(that is, $A$ is dense in $R^2$). Moreover, $S(\lambda)$ defined by (2), (3) is unidentifiable, since all $\lambda$ which give the same combinations.

\[\delta_1 = \lambda_{11}, \quad \delta_2 = \lambda_{22}, \quad \delta_3 = \lambda_{12} + \lambda_{21}, \quad \delta_4 = \lambda_{13} + \lambda_{31}, \quad \delta_5 = \lambda_{23} + \lambda_{32}, \quad \delta_6 = \lambda_{34} + \lambda_{43} \]

also gives the same input-output relation [4].

Since (2) is properly unidentifiable (as can be immediately verified by use of the proposition of Section 2) the evaluation of both $x_1$ and $x_4$ of great interest from a pathophysiological point of view, is not possible (since $x_4$ is clearly reconstructable, $x_1$ is the unknown that cannot be known). A feasible experimental condition can be obtained by adding a new input $u(t)$, yielding the equation [5]

\[\dot{x}(t) = A(\lambda)x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u(t). \quad (2 \text{bis})\]

It can be easily verified that (2 bis), (3) is identifiable and then $x_1$ can be estimated by means of the proposed change of the input matrix.

On the Comparison of Optimal Control Systems

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Abstract—This correspondence presents conditions which permit the comparison of the optimal performance of control systems which have the same criterion but different dynamics. These conditions are applied to linear quadratic problems and to establishing performance bounds for suboptimal systems.

I. INTRODUCTION

Let us consider the maximization of the functional

\[J(x_0; u; \tau) = \int_{t_0}^{t} L(x, u; t) dt + \phi(x_t) \]

on the trajectories of the systems

\[\dot{x} = f(x, u; t), \quad x(t_0) = x_0 \]

and

\[\dot{x} = f(x_0; u), \quad x(t_0) = x_0 \]

where $x \in R^n$ is the state variable and $u \in R^n$ is the control variable. The question arises to find conditions for the modified system (3) to have better (worse) performance than the original system (2) with respect to the functional (1). We provide such conditions in this correspondence. The conditions will be expressed in terms of the optimal value function for system (2) and hence are of limited applicability as pointed out in [3], where conditions involving optimal value functions appear. The comparison conditions will be illustrated for the case of linear quadratic problems for which the value function can be obtained from the solution of Riccati type equations. Finally we obtain, under similar comparison conditions, some lower bounds for the performance of suboptimal controllers in [3].

II. COMPARISON OF OPTIMAL SYSTEMS

Proposition 1: Assume that the HJB equation

\[v + \max_{u} \left\{ L(x, u; t) + \int_{t}^{T} f(x, u; t) \right\} = 0, \quad v(x; T) = \phi(x) \quad (4)\]

for system (2) and

\[v + \max_{u} \left\{ L(x, u; t) + \int_{t}^{T} f(x, u; t) \right\} = 0, \quad v(x; T) = \phi(x) \quad (5)\]

for system (3). The subscripts denote partial differentiation while $T$ denotes transient. We assume that there exist differentiable functions $v(x; t; R^n \times (-\infty, T]) \rightarrow R$, $i = 1, 2$ which satisfy the HJB equations (4) and (5), as well as functions $u(x; t) = \arg\max_{u} \{ L(x, u; t) + \int_{t}^{T} f(x, u; t) \}$. Then $u_{i}$ are the optimal value and control for the corresponding system (2) or (3). (See [1, theorem 1, p. 192].)

We will need the following proposition.

\[v(x; t) = \max_{u} \left\{ L(x, u; t) + \int_{t}^{T} f(x, u; t) \right\} = 0, \quad v(x; t) = \phi(x) \]

has a solution satisfying the above assumptions. If for some function

\[u(x; t) = \arg\max_{u} \left\{ L(x, u; t) + \int_{t}^{T} f(x, u; t) \right\} \exists (x, x; T) \rightarrow R \text{ we have}

\[v(x; t) = \max_{u} \left\{ L(x, u; t) + \int_{t}^{T} f(x, u; t) \right\} < 0 \]

and $u(x; t) = \arg\max_{u} \left\{ L(x, u; t) + \int_{t}^{T} f(x, u; t) \right\}$ exists, then

\[v(x; t) = \phi(x) \quad (6)\]

for all $x \in R^n$ and $t \in (-\infty, T]$. Proof: Consider the payoff corresponding to an arbitrary control $u(x; t)$ applied to the system $\dot{x} = f(x, u; t)$. Adding and subtracting $\int_{t}^{T} U(x, u; t) dt$ to the payoff we get (if is computed along the trajectory of the system)

\[J(x_0, u; \tau) = \int_{t_0}^{t} L(x, u; t) dt + \phi(x_T) + \int_{t_0}^{T} \left[ -U(x, u; t) + U(x; u; t) \right] dt - \int_{t}^{T} U(x_0; t) dt + \phi(x_T) \]

\[\int_{t}^{T} U(x, u; t) dt \int_{t}^{T} U(x, u; t) dt \int_{t}^{T} U(x, u; t) dt \int_{t}^{T} U(x, u; t) dt \int_{t}^{T} U(x, u; t) dt \int_{t}^{T} U(x, u; t) dt \]

\[\int_{t}^{T} U(x, u; t) dt \int_{t}^{T} U(x, u; t) dt \int_{t}^{T} U(x, u; t) dt \int_{t}^{T} U(x, u; t) dt \int_{t}^{T} U(x, u; t) dt \int_{t}^{T} U(x, u; t) dt \]

If

\[J(x_0, u; \tau) = \int_{t_0}^{T} L(x, u; t) dt + \phi(x_T) \]

the final equation of (5) implies that for any control $u(x; t)$ a $U(x_0, t)$ new, $V(x_0; t) = \max_{u} J(x_0, u; \tau)$ and thus $V(x_0; t) < U(x_0, t)$ for $x \in R^n$ and $t \in T$. If

\[U(x; t) \max_{u} \left\{ L(x, u; t) + \int_{t}^{T} f(x, u; t) \right\} \]

consider the payoff corresponding to the control $\dot{x}(t; \tau) = \arg\max_{u} \int_{t}^{T} U(x, x; t) dt$.
there exist an optimal value $V_i(x; t)$ and an optimal control $u_i(x; t)$ satisfying the HJB relations

$$V_i(x; t) = \max_{u_i} \left[ L(x, u_i; t) + V_{i+1}(f(x, u_i; t)) \right] - 0, \quad V(0; t) - 0 \tag{13}$$

Furthermore, it is assumed that $f_0(0, u_i; t) = 0$ for all $u_i$. Then $x = 0$ is an equilibrium point of (12). Reference [3] considers what will happen if the control $u_i(x; t)$ is applied to a different dynamical system, namely,

$$\dot{x} = f(x; u_i; t) + g(x; u_i; t), \quad x(0) = x_0 \tag{14}$$

with the restriction on $g$ that $g(0, u_i; t) = 0$; $x = 0$ is an equilibrium for (14) as well. It is shown that if the condition

$$L(x, u_i; t) - V_i(x; t) g(x; u_i; t) > 0 \tag{15}$$

holds for all $x, t$ then

1) The suboptimal control $u_i(x; t)$ when applied to system (14) results in a trajectory that converges to the equilibrium $0$.  

2) Denote by $V_i(x_0; t)$ the payoff $\int_0^T L(x, u_i; t) dt$ corresponding to the trajectory of system (14) when the control $u_i(x; t)$ is used. Then

$$V_i(x; t) < q V_i(x; t)$$

where

$$q = \sup_{x \in R^n} \frac{L(x, u_i; t)}{L(x, w_i; t) - V_i(x, u_i; t)}.$$ 

We now derive conditions for the existence of a lower bound for $V_i(x; t)$, the need for which has been noted in [3]. We write $V_i(x_0, t)$ as

$$V_i(x_0, t) = \int_0^T L(x, u_i, t) dt + \int_0^T \left[ -V_i + V_{i+1}(f(x, u_i, t) + g(x, u_i, t)) \right] dt$$

$$= \lim_{T \to \infty} \left( V_i(x_0, T) - V_i(x; T) \right)$$

$$= \lim_{T \to \infty} \left[ V_i(x_0; t) + L(x, u_i; t) + V_{i+1}(f(x, u_i; t) + g(x, u_i; t)) dt \right]$$

$$= V_i(x_0; t) + \int_0^T V_{i+1}(x, t) g(x, u_i; t) dt.$$

The last equality follows from HJB equation (13) and its boundary condition, and the fact that if (15) holds, $\lim_{T \to \infty} x_T = 0$ (and $V_i$ is continuous). Therefore if $V_{i+1}(x, u_i; t) > 0$ holds in addition to (15), i.e.,

$$L(x, u_i; t) > V_i(x, u_i; t) g(x, u_i; t) > 0$$

we obtain for all $x, t$

$$q V_i(x; t) = V_i(x; t) > V_i(x; t)$$

which provides both an upper and a lower bound for the performance of $u_i$.

**References**


