Values and Strategies for Infinite Time
Linear Quadratic Games

EVANGELOS F. MAGEIROU, STUDENT MEMBER, IEEE

Abstract—For a class of infinite time linear quadratic games it is shown that an appropriate solution of an algebraic Riccati type equation determines the value of the game but not necessarily any equilibrium strategies. In the case of nonexistence of equilibrium strategies, ε-optimal strategies are constructed through the solutions of a differential Riccati equation.

I. INTRODUCTION AND PROBLEM STATEMENT

Let us consider a zero-sum linear quadratic differential game with value functional

\[ J_T(u, v; x_0) = \int_0^T w(x(t), u(t), v(t)) dt = \int_0^T [x(t)' Q x(t) + u(t)' R u(t) - v(t)' S v(t)] dt \] (1a)

and dynamics

\[ \dot{x} = Ax + Bu + Cu, \quad x(0) = x_0. \] (1b)

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The author is with the Decision and Control Group, Harvard University, Cambridge, MA 02138.

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A minimizing player controls $u_i$ and a maximizing player $v_i$. At the end of the game, the maximizing’s utility increases by $J_T(u_i; v_i)$ while the minimizer’s decreases by the same amount. It is well known (see Lemma A1 and Proposition A1 in the Appendix) that if the Riccati equation

$$\dot{K} + A'K + KA + Q = K(BR^{-1}B' - CS^{-1}C')K$$

with boundary condition $K(T) = 0$ has a solution on $[0, T]$, denoted by $K(t; T)$, then the strategy pair

$$u_i^*(x) = -R^{-1}B'K(t; T)x$$
$$v_i^*(x) = S^{-1}C'K(t; T)x$$

is in equilibrium. The value of the game is

$$V_T(x_0) = x_0K(0; T)x_0$$

This short paper examines the infinite duration version of game (1), namely, a differential game with function value

$$J_\infty(u, v; x_0) = \int_0^\infty w(x(t), v(t), x_0)dt$$

and dynamics as in (1b). We will henceforth assume that $Q, R, S$ are symmetric positive definite matrices, an assumption that will greatly facilitate our analysis. We will consider the situation where the solution of (2) converges, i.e., $\lim_{t \to \infty} K(t; T) = K^*$. It is then reasonable to conjecture in view of (3) and (4) that the value of the game with function value (5) is

$$V_\infty(x_0) = x_0K^*x_0$$

and that the strategies

$$u_i^*(x) = -R^{-1}B'K^*x$$
$$v_i^*(x) = S^{-1}C'K^*x$$

are in equilibrium. This conjecture is very much in line with the well-known results on the regulator problem of optimal control (see [1, sec. 23]). It is surprising that this conjecture is not true in general. In fact, the following scalar example shows that the strategy pair in (7) need not be an equilibrium.

**Example:** Consider the scalar game with value functional

$$J_\infty(u, v; x_0) = \int_0^\infty (x^2 + u^2 - 2u^2)dt$$

and dynamics

$$\dot{x} = x + u + v_i, \quad x(0) = x_0$$

The Riccati equation for this game is

$$\dot{k} = (1/2)k^2 - 2k - 1, \quad k(T; T) = 0$$

which can be integrated to give

$$k(t; T) = \sqrt{6} \tanh \left[ \sqrt{6}(T - t) + \tanh^{-1}(-2/\sqrt{6}) \right] + 2.$$ 

Furthermore, $\lim_{t \to \infty} k(0; T) = 2 + \sqrt{6}$, and hence the strategies in (7) are

$$u_i^*(x) = -(2 + \sqrt{6})x$$
$$v_i^*(x) = \frac{(2 + \sqrt{6})}{2}x.$$ 

These strategies are not in equilibrium. If the minimizer uses the strategy $v_i^*(x)$ the minimizer will be faced with the least squares problem

$$\min_{u_i} \int_0^\infty [x^2 + u^2 - 2v_i(x)u_i]dt = \int_0^\infty \left[ -4 + 2\sqrt{6} \right] x^2 + u^2 dt$$

with dynamical constraints

$$\dot{x} = x + u + v_i(x) - \frac{4 + \sqrt{6}}{2}x + u_i, \quad x(0) = x_0.$$ 

The minimizer’s optimal response is not $v_i^*(x)$. In fact, by playing $v_i(x) = 0$ one can obtain a value $J_\infty(0, v_i^*(x)) = -\infty$ with $v_i^*(x) = 0$.

The purpose of this short paper is to show that even though the strategy pair $u_i^*, v_i^*$ need not be in equilibrium, the quantity $x_0K^*x_0$ in (6) is indeed the value of the infinite duration game (5). We will show that either player can guarantee a value of $J_\infty$ arbitrarily close to the quantity $x_0K^*x_0$, therefore, the quantity $x_0K^*x_0$ satisfies the following general definition of the value of a game.

**Definition** ([2, p. 77]): Consider a two player game where player 1 can use strategies $u$ from a set $U$ and player 2 can use strategies $v$ from a set $V$. When the strategy pair $(u, v)$ is used, player 1’s utility increases by $J(u, v)$ while player 2’s decreases by the same amount. We say that player 2 is the minimizer while player 1 is the maximizer. A number $p$ is the value of the game if, for any $v_i \geq 0$, there exist strategies $u_i, v_i$ such that

$$J(u_i, v_i) = p + e_i, \quad \text{for all } e_i \in V$$
$$J(u_i, v_i) = p - e_i, \quad \text{for all } e_i \in U.$$ 

The strategies $u_i, v_i$ are called $\epsilon$-optimal.

A strategy pair $(u_i^*, v_i^*)$ is said to be in equilibrium if

$$J(u_i^*, v_i^*) < J(u_i^*, v_i^*) < J(u, v_i^*), \quad \text{for all } u, v_i.$$ 

It is easily seen that if $u_i^*, v_i^*$ is in equilibrium, then $J(u_i^*, v_i^*)$ is the value of the game. This is perhaps the most satisfactory way of showing that a value exists, since it is then possible to make the normative statement that $u_i^*, v_i^*$ represent the best play for the minimizer and the maximizer, respectively. In the case where only $\epsilon$-optimal strategies are available for establishing a value, we can only advise the players how to guarantee a utility arbitrarily close to the value of the game. The determination of "how close" is up to the players. Despite its weaker normative implications, the concept of $\epsilon$-optimal strategies is standard in game theory. In the next section we construct appropriate $\epsilon$-optimal strategies for the game (5).

**II. STRATEGIES FOR THE INFINITE DURATION GAME**

Let us first note that since we are dealing with time-invariant games the solution $K(t; T)$ of (2) is a function of $t - T$ only. We write it, henceforth, as either $K(t; T)$ or as $K(T-t)$. Note the time reversal implied in the last notation. The boundary condition $K(T; T) = 0$ is expressed in this notation as $K(0) = 0$. We are interested in the behavior of $K(0; T)$ as $T \to \infty$, or equivalently, of $K(t)$ as $t \to \infty$.

Our assumption that $Q, R, S$ are symmetric positive definite matrices guarantees a very simple behavior of $K(t)$ as a function of $t$. We show in Lemma A2 of the Appendix that $K(t)$ is increasing in $t$ in the sense of positive definite matrices. More precisely, if $K(t_0) \leq K(t_1)$ for $t_0 < t_1$, then $K(t_1) \leq K(t_0)$; it is therefore evident that if $K(t)$ does not exist for $t \geq t_0$, it must increase without bound as $t$ approaches $t_0$ from below. On the other hand, if one can find a uniform bound for $K(t)$, a solution to (2) is guaranteed to exist for all $t$, as it cannot "blow up" for any finite $t$.

Let us now consider the algebraic Riccati equation

$$A'K + KA + Q = K(BR^{-1}B' - CS^{-1}C')K.$$  

In Lemma A3 of the Appendix we show that if $K = K^*$ is a real symmetric nonnegative definite solution of (8), then $K^* \leq K$ for all $t$. The existence of such a solution (8) is a sufficient condition for the existence of a solution $K(t)$ of (2) for all $t$. Now, if $K(t)$ is increasing in $t$ and bounded, it must converge to some matrix $K^*$ which satisfies (8). Furthermore, $K(t) > K(0) > 0$ and thus $K^* = \lim_{t \to \infty} K(t) > 0$ is a non-negative definite solution of (8). In fact, $K^*$ is the smallest such solution of (8). If $K$ is another nonnegative solution of (8) we have by Lemma A3
that $K > K(t)$ and hence $K > \lim_{t \to \infty} K(t) = K^+$. This minimality property of $K^+$ can be used to show (Lemma A4 in the Appendix) that $K^+$ is feedback stabilizing, i.e.,

$$\text{Re}(\lambda - B R^{-1} B' K^+ + CS^{-1} C' K^+) < 0.$$  

We summarize the above discussion in Proposition 1.

Proposition 1: Consider the solution $K(T - t)$ of (2). Then $\lim_{t \to \infty} K(t)$ exists if and only if there exists a positive definite solution to (8). Furthermore, the matrix $K^+ = \lim_{t \to \infty} K(t)$ is feedback stabilizing.

Proof: We have discussed all statements of the proposition except the strict positivity of $K^+$. Note that every real symmetric solution of (8) is invertible. If $K = K^+$ is such a solution of (8) and $K x = 0$ for some $x \neq 0$, we obtain by premultiplying and postmultiplying (8) by $x'$ and $x$ that $x' Q x = 0$. But $Q > 0$ and thus $x = 0$, showing that $K$ is invertible. We know already that $K^+ > 0$. The preceding discussion shows that $K^+ > 0$.

Remark: The explicit solution of the Riccati equation in the example of the introduction is consistent with the above discussion: it is increasing as a function of $T - t$ and converges to the only positive solution of the algebraic Riccati equation for that game, $(1/(2K^2 - 2K - 1))$. It is interesting to note that this example provides a counterexample to [3, proposition 3.2.6.10] where it is stated that the strategies $u^0, v^0$ in (7) are in equilibrium if $K^+$ is feedback stabilizing and the algebraic Riccati equation has a positive and a negative solution. All these assumptions are satisfied but the strategies $u^0, v^0$ are not in equilibrium.

On the basis of the above results we can now establish that $x_0 K^+ x_0$ is the value of the game (5). This is done in Proposition 2.

Proposition 2: Let us assume that there exists a positive definite solution of (8). Then $\lim_{t \to \infty} K(t) = K^+$ exists and the value of the game (5) is $x_0 K^+ x_0$.

Proof: Proposition 1 shows that $K^+$ exists. It remains to show that either player can guarantee a value arbitrarily close to $x_0 K^+ x_0$. We first consider the minimizing player. We will prove that if he uses the strategy $u^0(x) = - B R^{-1} B' K^+ x$, he can achieve a loss smaller than $x_0 K^+ x_0$. When $u^0$ is used, the maximizer faces the linear quadratic problem

$$\max_v J_m(u^0, v; x_0) = \int_0^\infty (x' Q x + (u^0 y R u^0 - v' Ru) dt$$

or equivalently

$$\max_v \int_0^\infty (x' Q x + (u^0 y R u^0 - v' Ru) dt$$

The algebraic Riccati equation for (9) is

$$(A - B R^{-1} B' K^+ ) M + M (A - B R^{-1} B' K^+) + Q - K^+ B R^{-1} B' K^+ = M C S^{-1} C' M.$$  

We note that for $M = K^+$ (10) becomes identical to (8). Thus, $M = K^+$ is a real symmetric negative definite solution of (10) and, in addition, feedback stabilizing. It follows from [4, theorem 8] that the minimum value of (9) is $x_0 K^+ x_0$ and hence $\min_{u^0} J_m(u^0, v; x_0) = x_0 K^+ x_0$, which we had to prove.

We now consider the maximizing player. Given an arbitrary $\epsilon > 0$ we must find a strategy $v^0$ such that

$$x_0 K^+ x_0 - \epsilon < J_m(u, v; x_0),$$

for all $u$. (11)

To construct $v^0$, we work as follows. For a given $\epsilon$ there exists a $T$ such that

$$x_0 K^+ x_0 - \epsilon < x_0 K(0; T) x_0.$$  

This follows from $\lim_{t \to \infty} K(0; T) = K^+$. Consider now the strategy

$$v(x) = \begin{cases} S^{-1} C' K(t; T) x_t, & t \in [0, T) \\ 0, & t \in [T, \infty) \end{cases}.$$  

We write the value functional as

$$J_m(u, v; x_0) = \int_0^\infty w(x, u, v) dt$$

$$= \int_0^T w(x, u, v) dt + \int_T^\infty w(x, u, v) dt.$$  

According to the definition of $v$, the second integral in (13) becomes $\int_T^\infty w(x, u, v) dt > 0$. Furthermore, using Lemma A1 we can express the first integral as

$$\int_0^T w(x, u, v) dt = x_0 K(0; T) x_0 + \int_0^T \| u_t + R^{-1} B' K(t; T) x_t \|_2^2 dt$$

$$- \int_0^T \| v(x) - S^{-1} C' K(t; T) x_t \|_2^2 dt.$$  

The last integral in (14) vanishes, according to the definition of $v(x)$, and thus $\int_0^\infty w(x, u, v) dt > x_0 K(0; T) x_0$. It follows from the above remarks on (13) and (14) that

$$J_m(u, v; x_0) > x_0 K(0; T) x_0 - \epsilon$$

Combining (12) and (15) we obtain $J_m(u, v; x_0) > x_0 K^+ x_0 - \epsilon$ which is exactly what had to be proved, i.e., (11).

Remark: If the maximizer commits himself to a linear strategy, say $v(x) = K x$, for all $x$, it might be possible for the minimizer to drive $x$ to a region of the state space where the value functional is negative and large in absolute value. The strategies $v(x)$ guard against such an occurrence by stipulating $v(x) = 0$ for $x > T$. The minimizer cannot risk driving $x$ very far from the origin at time $T$ as this will result in the functional $\int_0^T \{ x' Q x + u' R u \} dt$ reaching a very large and positive value regardless of the choice of $u$.

III. CONCLUSIONS

We showed that the solution of (8) which satisfies $\lim_{t \to \infty} K(t) = K^+$ determines the value of the infinite duration game. (5). In contrast to what might be expected from the theory of the linear regulator problem, one cannot guarantee the existence of linear time-invariant strategies as in (7). In Proposition 2 we show that $x_0 K^+ x_0$ is the value of the game by constructing appropriate e-optimal strategies. It is an interesting problem to find conditions for the strategy pair $u^0, v^0$ to be in equilibrium. If $K^+$ is strictly feedback stabilizing, i.e.,

$$\text{Re}(\lambda - B R^{-1} B' K^+ + CS^{-1} C' K^+) < 0,$$

it can be shown that the optimal response to $u^0$ is $v^0$. The proof is almost identical to the one presented in Proposition 2. It suffices therefore to find conditions for $u^0$ to be the optimal response to $v^0$. The problem faced by the maximizing player when the maximizer uses $u^0$ is

$$\min_{u^0} J_m(u, v^0, x_0) = \int_0^\infty (x' Q - K^+ CS^{-1} C' x + u' R u) dt$$

$$x = (A + CS^{-1} C' K^+) x + Bu, \quad x(0) = x_0$$

and its corresponding Riccati equation is

$$(A + CS^{-1} C' K^+) M + M (A + CS^{-1} C' K^+) + Q - K^+ C S^{-1} C' K^+ = M B R^{-1} B' M.$$  

It is easy to see that $M = K^+$ is a real symmetric stabilizing solution of (17). For this to imply that the optimal control for $u^0$ is $v^0 = - R^{-1} B' K^+ x$, the Riccati equation (17) must be shown to possess a negative definite solution [4, theorem 8], or the differential Riccati equation corresponding to (16) must have no focal points on any interval.
APPENDIX

Lemma A1 provides alternative expressions for the value functional.

**Lemma A1:** 1) Assume that (2) has a solution $K(t; T)$ on $[0, T]$. Then for any controls $u_0, v_0$ on $[0,T]$ we have

$$
J_T(u; v; x_0) = x_0^T K(0; T)x_0 + \int_0^T \left( ||u_t + R^{-1}B'K(t; T)x_t||_K^2 - ||v_t - S^{-1}C'Kx_t||_R^2 \right) dt
$$

and

$$
J_T(u; v; x_0) = x_0^T K(t; T)x_0 + \int_0^T \left( ||u_t + R^{-1}B'K(t; T)x_t||_K^2 - ||v_t - S^{-1}C'Kx_t||_R^2 \right) dt
$$

2) Assume that $K$ is a real symmetric solution of (8). Then for any two controls $u_0, v_0$ on $[0,T]$ we have

$$
J_T(u; v; x_0) = x_0^T K(0; T)x_0 - \int_0^T ||u_t + R^{-1}B'Kx_t||_K^2 dt
$$

and

$$
J_T(u; v; x_0) = x_0^T K(t; T)x_0 - \int_0^T ||u_t + R^{-1}B'K(t; T)x_t||_K^2 dt
$$

The proof is a "completing the square argument." See [3, lemma 3.2.6]. An immediate consequence is Proposition A1.

**Proposition A1:** Under the assumptions of Lemma A1 the strategies

$$
u_t^f(x) = -R^{-1}B'K(t; T)x$$
$$v_t^f(x) = -S^{-1}C'K(t; T)x$$

are in equilibrium for game (1). The value of the game is $x_0^T K(0; T)x_0$.

The assumption $Q > 0$ guarantees a nice behavior of $K(t; T)$.

**Lemma A2:** Under the assumptions of Lemma A1, $K(t; T)$ increases in the sense of positive definite matrices with increasing $t$.

**Proof:** A purely algebraic proof is possible on the basis of [1, exercise 1, p. 167]. Here we give a game theoretic one. In view of Proposition A1 the value of a game (1) of duration $T - t$ is $V_{T-t}(x_0) = x_0^T K(t; T)x_0$. Consider now the game of duration $T-t+\Delta t$ and consider the following strategy for $v$:

$$
\tilde{v}(x_t) = \begin{cases} 
S^{-1}C'K(t; T)x_t, & t \in [0, T-t) \\
0, & t \in [T-t, T-t+\Delta t].
\end{cases}
$$

For $t \in [T-t, T-t+\Delta t]$ the integrand of the value functional is non-negative, and for $t \in [0, T-t)$ it can be written using Lemma A1 as

$$
\int_0^{T-t} w(x, u, v) dt = x_0^T K(t; T)x_0
$$

The definition of $\tilde{v}(x_t)$ was used to eliminate the second integrand of the expression in Lemma A1. The above equation shows that the maximizer can guarantee a payoff greater than $V_{T-t}(x_0)$ and thus

$$
V_{T-t}(x_0) = x_0^T K(t; T)x_0 < V_{T-t+\Delta t}(x_0) = x_0^T K(t; T-x_0) Q.E.D.
$$

The following lemma provides an upper bound for $K(t; T)$ for all $t$.

**Lemma A3:** Let $K = K^*$ be a nonnegative solution of (8) and let $K(t; T)$ exist for $t \in [0, T]$. Then $K > K(t; T)$.

**Proof:** We can use Lemma A1 to derive the expressions

$$
J(u; v; x_0) = \int_0^T w(x, u, v) dt = x_0^T Kx_0 - x_0^T Kx_0
$$

and

$$
J(u; v; x_0) = x_0^T K(t; T)x_0 + \int_0^T \left( ||u_t + R^{-1}B'K(t; T)x_t||_K^2 - ||v_t - S^{-1}C'Kx_t||_R^2 \right) dt
$$

where the initial condition is $x(t) = x_0$. Since (18) and (19) hold for any $u, v$ we can substitute the strategy pair $\tilde{v}(x_t) = S^{-1}C'K(t; T)x_t$ and $\tilde{u}(x_t) = -R^{-1}B'Kx_t$. Subtracting (19) from (18) we get

$$
x_0^T [K(t; T) - K(t; T)]x_0 = x_0^T Kx_0 + \int_0^T \left( ||u_t - S^{-1}C'Kx_t||_R^2 - ||v_t - S^{-1}C'Kx_t||_R^2 \right) dt
$$

This completes the proof as $K, R, S$ are nonnegative definite matrices.

Q.E.D.

We consider now the solution of (8) which satisfies $K^* - \lim_{t \to \infty} K(t; T)$. We know that $K^* > 0$ and is the smallest positive definite solution of (8) (see Proposition 1). The following lemma uses this information to show that $K^*$ is feedback stabilizing.

**Lemma A4:** Let $K^*$ be defined as above. Then $\Re\lambda(A - BR^{-1}B'K^* + CS^{-1}C'K^*) < 0$.

**Proof:** Since $K^* > 0$, $(K^*)^{-1}$ exists and satisfies

$$
-AM - MA' + (BR^{-1}B' - CS^{-1}C') = MQM
$$

which is obtained by operating on both sides of (8) by $(K^*)^{-1}$. It is shown in [4] that (21) has a maximal solution $M^*$ which satisfies

$$
\Re\lambda(A' - MQM) < 0.
$$

Since $M^*>K^*>0$ it follows that $\Re\lambda(M^*) < 0$. But $(M^*)^{-1}$ is a solution of (8) contradicting the minimality of $K^*$ unless $M^* = (K^*)^{-1}$. By rearranging (21) we get

$$
-K^* - (Q(K^*)^{-1}) = K^* (A - BR^{-1}B'K^* + CS^{-1}C'K^*) (K^*)^{-1}.
$$

Using the fact that similar matrices have identical eigenvalues we get from (22)

$$
\Re\lambda(A' - Q(K^*)^{-1}) - \Re\lambda(A - BR^{-1}B'K^* + CS^{-1}C'K^*) < 0.
$$

Q.E.D.

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