \( C(t, S_t, V_t) \) is infinitely differentiable with respect to \( S \). Substituting equation (49) into the last relationship and taking a Taylor’s series expansion of the discontinuous component of the contingent claim price with respect to stock price \( S \) yields

\[
\begin{align*}
\frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial V} dV_t + 1 \frac{\partial^2 C}{2 \partial S^2} S_t^2 V_t dt + 1 \frac{\partial^2 C}{2 \partial V^2} \sigma^2 V_t dt + \frac{\partial C}{\partial S} \rho \sigma S_t V_t dt & + \lim_{N \to \infty} \left( \sum_{n=2}^N \left( \frac{\partial^{(n)} C}{\partial \ln S} - \frac{\partial C}{\partial \ln S} \right) \frac{\tilde{J}_n^t}{n!} \right) dN_t \\
= \phi_t^S dS_t + \phi_t^B dB_t + \phi_t^{FV} \left( V_t + \frac{\partial FV}{\partial t} + \frac{\partial G}{\partial t} \right) dt + \phi_t^{FV} \frac{\partial FV}{\partial V} dV_t + \phi_t^{FV} \tilde{\tau}_t^2 dN_t + & \\
+ \lim_{N \to \infty} \sum_{n=2}^N \phi_t^{F(n)} \left( \tilde{J}_n^t dN_t - \lambda \mu_Q^{(n)} dt \right) & + \delta_t \phi_t^S S_t dt.
\end{align*}
\]

Setting \( \phi_t^S = \partial C/\partial S, \phi_t^{FV} = \partial C/\partial V, \phi_t^{F(2)} = \frac{1}{2} \left( \frac{\partial^{(2)} C}{\partial \ln S} - \frac{\partial C}{\partial \ln S} \right) - \partial C/\partial FV \) and \( \phi_t^{F(j)} = \frac{1}{n!} \left( \frac{\partial^{(n)} C}{\partial \ln S^{(n)}} - \frac{\partial C}{\partial \ln S} \right), \) for \( j > 2 \), into the last relationship and using equations (17) and (44) yields

\[
\lim_{N \to \infty} \left[ \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 V_t + \frac{\partial C}{\partial S} S_t (r_t - \delta_t) - C_t r_t + \frac{\partial C}{\partial S} \rho \sigma S_t V_t + \frac{1}{2} \frac{\partial^2 C}{\partial V^2} \sigma^2 V_t + \frac{\partial C}{\partial V} \kappa_Q (\theta_Q - V_t) + \sum_{n=2}^N \frac{1}{n!} \left( \frac{\partial^{(n)} C}{\partial \ln S^{(n)}} - \frac{\partial C}{\partial \ln S} \right) \lambda \mu_Q^{(n)} \right] = 0. \quad (51)
\]
The last formula can be simplified by taking the risk-neutral conditional expectation of equation (49), which implies

$$\lim_{N \to \infty} \sum_{n=2}^{N} \frac{1}{n!} \left( \frac{\partial^{(n)} C}{\partial \ln S^{(n)}} - \frac{\partial C}{\partial \ln S} \right) \lambda \mu_t^Q = \lambda E_t^Q [\Delta C_t] - \frac{\partial C}{\partial S} S_t \lambda \mu_t^Q. \tag{52}$$

Then, (51) can be written as

$$\frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 V_t + \frac{\partial C}{\partial S} S_t (r_t - \delta_t - \lambda \mu_t^Q) - C_t r_t + \frac{\partial C}{\partial S} \rho \sigma S_t V_t + \frac{1}{2} \frac{\partial^2 C}{\partial V^2} \sigma^2 V_t$$

$$+ \frac{\partial C}{\partial V} \kappa^Q (\theta^Q - V_t) + \lambda E_t^Q [\Delta C_t] = 0, \tag{53}$$

which is the PDE derived Bates (1996) under equilibrium approach. As with the SV model, this PDE now is derived by forming a portfolio that approximately replicates contingent claim price $C_t$. Equation (53) implies that the drift term of $dC_t$ is given as

$$r_t C_t - \frac{\partial C}{\partial S} S_t (r_t - \delta_t - \lambda \mu_t^Q) - \frac{\partial C}{\partial V} \kappa^Q (\theta^Q - V_t) - \lambda E_t^Q [\Delta C_t],$$

which implies that the stochastic process of $dC_t$ can be written as

$$dC_t = \left( r_t C_t + \frac{\partial C}{\partial S} S_t \gamma_S V_t + \frac{\partial C}{\partial V} \gamma_V V_t - \lambda E_t^Q [\Delta C_t] \right) dt + \frac{\partial C}{\partial S} S_t \sqrt{V_t} dW_t^{(1)} +$$

$$+ \frac{\partial C}{\partial V} \sigma \sqrt{V_t} dW_t^{(2)} + \Delta C_t dN_t. \tag{54}$$

The last equation gives

$$E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln S} \gamma_S V_t dt + \frac{\partial \ln C}{\partial \ln V} \gamma_V dt + \lambda \left( E_t \left[ \frac{\Delta C_t}{C_t} \right] - E_t^Q \left[ \frac{\Delta C_t}{C_t} \right] \right) dt.$$

Using $\gamma_S V_t = \mu^S_t + \delta_t - r_t + \lambda (\bar{\mu} - \bar{\mu})$, where $\mu^S_t$ is the expected return of the stock at $t$ and equation (25), we can write the last relationship as

$$E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln S} (\mu^S_t + \delta_t - r_t) dt + \frac{\partial \ln C}{\partial \ln V} \left( \frac{1}{\partial \ln V / \partial \ln V} \mu^F_t - \frac{1}{\partial \ln V / \partial \ln V} F_t \right) dt$$

$$+ \lambda \left( E_t \left[ \frac{\Delta C_t}{C_t} \right] - E_t^Q \left[ \frac{\Delta C_t}{C_t} \right] + \frac{\partial \ln C}{\partial \ln S} (\bar{\mu}^Q - \bar{\mu}) \right) dt. \tag{55}$$
Using (52), the last term of the right-hand-side of the last formula can be written as

\[
\lambda \left( E_t \left[ \frac{\Delta C_t}{C_t} \right] - E_t^Q \left[ \frac{\Delta C_t}{C_t} \right] + \frac{\partial \ln C}{\partial \ln S} \left( F^Q - \bar{F} \right) \right) = \frac{1}{2} \left( \frac{\partial (C/C)}{\partial \ln S} - \frac{\partial \ln C}{\partial \ln S} \right) \mu_t^{F(2)} + \lim_{N \to \infty} \sum_{j=3}^N \frac{1}{j!} \left( \frac{\partial (C/C)}{\partial \ln S} - \frac{\partial \ln C}{\partial \ln S} \right) \mu_t^{F(j)}
\]

Given this, equation (55) can be written as

\[
E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln S} \left( \mu_t^S + \delta_t - r_t \right) dt + \frac{\partial \ln C}{\partial \ln FV} \mu_t^{FV} dt + \left( \frac{1}{2} \left( \frac{\partial (C/C)}{\partial \ln S} - \frac{\partial \ln C}{\partial \ln S} \right) - \frac{\partial \ln C}{\partial \ln FV} \right) \mu_t^{F(2)} dt + \lim_{N \to \infty} \sum_{j=3}^N \frac{1}{j!} \left( \frac{\partial (C/C)}{\partial \ln S} - \frac{\partial \ln C}{\partial \ln S} \right) \mu_t^{F(j)} dt
\]

which proves formula (29) of Proposition 7.  ■

6.8 Proof of equation (30)

From the proof of Proposition 1, we have that

\[
-\frac{2}{\tau} \sum_{j=3}^\infty \frac{1}{j!} E_t^Q \left[ \sum_{t<u\leq T} (\Delta X_u)^j \right] = \frac{2}{\tau} \left( E_t^Q \left[ \sum_{t<u\leq T} \Delta X_u \right] + \frac{1}{2} E_t^Q \left[ \sum_{t<u\leq T} (\Delta X_u)^2 \right] - E_t^Q \left[ \sum_{t<u\leq T} \frac{S_u - S_{u-}}{S_{u-}} \right] \right),
\]

where \( \tau = T - t \). Under the SVJ model, the discontinuous component of process \((S_u)_{u \in [t,T]}\) is given as \( J_t dN_t \), where \( J_t \) is the random percentage jump conditional on a jump occurring with probability \( \ln(1 + J_t) \sim N \left( \mu_j^Q, \left( \sigma_j^Q \right)^2 \right) \) and \( N_t \) is a Poisson process with intensity \( \lambda \) under measure \( Q \). Given these definitions the above relationship can be written as

\[
-\frac{2}{\tau} \sum_{j=3}^\infty \frac{1}{j!} E_t^Q \left[ \sum_{t<u\leq T} (\Delta X_u)^j \right] = \frac{2}{\tau} \left( E_t^Q \left[ \int_t^T \ln(1 + J_u) dN_u + \frac{1}{2} \int_t^T \ln(1 + J_u)^2 dN_u - \int_t^T J_u dN_u \right] \right.
\]

\[
= 2\lambda \left( \mu_j^Q + \frac{1}{2} \left( \sigma_j^Q \right)^2 + \frac{1}{2} \left( \mu_j^Q \right)^2 - e^{\mu_j^Q + (\sigma_j^Q)^2/2} + 1 \right),
\]

which proves equation (30).
References


Table 1: Estimates of the SVJ model parameters

<table>
<thead>
<tr>
<th>Estimation periods</th>
<th>$\kappa^Q$</th>
<th>$\mu^Q$</th>
<th>$\sigma_J^Q$</th>
<th>RMSE</th>
<th>RMSPE (in $)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30/3/2007-29/10/2010</td>
<td>0.72</td>
<td>-0.07</td>
<td>0.05</td>
<td>0.05</td>
<td>2.83</td>
</tr>
<tr>
<td></td>
<td>(0.043)</td>
<td>(0.002)</td>
<td>(0.001)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30/3/2007-3/9/2008</td>
<td>0.55</td>
<td>-2.08 × 10^{-6}</td>
<td>0.0001</td>
<td>0.07</td>
<td>2.49</td>
</tr>
<tr>
<td></td>
<td>(0.052)</td>
<td>(2.85 × 10^{-7})</td>
<td>(1.58 × 10^{-5})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4/9/2008-29/10/2010</td>
<td>0.83</td>
<td>-0.12</td>
<td>0.08</td>
<td>0.04</td>
<td>3.05</td>
</tr>
<tr>
<td></td>
<td>(0.062)</td>
<td>(0.003)</td>
<td>(0.002)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: This table presents average values of the cross-section estimates of the elements of vector $\Theta = (\kappa^Q, \mu^Q, \sigma_J^Q)'$, for all time points (days) of the sample. Standard errors of these values are given in parentheses. RMSE and RMSPE denote the root mean square and the root mean square pricing errors of the put option prices implied by the SVJ model, respectively. The table reports values of vector $\Theta$ for the whole sample period (March 30, 2007 to October 29, 2010), and for the following subsamples: March 30, 2007 to September 3, 2008, and September 4, 2008 to October 29, 2010.
Table 2: Average estimates of \( V_{o,t} \) with their error bounds

<table>
<thead>
<tr>
<th>Maturity intervals</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( V_{o,t} ) (in $)</td>
<td>0.0608</td>
<td>0.0591</td>
<td>0.0578</td>
<td>0.0551</td>
</tr>
<tr>
<td></td>
<td>Error bounds</td>
<td>0.0004</td>
<td>0.0003</td>
<td>0.0003</td>
<td>0.0004</td>
</tr>
</tbody>
</table>

|                    | \( V_{o,t} \) (in $) | 0.0803 | 0.0791 | 0.0781 | 0.0758 | 0.0748 |
|                    | Error bounds | 0.0242 | 0.0377 | 0.0353 | 0.0458 | 0.0473 |

Notes: The table presents average values of the estimates of \( V_{o,t} \) over the whole sample and their approximation error bounds. The sample estimates of \( V_{o,t} \) are calculated on weekly basis (every Wednesday). This is done for 5 different maturity intervals, i.e., for \( \tau = 1, 2, 3, 6, 9 \) months. The table presents two sets of results. The first assumes no extrapolation, i.e., it sets \( K_0 = K_{\min} \) and \( K_\infty = K_{\max} \), while the second it assumes constant extrapolation.
Table 3: Descriptive Statistics of $\widetilde{FV}_t - \widetilde{V}_{o,t}$

<table>
<thead>
<tr>
<th>Maturity intervals</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole sample 30/3/2007 - 29/10/2010</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.0258</td>
<td>0.0273</td>
<td>0.0288</td>
<td>0.0303</td>
<td>0.0311</td>
</tr>
<tr>
<td>t-stat</td>
<td>(13.11)</td>
<td>(14.16)</td>
<td>(15.24)</td>
<td>(17.82)</td>
<td>(19.86)</td>
</tr>
<tr>
<td>Mean</td>
<td>0.011</td>
<td>0.0121</td>
<td>0.0129</td>
<td>0.0144</td>
<td>0.0153</td>
</tr>
<tr>
<td>t-stat</td>
<td>(20.27)</td>
<td>(22.07)</td>
<td>(22.72)</td>
<td>(23.23)</td>
<td>(24.46)</td>
</tr>
<tr>
<td>Subsample 4/9/2008 - 29/10/2010</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.0356</td>
<td>0.0373</td>
<td>0.0393</td>
<td>0.0409</td>
<td>0.0416</td>
</tr>
</tbody>
</table>

Notes: The table presents average values of difference $\widetilde{FV}_t - \widetilde{V}_{o,t}$. The latter are based on weekly (every Wednesday) estimates of $V_{o,t}$. These are obtained based on the interpolation-extrapolation scheme that sets $K_0 = K_{\min}$ and $K_\infty = K_{\max}$, i.e. it chooses not to extrapolate the implied volatility function. In parentheses, we present values of the test statistic (denoted t-stat) if these values are different than zero.
Table 4: Regression analysis of $LNFVO_t$

<table>
<thead>
<tr>
<th>Maturity intervals</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A:</strong> Regression model $LNFVO_t = \beta_0 + \beta_1 LNJT_t + \varepsilon_t$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LS estimates</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>0.40</td>
<td>0.18</td>
<td>-0.14</td>
<td>-0.50</td>
<td>-0.80</td>
</tr>
<tr>
<td></td>
<td>(0.08)</td>
<td>(0.09)</td>
<td>(0.10)</td>
<td>(0.10)</td>
<td>(0.11)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.69</td>
<td>0.63</td>
<td>0.56</td>
<td>0.48</td>
<td>0.42</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.02)</td>
<td>(0.02)</td>
<td>(0.02)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>$\bar{R}^2$</td>
<td>0.94</td>
<td>0.93</td>
<td>0.90</td>
<td>0.87</td>
<td>0.83</td>
</tr>
<tr>
<td>FMLS estimates</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>0.37</td>
<td>0.13</td>
<td>-0.28</td>
<td>-0.68</td>
<td>-1.04</td>
</tr>
<tr>
<td></td>
<td>(0.11)</td>
<td>(0.18)</td>
<td>(0.18)</td>
<td>(0.20)</td>
<td>(0.22)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.68</td>
<td>0.63</td>
<td>0.54</td>
<td>0.45</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.03)</td>
<td>(0.03)</td>
<td>(0.03)</td>
<td>(0.04)</td>
</tr>
<tr>
<td>$Z_t$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.02</td>
<td>0.08</td>
</tr>
<tr>
<td>$Z_\alpha$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.02</td>
<td>0.07</td>
</tr>
<tr>
<td><strong>Panel B:</strong> Regression model $LNFVO_t = \beta_0 + \beta_1 LN\mu_{3,t} + \varepsilon_t$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LS estimates</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>2.24</td>
<td>1.22</td>
<td>0.35</td>
<td>-0.74</td>
<td>-1.18</td>
</tr>
<tr>
<td></td>
<td>(0.15)</td>
<td>(0.12)</td>
<td>(0.11)</td>
<td>(0.10)</td>
<td>(0.12)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.68</td>
<td>0.64</td>
<td>0.56</td>
<td>0.45</td>
<td>0.41</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.01)</td>
<td>(0.01)</td>
<td>(0.02)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>$\bar{R}^2$</td>
<td>0.90</td>
<td>0.91</td>
<td>0.89</td>
<td>0.82</td>
<td>0.70</td>
</tr>
<tr>
<td>FMLS estimates</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>2.29</td>
<td>1.30</td>
<td>0.45</td>
<td>-0.17</td>
<td>-0.78</td>
</tr>
<tr>
<td></td>
<td>(0.18)</td>
<td>(0.21)</td>
<td>(0.23)</td>
<td>(0.30)</td>
<td>(0.58)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.69</td>
<td>0.65</td>
<td>0.57</td>
<td>0.55</td>
<td>0.49</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.03)</td>
<td>(0.03)</td>
<td>(0.05)</td>
<td>(0.09)</td>
</tr>
<tr>
<td>$Z_t$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$Z_\alpha$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>
Notes: The table presents LS and FMLS estimates of the regression coefficients of $LNFV_{O_t}$ (log-transform of $FV_{t} - \bar{V}_{a,t}$) on $LNJT_t$ (log-transform of $JT_t$) (see Panel A), and of $LNFV_{O_t}$ on $LN\mu_{3,t}$ (log-transform of $-\mu_{3,t}$) (see Panel B). The estimation period of the first regression is from 4/9/2008 to 29/10/2010, while of the second is the whole sample. Standard errors are in parentheses. $\bar{R}^2$ is the coefficient of determination of the LS regression. $Z_t$ and $Z_\alpha$ are Phillips-Ouliaris’ test statistics for cointegration. Their $p$-values are reported in the table.
### Table 5A: Evaluation of alternative hedging strategies (30/3/2007 - 29/10/2010)

<table>
<thead>
<tr>
<th>$K/S_t$</th>
<th>RMSE (in $)</th>
<th>MAE (in $)</th>
<th>$\sum$ (in $)</th>
<th>(J)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>15-35</td>
<td>40-60</td>
<td>100-160</td>
<td>15-35</td>
</tr>
<tr>
<td>0.8 (put)</td>
<td>$HP_1$</td>
<td>1.02</td>
<td>1.29</td>
<td>1.78</td>
</tr>
<tr>
<td></td>
<td>$HP_2$</td>
<td>0.98</td>
<td>1.11</td>
<td>1.18</td>
</tr>
<tr>
<td></td>
<td>$HP_3$</td>
<td>0.98</td>
<td>1.08</td>
<td>1.17</td>
</tr>
<tr>
<td>0.9 (put)</td>
<td>$HP_1$</td>
<td>1.95</td>
<td>2.14</td>
<td>2.37</td>
</tr>
<tr>
<td></td>
<td>$HP_2$</td>
<td>1.83</td>
<td>1.80</td>
<td>1.56</td>
</tr>
<tr>
<td></td>
<td>$HP_3$</td>
<td>1.73</td>
<td>1.71</td>
<td>1.54</td>
</tr>
<tr>
<td>1 (call)</td>
<td>$HP_1$</td>
<td>3.72</td>
<td>3.60</td>
<td>3.40</td>
</tr>
<tr>
<td></td>
<td>$HP_2$</td>
<td>3.44</td>
<td>3.05</td>
<td>2.46</td>
</tr>
<tr>
<td></td>
<td>$HP_3$</td>
<td>2.95</td>
<td>2.77</td>
<td>2.35</td>
</tr>
<tr>
<td>1.1 (call)</td>
<td>$HP_1$</td>
<td>1.99</td>
<td>2.85</td>
<td>3.49</td>
</tr>
<tr>
<td></td>
<td>$HP_2$</td>
<td>1.80</td>
<td>2.35</td>
<td>2.51</td>
</tr>
<tr>
<td></td>
<td>$HP_3$</td>
<td>1.99</td>
<td>2.22</td>
<td>2.42</td>
</tr>
<tr>
<td>1.2 (call)</td>
<td>$HP_1$</td>
<td>1.05</td>
<td>1.93</td>
<td>3.18</td>
</tr>
<tr>
<td></td>
<td>$HP_2$</td>
<td>0.96</td>
<td>1.64</td>
<td>2.37</td>
</tr>
<tr>
<td></td>
<td>$HP_3$</td>
<td>1.15</td>
<td>1.75</td>
<td>2.34</td>
</tr>
</tbody>
</table>

Notes: The table presents values (in US dollars) of the RMSE and MAE metrics, as well as average over the whole sample period values of delta-hedged gains (denoted $\sum$) of the following hedging strategies: $HP_1$, $HP_2$ and $HP_3$. $HP_1$ considers only a short position in the underlying stock. $HP_2$ is a two-instruments strategy, which considers short positions in the underlying stock and a variance futures contract. $HP_3$ is a three-instruments strategy, which considers short positions in the underlying stock, a variance futures contract and a put option.
these two instruments, includes the bipower and higher-order moment futures in the hedging portfolio. The number of instruments are based on the daily estimates of the SVJ model, presented in Section 4.2. We consider a daily rebalancing, calculating hedging errors $H(t + \Delta t)$ at each time-point (day) of the sample. The $p$-values reported in the table are of statistic $J$-stat. This test the null hypothesis that there are no hedging benefits between a pair of hedging strategies $HP_k$ and $HP_l$, for $k \neq l$. It is distributed as $\chi^2$ with one degree of freedom.
Table 5B: Evaluation of alternative hedging strategies (30/3/2007 - 3/9/2008)

<table>
<thead>
<tr>
<th></th>
<th>RMSE (in $)</th>
<th>MAE (in $)</th>
<th>Σ (in $)</th>
</tr>
</thead>
<tbody>
<tr>
<td>K/S_t</td>
<td>15-35 40-60</td>
<td>100-160</td>
<td>15-35 40-60</td>
</tr>
<tr>
<td>0.8 (put)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HP_1</td>
<td>0.46 0.90 1.58</td>
<td>0.24 0.59 1.11</td>
<td>-0.15 -0.21 0.02</td>
</tr>
<tr>
<td>HP_2</td>
<td>0.44 0.81 1.13</td>
<td>0.23 0.53 0.79</td>
<td>-0.14 -0.19 0.06</td>
</tr>
<tr>
<td>HP_3</td>
<td>0.43 0.80 1.12</td>
<td>0.23 0.52 0.78</td>
<td>-0.17 -0.25 0.01</td>
</tr>
<tr>
<td>0.9 (put)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HP_1</td>
<td>1.38 1.84 2.26</td>
<td>0.86 1.28 1.62</td>
<td>-0.57 -0.35 0.08</td>
</tr>
<tr>
<td>HP_2</td>
<td>1.30 1.60 1.60</td>
<td>0.80 1.10 1.13</td>
<td>-0.51 -0.32 0.09</td>
</tr>
<tr>
<td>HP_3</td>
<td>1.23 1.54 1.58</td>
<td>0.76 1.08 1.12</td>
<td>-0.76 -0.57 -0.02</td>
</tr>
<tr>
<td>1 (call)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HP_1</td>
<td>2.99 3.03 3.06</td>
<td>2.21 2.25 2.24</td>
<td>-0.54 -0.32 0.10</td>
</tr>
<tr>
<td>HP_2</td>
<td>2.78 2.63 2.33</td>
<td>2.07 1.97 1.70</td>
<td>-0.37 -0.31 0.08</td>
</tr>
<tr>
<td>HP_3</td>
<td>2.45 2.46 2.27</td>
<td>1.88 1.89 1.67</td>
<td>-1.51 -0.93 -0.15</td>
</tr>
<tr>
<td>1.1 (call)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HP_1</td>
<td>1.30 1.91 2.62</td>
<td>0.76 1.28 1.83</td>
<td>0.07 -0.04 0.06</td>
</tr>
<tr>
<td>HP_2</td>
<td>1.21 1.61 1.85</td>
<td>0.71 1.07 1.27</td>
<td>0.12 -0.03 0.04</td>
</tr>
<tr>
<td>HP_3</td>
<td>1.23 1.57 1.79</td>
<td>0.70 1.03 1.24</td>
<td>-0.32 -0.60 -0.28</td>
</tr>
<tr>
<td>1.2 (call)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HP_1</td>
<td>0.52 0.86 1.79</td>
<td>0.20 0.48 1.14</td>
<td>-0.01 0.13 0.01</td>
</tr>
<tr>
<td>HP_2</td>
<td>0.50 0.71 1.21</td>
<td>0.19 0.41 0.73</td>
<td>0.01 0.11 -0.01</td>
</tr>
<tr>
<td>HP_3</td>
<td>0.48 0.71 1.22</td>
<td>0.19 0.40 0.72</td>
<td>-0.10 -0.16 -0.30</td>
</tr>
</tbody>
</table>

Notes: See Table 5A. The sample interval is 30/3/2007 - 3/9/2008.
Table 5C: Evaluation of alternative hedging strategies (4/9/2008-29/10/2010)

<table>
<thead>
<tr>
<th>$K/S_t$</th>
<th>15-35</th>
<th>40-60</th>
<th>100-160</th>
<th>15-35</th>
<th>40-60</th>
<th>100-160</th>
<th>15-35</th>
<th>40-60</th>
<th>100-160</th>
<th>$J$-stat ($p$-values)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8 (put)</td>
<td>$HP_1$</td>
<td>1.26</td>
<td>1.49</td>
<td>1.90</td>
<td>0.56</td>
<td>0.86</td>
<td>1.26</td>
<td>-0.15</td>
<td>-0.15</td>
<td>-0.13</td>
</tr>
<tr>
<td></td>
<td>$HP_2$</td>
<td>1.20</td>
<td>1.27</td>
<td>1.20</td>
<td>0.54</td>
<td>0.73</td>
<td>0.75</td>
<td>-0.15</td>
<td>-0.13</td>
<td>-0.06</td>
</tr>
<tr>
<td></td>
<td>$HP_3$</td>
<td>1.21</td>
<td>1.22</td>
<td>1.19</td>
<td>0.50</td>
<td>0.70</td>
<td>0.74</td>
<td>-0.01</td>
<td>-0.01</td>
<td>-0.02</td>
</tr>
<tr>
<td>0.9 (put)</td>
<td>$HP_1$</td>
<td>2.24</td>
<td>2.31</td>
<td>2.44</td>
<td>1.21</td>
<td>1.46</td>
<td>1.59</td>
<td>-0.18</td>
<td>-0.29</td>
<td>-0.20</td>
</tr>
<tr>
<td></td>
<td>$HP_2$</td>
<td>2.10</td>
<td>1.91</td>
<td>1.53</td>
<td>1.12</td>
<td>1.19</td>
<td>0.94</td>
<td>-0.14</td>
<td>-0.25</td>
<td>-0.13</td>
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<td>$HP_3$</td>
<td>1.99</td>
<td>1.81</td>
<td>1.52</td>
<td>1.01</td>
<td>1.12</td>
<td>0.93</td>
<td>0.78</td>
<td>0.04</td>
<td>-0.09</td>
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<td>1 (call)</td>
<td>$HP_1$</td>
<td>4.12</td>
<td>3.92</td>
<td>3.60</td>
<td>2.46</td>
<td>2.44</td>
<td>2.19</td>
<td>-0.06</td>
<td>-0.26</td>
<td>-0.22</td>
</tr>
<tr>
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<td>$HP_2$</td>
<td>3.80</td>
<td>3.29</td>
<td>2.54</td>
<td>2.24</td>
<td>2.00</td>
<td>1.39</td>
<td>0.11</td>
<td>-0.22</td>
<td>-0.16</td>
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<tr>
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<td>$HP_3$</td>
<td>3.24</td>
<td>2.95</td>
<td>2.40</td>
<td>2.18</td>
<td>1.90</td>
<td>1.35</td>
<td>0.20</td>
<td>0.14</td>
<td>-0.15</td>
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<tr>
<td>1.1 (call)</td>
<td>$HP_1$</td>
<td>2.32</td>
<td>3.31</td>
<td>3.95</td>
<td>1.00</td>
<td>1.74</td>
<td>2.31</td>
<td>0.32</td>
<td>0.06</td>
<td>-0.17</td>
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<td>$HP_2$</td>
<td>2.09</td>
<td>2.72</td>
<td>2.86</td>
<td>0.89</td>
<td>1.37</td>
<td>1.47</td>
<td>0.33</td>
<td>0.03</td>
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</tr>
<tr>
<td></td>
<td>$HP_3$</td>
<td>2.34</td>
<td>2.55</td>
<td>2.74</td>
<td>1.20</td>
<td>1.38</td>
<td>1.44</td>
<td>0.41</td>
<td>-0.94</td>
<td>-0.20</td>
</tr>
<tr>
<td>1.2 (call)</td>
<td>$HP_1$</td>
<td>1.28</td>
<td>2.38</td>
<td>3.81</td>
<td>0.37</td>
<td>0.91</td>
<td>1.93</td>
<td>-0.02</td>
<td>0.01</td>
<td>-0.09</td>
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<td>$HP_2$</td>
<td>1.16</td>
<td>2.03</td>
<td>2.87</td>
<td>0.34</td>
<td>0.75</td>
<td>1.26</td>
<td>-0.04</td>
<td>-0.03</td>
<td>-0.08</td>
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<td>$HP_3$</td>
<td>1.42</td>
<td>2.16</td>
<td>2.84</td>
<td>0.41</td>
<td>0.81</td>
<td>1.25</td>
<td>-0.12</td>
<td>-0.24</td>
<td>-0.25</td>
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</tbody>
</table>

Notes: See Table 5A. The sample period is 4/9/2008 - 29/10/2010.
Table 6: Delta-hedged gains, volatility and jump risk premia

<table>
<thead>
<tr>
<th>Maturity</th>
<th>$\Delta \hat{\Sigma}_t^{JP1}/S_t$</th>
<th>$\Delta \hat{\Sigma}_t^{JP2}/S_t$</th>
<th>$\Delta \hat{\Sigma}_t^{JP3}/S_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15-35</td>
<td>$-0.0001$</td>
<td>$-0.026$</td>
<td>$-0.71$</td>
</tr>
<tr>
<td></td>
<td>(0.0002)</td>
<td>(0.008)</td>
<td>(0.12)</td>
</tr>
<tr>
<td>40-60</td>
<td>$5.6 \times 10^{-5}$</td>
<td>$-0.029$</td>
<td>$-0.75$</td>
</tr>
<tr>
<td></td>
<td>(0.0005)</td>
<td>(0.01)</td>
<td>(0.14)</td>
</tr>
<tr>
<td>100-160</td>
<td>$-2.2 \times 10^{-5}$</td>
<td>$-0.015$</td>
<td>$-0.8$</td>
</tr>
<tr>
<td></td>
<td>(0.0003)</td>
<td>(0.007)</td>
<td>(0.17)</td>
</tr>
</tbody>
</table>

Notes: The table presents GMM estimates of regression models (37), for $j = 1, 2, 3$, based on ATM options. This is done for the following maturity intervals: 15-35, 40-60 and 100-160 days, using the whole sample period 30/3/2007 - 29/10/2010. Newey-West standard errors correcting for heteroscedasticity and serial correlation are in parentheses. The instruments used in the GMM estimation procedure are: the constant, and two-periods back lagged values of the dependent variable and the two independent variables.
Figure 1: Values of function $f(y)$, defined by (9).
Figure 2: This figure presents values of $\phi^{(2)}$, $\phi^{(3)}$ and $\phi^{(4)}$ with respect to strike price $K$, for maturity intervals of 1 and 6 months. The vertical line in the middle of graphs indicates current stock price.
Figure 3: This figure presents estimates of $\pi^Q$ (top graph) and $\sigma^Q$ (bottom graph) from March 3, 2007 to October 29, 2010. The vertical line corresponds to September 4, 2008.
Figure 4: Weekly estimates of $\widehat{FV}_t - \widehat{V}_{o,t}$ (solid line) and $JT_t$ (dashed line), for $\tau = 1$ month.