The primal versus the dual approach to the Ramsey tax problem:  
A note

by

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Abstract: It is known that there are two solution approaches to the dynamic optimal taxation (Ramsey) problem: the primal and the dual. All papers that have fully solved the Ramsey problem (by full solution, we also mean a quantitative solution of the whole optimal path) have used the primal approach; this is because it is simpler than the dual. This short paper fully solves a Ramsey tax policy problem by applying both approaches, and compares them.

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1. Introduction

The dynamic optimal taxation problem, referred as the Ramsey problem, is one of the most fundamental and influential policy problems. In this problem, the government chooses its tax policy to maximize households’ welfare by taking into account the equilibrium reaction of private agents to the tax policy. A celebrated result is that the tax rate on (capital) income should be high in the initial periods and then roughly zero (Chamley, 1986, and Judd, 1985).¹

Two approaches have been used to solve this problem: the primal and the dual. In the primal, we eliminate taxes and prices, so that the government can be thought as directly using the quantities as controls. In the dual, the government uses the tax rates or prices as controls. All papers that have “fully” solved the Ramsey problem (by full solution, we mean not only the above celebrated qualitative result but also a quantitative solution of the whole optimal path)² have used the primal approach.³ We are not aware of a full solution to a dual problem. This is because, as is widely recognized (see e.g. Jones et al., 1997, p. 99), the primal is considerably simpler than the dual.

This short paper fully solves a Ramsey tax policy problem by applying both approaches, and compares them.⁴ To make the analysis clear and tractable, we choose a simple setup. The model is as in the literature except that we use a single income tax and a linear AK production technology. The former is not important because an income tax inherits the features of a capital income tax. The AK technology has the advantage that there are no transition dynamics within each tax policy regime; this reduces the complexity of the Ramsey problem without affecting the key points.

As is known, there are three tax policy regimes: an initial period during which the tax rate is exogenously given; the period(s) of relatively heavy chosen tax rates; and the later period(s) of chosen zero tax rates. We will therefore present three subsets of equations associated with these

¹ See Chari and Kehoe (1999), Lansing (1999), Guo and Lansing (1999) and Ljungqvist and Sargent (2000, chapter 12) for reviews of the literature. The optimal long-run tax rate can be different from zero if there are imperfections. But the key logic remains: it is optimal to heavily tax inelastically supplied inputs.
² As Ljungqvist and Sargent (2000, p. 349) point out, a qualitative analysis provides important insights but cannot yield definite results. What is the value of the non-zero tax rate(s) in the initial period(s)? What is for consumption, growth, etc, over time? These questions have to be studied numerically.
³ In the Ramsey problem, irrespectively of the solution approach used, one cannot solve first for the long run and in turn study the transition period, as is typically the case in growth models. Instead, one has to solve simultaneously for the long run and the whole optimal path. Chari et al. (1994) were the first ones who obtained numerical results by applying the primal approach. But, as far as we know, there are no analogous results by applying the dual approach (Chamley, 1986, p. 618, provided a sketch of a full dual solution).
⁴ Solving the problem also under the dual approach is not only for intellectual curiosity. In addition to the big number of papers that have used the dual approach to get qualitative results, there are cases (in richer setups) where it is not possible to reduce the constraints to the Ramsey problem into a simple implementability (budget) constraint and a resource equation. Also, as Glomm and Ravikumar (1997, p. 192) point out, existence and uniqueness in the dual problem are more difficult to obtain.
three regimes and show how the three subsets are interconnected to give a system of equations that characterizes the full Ramsey problem. Beyond this, we will solve the Ramsey system numerically. We will do so under both the primal and the dual approach.

The main results are as follows. The dual problem is indeed more complex to solve (in terms of equations and unknowns). It is hence harder to find ranges of parameter values, initial conditions and exogenous government spending that yield a dual solution. The primal solution is easier to obtain. This can partly explain the lack of full, numerical solutions to the dual problem. Nevertheless, given a solution, the two approaches give identical results along the whole optimal path as it should be expected.

The rest of the paper is as follows. Section 2 presents the economy and solves for its competitive equilibrium. Section 3 solves the Ramsey policy problem by using the primal approach, while Section 4 uses the dual approach. Section 5 concludes.

2. Competitive equilibrium

2.1 Description of the economy and how we are going to work

Consider a closed economy with an individual agent and a government. The individual consumes, saves in the form of capital and government bonds, and produces a single good according to a linear $AK$ production technology. The government imposes an income tax and issues bonds to finance public services, where the latter enter the individual’s utility function. We assume discrete time, infinite time-horizons and perfect foresight. The government is benevolent and chooses the path of the tax rate once-and-for-all at time 0 by taking into account the competitive equilibrium.5

After we present the competitive equilibrium, we solve the primal problem. The dual will follow next. This is for convenience.

2.2 Individuals

The representative individual maximizes:

$$\sum_{t=0}^{\infty} \beta^t [\nu \ln C_t + (1-\nu) \ln H_t]$$

(1)

---

5 Following most of the literature, we assume that government expenditure is exogenous. We report that our main results do not change if government expenditure is chosen optimally jointly with the tax rates.
where \( C_t \) and \( H_t \) are respectively private consumption and public consumption at \( t \), \( 0 < \beta < 1 \) is the discount rate and \( 0 < \nu < 1 \) is the weight given to private relative to public consumption. We use a log-linear utility function for simplicity (see subsection 3.1 below).

The within-period budget constraint of the individual is:

\[
K_{t+1} - K_t + B_{t+1} - B_t + C_t = (1 - \tau_t) A(K_t + B_t)
\]

(2)

where \( K_{t+1} \) is end-of-period capital, \( B_{t+1} \) is end-of-period bonds, \( 0 < \tau_t < 1 \) is the tax rate at \( t \) and \( A > 0 \) is a parameter. We assume for simplicity that capital and bonds pay the same gross return, \( A \), and are taxed at the same rate, \( \tau_t \). We also assume zero capital depreciation. The initial stocks, \( K_0 \) and \( B_0 \), are given. The household chooses the paths of \( \{C_t, K_{t+1}, B_{t+1}\}_{t=0}^\infty \) to maximize (1) subject to (2). In doing so it takes policy variables as given.

2.3 Government budget constraint

To finance its expenditure, \( H_t \), the government taxes all types of income at a rate \( 0 < \tau_t < 1 \) and issues bonds. The within-period government budget constraint is:

\[
B_{t+1} - B_t = H_t + AB_t - \tau_t A(K_t + B_t)
\]

(3)

2.4 Competitive equilibrium (CE)

Given the paths of the independent policy instruments \( \{\tau_t, H_t\}_{t=0}^\infty \) and initial conditions for \( K_0 \) and \( B_0 \), a CE is an allocation \( \{C_t, K_{t+1}, B_{t+1}\}_{t=0}^\infty \) such that the individual’s problem is solved, markets clear and budget constraints are satisfied.

There are two equivalent ways of presenting the CE. In the first, the CE is summarized by the resource constraint holding in each period and a single implementability (budget) constraint in period 0 (see Appendix A that also compares our implementability constraint (4b) to the literature):\(^6\)

\[
C_t + H_t + K_{t+1} - K_t = AK_t
\]

\[
\frac{1}{1 - \beta} = \frac{[1 + (1 - \tau_0)A] (K_0 + B_0)}{C_0}
\]

(4a)

(4b)

\(^6\) \( K_0 \) and \( B_0 \) are exogenously given initial stocks, while period-0 consumption, \( C_0 \), is endogenous. Thus, in our setup, \( C_0 \) follows from (4b); see also below.
where, in (4a)-(4b), we have eliminated prices and taxes apart from \( \tau_0 \), which is exogenously given to make the policy problem nontrivial (see below).

Secondly, and equivalently, the CE can be summarized by the individual Euler equation, the resource constraint and government budget constraint in each period (see Appendix A):

\[
\frac{C_{t+1}}{C_t} = \beta (1 + \bar{r}_{t+1}) \tag{5a}
\]

\[
C_t + H_t + K_{t+1} - K_t = AK_t \tag{5b}
\]

\[
B_{t+1} - B_t = H_t + \bar{r}_t B_t + \bar{r}_t K_t - AK_t \tag{5c}
\]

where \( \bar{r}_t = (1 - \tau_t)A \) is the net (after tax) return to assets.

Concerning the exogenous policy instruments, we assume that the government sets its expenditure as an exogenous fraction of the beginning-of-period capital stock, \( \frac{H_t}{K_t} = h_t \) (this satisfies stationarity). For simplicity, in the computations below, we will assume that \( h_t = h \) is constant over time. Also, the period-0 tax rate, \( 0 \leq \tau_0 < 1 \), will be taken as given; otherwise the government would use it as a lump-sum tax which makes the policy problem first-best and hence trivial (see e.g. Ljungqvist and Sargent, 2000, pp. 323-4).

3. The primal approach to the Ramsey problem

In the primal approach, the government chooses the paths of \( \{C_t, K_{t+1}\}_{t=0}^\infty \) to maximize (1) subject to (4a)-(4b). To make our results easily comparable, we follow Chari et al. (1994) and Ljungqvist and Sargent (2000, pp. 321-2).\(^7\) The Lagrangian is:

\[
\sum_{t=0}^{\infty} \beta^t \left[ \nu \ln C_t + (1 - \nu) \ln H_t + \lambda_t [AK_t - C_t - H_t - K_{t+1} + K_t] \right] + \\
+ \frac{\xi}{1 - \beta} \left[ \frac{1}{C_0} - \frac{(K_0 + B_0)[1 + (1 - \tau_0)A]}{C_0} \right] \tag{6}
\]

\(^7\) Well-known papers that use the primal approach include Lucas and Stokey (1983), Lucas (1990), Chari et al. (1994) and Jones et al. (1997).
where \( \xi \geq 0 \) is an atemporal multiplier associated with (4b) and \( \lambda_0 \geq 0 \) is a dynamic multiplier associated with (4a).

As is known (see e.g. Chari et. al, 1994, p. 625, and Ljungqvist and Sargent, 2000, p. 322), the period-0 allocations differ from the same rules governing behavior from period 1 onward. This is because the period-0 first-order conditions include terms related to the initial stock of assets, \( K_0 \) and \( B_0 \). Specifically, at \( t = 0 \) the first-order conditions for \( C_0 \) and \( K_1 \) are:

\[
\lambda_0 = \frac{\nu}{C_0} + \frac{\xi \left( K_0 + B_0 \right) \left[ 1 + (1 - \tau_0) A \right]}{C_0^2} \quad (7a)
\]

\[
\frac{\lambda_0}{\lambda_1} = \beta (1 + A) \quad (7b)
\]

while, at \( t \geq 1 \) the first-order conditions for \( C_t \) and \( K_{t+1} \) are:

\[
\lambda_t = \frac{\nu}{C_t} \quad (8a)
\]

\[
\frac{\lambda_t}{\lambda_{t+1}} = \beta (1 + A) \quad (8b)
\]

In addition, in all periods \( t \geq 0 \), the first-order conditions include the constraints to the government’s problem, namely (4a-b).

3.1 Qualitative features

In the absence of exogenous upper bounds on the tax rate, there can be one period only with nonzero taxation, and this is at \( t = 1 \). Actually, it is straightforward to show by working as in Chari et al. (1994, pp. 629-630) that, in this class of utility functions (see equation (1) above), the optimal tax rate is zero at \( t = 2 \) onward. Thus, there are three tax policy regimes that correspond to \( t = 0 \), \( t = 1 \) and \( t \geq 2 \), where \( \tau_0 \) is exogenously given, \( \tau_1 > 0 \) and \( \tau_t = 0 \) for \( t \geq 2 \). This is confirmed below.

3.2 The full Ramsey system

We now present the full system. We work in two steps. First, we combine the first-order conditions - equations (7a-b), (8a-b) and the constraints (4a-b) - so as to satisfy continuity across policy
regimes. Recall that there are three distinct policy regimes, which correspond to periods \( t = 0 \), \( t = 1 \) and \( t \geq 2 \). Second, since the model allows for long-term growth, we transform the variables to make them stationary. In particular, we define \( c_t \equiv \frac{C_t}{K_t} \) at all \( t \), which is a jump variable. Thus, as in the basic \( AK \) model, after period 2 there are no transitional dynamics; as soon as the zero tax rate regime starts at \( t = 2 \) with given values of capital and bonds, \( K_2 \) and \( B_2 \), all stationary variables jump to their long-run values where all quantities grow at the same constant rate.\(^9\) Therefore, we have (see Appendix B for details):

\[
\begin{align*}
\Gamma_0 c_t [1 + A - c_t - h_t] &= \beta v (1 + A) \quad \text{First policy regime, } t = 0 \\
\frac{c}{c_1} &= \frac{\beta (1 + A)}{1 + A - c_1 - h_1} \quad \text{Second policy regime, } t = 1 \\
1 &= \frac{\beta (1 + A)}{1 + A - c - h} \quad \text{Third policy regime, } t \geq 2
\end{align*}
\]

where \( \Gamma_0 \equiv \frac{v}{c_0} \left( \frac{1 + b_0}{c_0^2} \right) \left[ 1 + (1 - \tau_0)A \right] \). Throughout, numbers in subscripts denote time periods, while variables without time subscripts denote long-run values (here the long run is reached at \( t = 2 \)).

We also have the implementability constraint (4b) rewritten in stationary form as:

\[
\frac{1}{1 - \beta} = \frac{(1 + b_0)[1 + (1 - \tau_0)A]}{c_0}
\]

(9a-d) summarize the Ramsey problem. We have four equations in four unknowns which are \( c_0, c_1, c, \xi \). This is given the path of \( h_t \), the initially given \( b_0 \equiv \frac{B_0}{K_0} \) and the period-0 tax rate, \( \tau_0 \).

In turn, the implied optimal tax rate \((\tau_t \text{ for } t \geq 1)\) can follow from the individual’s optimality conditions (see Appendix B). This is a simple system that can be solved even analytically.

### 3.3 Numerical solution

We solve the system (9a-d) numerically to make our results comparable to those in section 4 below. As a baseline case, we set the following values for parameters, \( v = 0.85 \), \( A = 1 \),

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\(^9\) See Chari et al. (1994, pp. 632-4) for a richer model with transition dynamics within each regime.
\( \beta = 0.9; \) initial conditions, \( b_0 = \frac{B_0}{K_0} = 0.25; \) and exogenous variables, \( \tau_0 = 0.4 \) and \( h_t = \frac{H_t}{K_t} = 0.0667. \)

Solutions are in Table 1. We also report the implied values of the gross growth rates of consumption and capital, \( C_{t+1}/C_t \) and \( K_{t+1}/K_t \), and the tax rate, \( \tau_t \).

<table>
<thead>
<tr>
<th>endogenous variables</th>
<th>( c_t )</th>
<th>( \tau_t )</th>
<th>( \frac{C_{t+1}}{C_t} )</th>
<th>( \frac{K_{t+1}}{K_t} )</th>
<th>( \xi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>tax policy regimes</td>
<td>( t = 0 )</td>
<td>0.20</td>
<td>0.4</td>
<td>1.15</td>
<td>1.73</td>
</tr>
<tr>
<td></td>
<td>( t = 1 )</td>
<td>0.13</td>
<td>0.71</td>
<td>1.80</td>
<td>1.80</td>
</tr>
<tr>
<td></td>
<td>( t \geq 2 )</td>
<td>0.13</td>
<td>0</td>
<td>1.80</td>
<td>1.80</td>
</tr>
</tbody>
</table>

Note: We use Matlab 7.0.

All values make sense. For instance, the tax rate in the first period is positive, \( \tau_1 = 0.71 \), while it is zero in the long run (at \( t \geq 2 \)). The value of the atemporal multiplier associated with the implementability constraint, \( \xi \), is positive. It is worth pointing out that, except for the tax rate, all other variables jump to their long run values in the first period (this is a property of the \( AK \) model). We report that comparative static exercises give intuitive results. For instance, a higher \( \tau_0 \) leads to a fall in \( \tau_t \) and an increase in the growth rate of consumption and capital in period 1 onward. A higher \( b_0 \) leads to opposite effects. We finally report that our results are robust to changes in the values of parameters, initial conditions and exogenous variables.

4. The dual approach to the Ramsey problem

In the dual approach, the government chooses \( \{\tau_t, C_t, K_t, B_t^{\infty}\}_{t=0} \) to maximize (1) subject to (5a)-(5c). We now follow Chamley (1986) and Ljungqvist and Sargent (2000, p. 316). Since \( 0 \leq \tau_0 < 1 \) is taken as given to make the policy problem nontrivial, the government can choose \( C_0, K_1, B_1 \) only at

\( ^{10} \) The values of \( b_0 \) and \( h_t \) are the means of the US economy.
After this period, \( t \geq 1 \), the government chooses \( \tau_t, C_t, K_{t+1}, B_{t+1} \), or equivalently \( \tau_t, C_t, K_{t+1}, B_{t+1} \), where \( \tau_t = (1 - \tau_t)A \). The Lagrangean is:

\[
\sum_{t=0}^{\infty} \beta^t \left[ \nu \ln C_t + (1 - \nu) \ln H_t + \lambda_t^k \left[ AK_t - C_t - H_t - K_{t+1} + K_t \right] + \lambda_t^b \left[ H_t + (1 + \bar{\tau}_t)B_t - B_{t+1} + \bar{\tau}_t K_t - AK_t \right] + \lambda_t^c \left[ \beta(1 + \bar{\tau}_t)C_t - C_{t+1} \right] \right] = 0 \quad (10)
\]

where \( \lambda_t^u, \lambda_t^b \) and \( \lambda_t^c \) are dynamic multipliers associated with (5a), (5b) and (5c) respectively.

The period-0 first-order conditions again differ from the same rules governing behavior from period 1 onward. Specifically, at \( t = 0 \) the first-order conditions for \( C_0, K_1, B_1 \) are respectively:

\[
\frac{\nu}{C_0} - \lambda_0^k + \lambda_0^c \beta(1 + \bar{\tau}_1) = 0 \quad (11a)
\]

\[
\lambda_0^k = \beta \lambda_1^k (1 + A) + \beta \lambda_1^c (\bar{\tau}_1 - A) \quad (11b)
\]

\[
\frac{\lambda_0^b}{\lambda_1^b} = \beta(1 + \bar{\tau}_1) \quad (11c)
\]

while, at \( t \geq 1 \) the first-order conditions for \( \tau_t, C_t, K_{t+1}, B_{t+1} \) are respectively:

\[
\lambda_t^k (B_t + K_t) + \lambda_{t+1}^c C_{t+1} = 0 \quad (12a)
\]

\[
\frac{\nu}{C_t} - \lambda_t^k + \lambda_t^c \beta(1 + \bar{\tau}_{t+1}) - \frac{\lambda_{t+1}^c}{\beta} = 0 \quad (12b)
\]

\[
\lambda_t^k = \beta \lambda_{t+1}^k (1 + A) + \beta \lambda_{t+1}^c (\bar{\tau}_{t+1} - A) \quad (12c)
\]

\[
\frac{\lambda_t^b}{\lambda_{t+1}^b} = \beta(1 + \bar{\tau}_{t+1}) \quad (12d)
\]

In addition, in all periods \( t \geq 0 \), the first-order conditions include the constraints to the government’s problem, namely (5a-c).

### 4.1 Qualitative features

In addition to the features discussed in the primal approach (see subsection 3.1), note that equation (5a) is linearly dependent with equations (11c) and (12d). In particular, at any time \( t \geq 0 \),
\( C_t^* = C_{t+1}^* = C_{t+2}^* = \ldots \), which means that \( C_t^* \equiv \phi \) is constant over time as in Chamley (1986, equation 34). Keep in mind that \( \phi \) is a new endogenous variable. Following Chamley (1986, p. 618), in what follows, we use \( \lambda_t^* = \frac{\phi}{C_t^*} \) to substitute \( \lambda_t^* \) out (the sign of \( \phi \) is the sign of \( \lambda_t^* \), which is expected to be negative in a second-best problem) and omit (11c) and (12d) from the system. All this also confirms that one has to solve simultaneously for the long run and the transition path including period 0, as was obviously the case in the primal approach (see below).\(^{11}\)

4.2 The full Ramsey system

We now present the full system working as in the primal approach. Thus, we first combine the first-order conditions - equations (11a-c), (12a-d) and the constraints (5a-c) - so as to satisfy continuity across the three policy regimes. In turn, we transform the variables to make them stationary; in particular, we define \( c_t \equiv C_t/K_t, m_t \equiv C_t/B_t, \Lambda_t^* \equiv \lambda_t^* C_t, \Lambda_t^* \equiv \lambda_t^* K_t \) at all \( t \geq 0 \), which are all jump variables.\(^{12}\) Therefore, we have (see Appendix C for details):

**First policy regime, \( t = 0 \)**

\[
\frac{c_1}{c_0} = \frac{\beta(1 + \bar{r}_1)}{1 + A - c_0 - h_0} \tag{13a}
\]

\[
\frac{m_1}{m_0} = \frac{\beta(1 + \bar{r}_1)}{m_0 h_0 + \bar{r}_0 \left( 1 + \frac{m_0}{c_0} \right) + 1 - A m_0} \tag{13b}
\]

\[
\Lambda_0^*(1 + A - c_0 - h_0) = \beta \Lambda_0^*(1 + A) + \frac{\beta \phi}{c_1}(\bar{r}_1 - A) \tag{13c}
\]

\[
\frac{\nu_1}{c_0} - \Lambda_0^* + \frac{\Lambda_0^* \beta(1 + \bar{r}_1)}{c_0} = 0 \tag{13d}
\]

**Second policy regime, \( t = 1 \)**

\[
\frac{c}{c_1} = \frac{\beta(1 + \bar{r})}{1 + A - c_1 - h_1} \tag{14a}
\]

\(^{11}\) While the need to solve simultaneously for the long run and the transition path is, by construction, the case in the primal approach, in the dual approach we could solve for the long run independently if there were no public debt (i.e. the government budget is balanced); see e.g. Park and Philippopoulos (2004) and Economides and Philippopoulos (2007). See Lansing (1999) for a detailed study of the case in which there is no public debt and the utility function is log linear.

\(^{12}\) We report that the solution is robust to the transformations used.
\[
\frac{m}{m_1} = \frac{\beta(1 + \bar{r})}{\frac{m h_i}{c_i} + \bar{r} \left(1 + \frac{m_i}{c_i}\right) + 1 - A \frac{m_i}{c_i}}
\]  
(14b)

\[
\Lambda_i^c [1 + A - c_i - h_i] = \beta \Lambda_i^c (1 + A) + \frac{\beta \phi}{c} (\bar{r} - A)
\]  
(14c)

\[
\frac{\phi}{m_i} + \frac{\phi}{c_i} + \Lambda_0^c = 0
\]  
(14d)

\[
\frac{\nu}{c_i} - \Lambda_1^c + \frac{\Lambda_c^c (1 + \bar{r})}{c_i} - \frac{\Lambda_0^c [1 + A - c_0 - h_0]}{\beta c_0} = 0
\]  
(14e)

**Third policy regime, \( t \geq 2 \)**

\[
1 = \frac{\beta(1 + \bar{r})}{1 + A - c - h}
\]  
(15a)

\[
1 = \frac{m h_i}{c} + \bar{r} \left(1 + \frac{m}{c}\right) + 1 - A \frac{m}{c}
\]  
(15b)

\[
\Lambda_i^c [1 + A - c - h ] = \beta \Lambda_i^c (1 + A) + \frac{\beta \phi}{c} (\bar{r} - A)
\]  
(15c)

\[
\frac{\phi}{m} + \frac{\phi}{c} + \Lambda_i^c = 0
\]  
(15d)

\[
\frac{\nu}{c} - \Lambda_i^c + \frac{\Lambda_c^c (1 + \bar{r})}{c} - \frac{\Lambda_i^c [1 + A - c_i - h_i]}{\beta c_i} = 0
\]  
(15e)

Equations (13a-d), (14a-e), (15a-e) and \( m_0 \equiv \frac{C_0}{B_0} = \frac{c_0}{B_0} \) summarize the Ramsey problem. We have fifteen equations in fifteen unknowns which are \( c_0, c_1, c, \bar{r}, \bar{r}, \Lambda_c, \Lambda_i, \Lambda^c, \Lambda_0, \Lambda_1, \Lambda^k, m_0, m_1, m, \phi \). This is given the path of \( h_0 \), initial conditions for \( K_0 \) and \( B_0 \), and the period-0 tax rate, \( \tau_0 \). In terms of the number of equations and unknowns, this is a more complex system to solve than the one in the primal approach (see (9a-d) above).

**4.3 Numerical solution**

To solve the above system, we use the same parameter values used in section 3. Actually, it was harder to find ranges of parameter values, initial conditions and exogenous variables that yield a dual solution; a solution to the primal problem was much easier to obtain. We thus started our

\[^{13} \text{As in the primal solution, since all transformed variables are jump, the economy is at its long run at } t = 2.\]
search for a solution from the dual. Numerical results for all endogenous variables are presented in Table 2. We also report the implied values of \( b_t \equiv \frac{B_t}{K_t} = \frac{c_t}{m_t} \), the gross growth rates of consumption and capital, \( \frac{C_{t+1}}{C_t} \) and \( \frac{K_{t+1}}{K_t} \), and the tax rate, \( \tau_t \).

Table 2: Solution of the dual problem

<table>
<thead>
<tr>
<th>endogenous variables</th>
<th>( c_t )</th>
<th>( m_t )</th>
<th>( b_t )</th>
<th>( \Lambda_t^c )</th>
<th>( \Lambda_t^k )</th>
<th>( \frac{C_{t+1}}{C_t} )</th>
<th>( \frac{K_{t+1}}{K_t} )</th>
<th>( \bar{t} )</th>
<th>( \tau_t )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t = 0 )</td>
<td>0.20</td>
<td>0.80</td>
<td>0.25</td>
<td>0.36</td>
<td>6.38</td>
<td>1.15</td>
<td>1.73</td>
<td>0.60</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>( t = 1 )</td>
<td>0.13</td>
<td>4.33</td>
<td>0.03</td>
<td>0.23</td>
<td>6.01</td>
<td>1.80</td>
<td>1.80</td>
<td>0.28</td>
<td>0.71</td>
<td>-0.04</td>
</tr>
<tr>
<td>( t \geq 2 )</td>
<td>0.13</td>
<td>-0.39</td>
<td>-0.33</td>
<td>0.23</td>
<td>6.01</td>
<td>1.80</td>
<td>1.80</td>
<td>1.00</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Note: We use Matlab 7.0.

The solutions of \( c_t, \frac{C_{t+1}}{C_t}, \frac{K_{t+1}}{K_t} \) and \( \tau_t \) are the same as in Table 1. In addition, here we have solutions for \( m_t, \Lambda_t^c, \Lambda_t^k \) and the constant value of \( \phi \), which are present in the dual problem only. All endogenous variables have the right sign. Regarding the solution for \( b_t \), the idea is that it is optimal for the government to raise all tax revenue through a time-1 capital levy, lend the proceeds to the private sector and finance government expenditure by using the interest from the loan (hence the negative value of \( b_2 \equiv b = -0.33 \)).

5. Conclusions

We solved the dynamic optimal taxation (Ramsey) problem by applying both the dual and the primal approach. As it should be expected, they yield the same results along the whole optimal path. To the extent that we solved a full dual problem, we believe we have contributed methodologically to the literature.
Appendix A: Competitive equilibrium

The first-order conditions of the individual’s problem include (2) and the Euler equation (5a). Using the government’s budget constraint (3) into the individual’s budget constraint (2), we get the resource constraint, (4a) or (5b). Equation (5c) is equation (3) rewritten as in Chamley (1986, equation 20).

To get the implementability constraint (4b), one can work as in Ljungqvist and Sargent (2000, chapter 12). Actually, our model with a log-linear utility function and an AK production technology is a special case of equations (12.31 and 12.32) in Ljungqvist and Sargent if: (i) we set \( u_t(C_t) = \frac{1}{C_t} \) (ii) we ignore their labor supply terms (iii) since here capital and bonds are taxed at the same rate and earn the same return, we use \( \frac{1 + (1 - \tau_0)A(K_0 + B_0)}{C_0} \) on the right-hand side of (12.32). This gives \( \sum_{t=0}^{\infty} \beta^t = \frac{1 + (1 - \tau_0)A(K_0 + B_0)}{C_0} \), which is (4b).

Appendix B: Primal solution to the Ramsey problem

Consider conditions at \( t = 0 \). If we use (7a) for \( \lambda_0 \) and (8a) for \( \lambda_1 \) into (7b), we have

\[
\beta^0 \frac{K_0}{C_0} (1 + A) = \nu \frac{K_0}{C_0} + \frac{\nu}{2} \left( 1 + \frac{B_0}{K_0} \right) \frac{K_0}{C_0} \left[ 1 + (1 - \tau_0)A \right] \equiv \Gamma_0, \text{ where } \frac{K_0}{K_0} = 1 + A - \frac{C_0}{K_0} - \frac{H_0}{K_0} \text{ from (4a). This gives (9a).}
\]

Consider conditions at \( t \geq 1 \). Equations (8a)-(8b) imply \( \frac{C_{t+1}}{C_t} \frac{K_{t+1}}{K_t} = 1 + A \) or

\[
\frac{c_{t+1}}{c_t} \frac{K_{t+1}}{K_t} = \beta(1 + A), \text{ where } \frac{K_{t+1}}{K_t} = 1 + \frac{1 - c_t - h_t}{1 + A - c_t - h_t} \text{ from (4a). Thus, at } t \geq 1, \frac{c_{t+1}}{c_t} = \frac{\beta(1 + A)}{1 + A - c_t - h_t}.
\]

When we are in the second policy regime at \( t = 1 \), which also means that at \( t = 2 \) the economy will be on its balanced growth path where variables remain constant and are denoted without a time subscript, this is written as \( \frac{c}{c_1} = \frac{\beta(1 + A)}{1 + A - c_1 - h_1} \), which is (9b). At \( t \geq 2 \), we are on the balanced growth path, so this is written as \( 1 = \frac{\beta(1 + A)}{1 + A - c - h} \), which is (9c).

Equation (9d) is equation (4b) rewritten in stationary form.
Finally, given the (primal) solution, the gross growth rate of consumption is
\[
\frac{C_{t+1}}{C_t} = \beta(1 + A) \quad \text{at } t \geq 1, \quad \text{while at } t = 0 \text{ we have } \frac{C_1}{C_0} = \frac{\beta\nu(1 + A)}{\Gamma_0 c_0}.
\] Also, at any \( t \geq 0 \), the gross growth rate of capital follows from the resource constraint, \( \frac{K_{t+1}}{K_t} = 1 + A - c_t - h_t \). Having solved for \( \frac{C_{t+1}}{C_t} \), the tax rate can follow from \( \frac{C_{t+1}}{C_t} = \beta[1 + (1 - \tau_{t+1})A] \).

Appendix C: Dual solution to the Ramsey problem

Consider (11a) which holds at \( t = 0 \) only. This is written as
\[
\frac{vK_0}{C_0} - \lambda_0^b K_0 + \lambda_0^c C_0 \frac{K_0}{C_0} \beta(1 + \bar{r}_t) = 0,
\]
which is (13d).

Consider (12a) and (12b) which hold at \( t \geq 1 \) only. Equation (12a) is rewritten as
\[
\lambda^b \frac{B_t}{K_t} \left(1 + \frac{K_t}{B_t}\right) + \lambda^c_{t-1} C_{t-1} = 0 \quad \text{or, by using } \lambda^b_t = \frac{PH}{C_t}, \quad \text{as } \left(\frac{PH}{C_t}\right) \left(1 + \frac{K_t}{B_t}\right) + \lambda^c_{t-1} C_{t-1} = 0.
\]
This gives (14d) at \( t = 1 \) and (15d) at \( t \geq 2 \). Equation (12b) is rewritten as
\[
\frac{vK_t}{C_t} - \lambda^b_t K_t + \lambda^c_t C_t \frac{K_t}{C_t} \beta(1 + \bar{r}_{t+1}) - \lambda^c_{t-1} C_{t-1} \frac{K_{t-1}}{K_t} - \frac{K_t}{K_{t-1}} K_{t-1} = 0,
\]
where
\[
\frac{K_t}{K_{t-1}} = 1 + A - \frac{C_{t-1}}{C_t} - h_t - \bar{r}_{t-1} K_{t-1} \quad \text{from (5b)}.
\]
This gives (14e) at \( t = 1 \) and (15e) at \( t \geq 2 \).

Consider now those conditions that hold at any time, \( t \geq 0 \). The Euler condition, (5a), and the resource constraint, (5b), imply
\[
\frac{c_{t+1}}{C_t} \equiv \frac{c_{t+1}}{C_t} \frac{K_t}{C_t} = \frac{\beta(1 + \bar{r}_{t+1})}{1 + A - c_t - h_t}.
\]
This gives (13a) at \( t = 0 \), (14a) at \( t = 1 \) and (15a) at \( t \geq 2 \). The government budget constraint, (5c), jointly with the Euler condition, (5a), imply
\[
\frac{m_{t+1}}{m_t} \frac{C_{t+1}}{C_t} \frac{B_{t+1}}{B_t} = \frac{\beta(1 + \bar{r}_{t+1})}{1 + A - m_t / c_t}.
\]
This gives (13b) at \( t = 0 \), (14b) at \( t = 1 \) and (15b) at \( t \geq 2 \). Finally, the Euler conditions for capital, (11b) or (12c), imply that at any \( t \geq 0 \)
\[
\lambda^b_t K_t = \beta \lambda^b_{t+1} K_{t+1} \frac{K_t}{K_{t+1}} (1 + A) + \beta \lambda^b_t K_t \frac{K_t}{K_{t+1}} (\bar{r}_{t+1} - A), \quad \text{where } \lambda^b_t = \frac{PH}{C_t} \quad \text{and } \frac{K_{t+1}}{K_t} = 1 + A - \frac{C_t}{K_t} - H_t.
\]
This gives (13c) at \( t = 0 \), (14c) at \( t = 1 \) and (15c) at \( t \geq 2 \).
REFERENCES


Economides G. and A. Philippopoulos (2007): Growth enhancing policy is the means to sustain the environment, forthcoming in Review of Economic Dynamics.


