Pricing and hedging contingent claims using variance and higher-order moment futures

by

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Abstract

This paper suggests perfect hedging strategies of contingent claims under stochastic volatility and/or random jumps of the underlying asset price. This is done by enlarging the market with appropriate futures contracts whose payoffs depend on higher-order sample moments of the underlying asset price process. It also derives a model-free relation between these higher-order moment futures contracts and the value of a composite portfolio of European options, which can be employed to perfectly hedge variance futures contracts. Based on the theoretical results of the paper and on options and variance futures contracts price data written on the S&P 500 index, it is shown that, first, random jumps are priced in the market and, second, hedging strategies for European options employing variance and higher-order moment futures considerably improves upon the performance of traditional delta hedging strategies. This happens because these strategies account for volatility and jump risks.

Keywords: Variance futures contracts, higher-order moments, volatility and jump risks, hedging strategies.

JEL: C14, G11, G13

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1 Introduction

Since the seminal papers of Black and Scholes (1973), and Merton (1973) on pricing and hedging contingent claims, there has been a vast amount of studies trying to develop parametric option pricing models based on more general assumptions of the underlying asset price stochastic process. Prominent examples of these studies include the stochastic volatility (SV) option pricing model of Heston (1993) and its extension allowing for random jumps in the underlying asset price (denoted as SVJ), suggested by Bates (1996), as well as its various extensions (see, e.g., Eraker, Johannes and Polson (2003)). Despite the plethora of studies extending the Black-Scholes (BS) model for option pricing, there are a few studies focused on hedging derivatives under more general assumptions about the stochastic process of the underlying asset price (see, e.g., Pham (2000) for a theoretical survey and Bakshi, Cao and Chen (1997) for an empirical application). This happens because hedging derivatives under more general assumptions of the asset process is not so straightforward, compared to the BS framework. Indeed, as the market is incomplete perfect hedging against all sources of risks and, in particular, volatility and jump risks associated with a position in derivatives is not feasible.

In this paper, we extend the BS framework to provide perfect hedging strategies of contingent claims under more general assumptions of the underlying asset price allowing for stochastic volatility and/or random jumps. This is done by enlarging the market with appropriate futures contracts, whose payoffs depend on higher-order sample moments of the asset price process. These contracts can efficiently hedge against volatility and jump risks. The paper contributes into the literature on many fronts. It derives new hedging ratio formulas of the above futures contracts, the underlying asset and the zero-coupon bond of the self-financing hedging portfolio for both SV and SVJ models. For the SV model, we demonstrate that enlarging the market with a variance futures (or swap) contract, which is nowadays traded in the market, can perfectly hedge positions in contingent claims written on the underlying asset and/or its volatility. This makes the market complete.

For the SVJ model, the paper shows that, in addition to variance futures, higher than second-order moment futures contracts should be included in the self-financing portfolio in

\footnote{These studies develop the so-called quadratic hedging strategies which are based on risk minimization and mean-variance criteria.}
order to hedge the exposure of a contingent claim price against random jumps. If this claim is a variance futures contract, we show that it can perfectly hedged by a composite portfolio of out-of-the-money (OTM) European call and put options, and appropriate positions in higher-order moment futures. Since the size of jumps is random, a sufficiently large number of higher-order moment futures contracts is required for perfect hedging. As this number goes to infinity, the paper shows that the value of the self-financing hedging portfolio converges to the price of the contingent claim, thus making the market approximately (or quasi) complete in the sense of Björk, Kabanov and Runggaldier (1997) and Jarrow and Madan (1999). In practice, someone can approximate the value of this hedging portfolio with a finite number of higher-order moment futures contracts.

Several recent studies have dealt with the problem of constructing perfect hedging strategies under an incomplete market model. Corcuera et al (2005) have suggested that the market can be completed if it is enlarged with power jump assets. Yip, Stephens and Olhede (2010) have noted that the values of power jump assets can not be observed in the market and, hence, cannot be traded. Thus, they propose hedging strategies based on higher-order moment futures. The latter depend on the increment of the underlying asset price, and can hence be observed and traded in the market. Our paper extends these studies in three directions. First, we consider a model with stochastic volatility and random jumps, whereas the above studies assumed a constant volatility with jumps. This implies that contingent claims, in addition to jump risk, are also exposed to volatility risk which is consistent with evidence provided in the literature. Second, we introduce perfect hedging strategies which imply well known option pricing formulas. Finally, we empirically investigate the performance of these hedging strategies using both simulated and actual data.

The results of our paper can find applications to different markets. These include the stock market (see Bakshi and Kapadia (2003), and Bollen and Whaley (2004)), the fixed-income security market (see Li and Zhao (2006) and Jarrow, Li and Zhao (2007)), the mortgage-backed security market (see Boudoukh, Whitelaw, Richardson and Stanton (1997)) and the credit default swap market (see Brigo and El-Bachir (2010)). In this paper, they are used to empirically address two questions, which have important portfolio management implications. The first is if random jumps are priced in the market. According to theory, the price of jump risk must be reflected in the difference between the price of a variance futures contract and the value of the composite options portfolio. This will be
examined without relying on any parametric model of the market. The second question is if a position in a European call, or put, can be efficiently hedged based on a two-instruments hedging strategy, which considers a variance futures contract as a hedging vehicle, or it requires hedging strategies which rely on higher-order moment futures. This can shed light on the ability of variance futures contracts to hedge against volatility risk and can reveal the number of higher-order moment futures contracts needed for approximately completing the market in the presence of random jumps. To answer the above questions, we rely on options and variance futures price data written on the S&P 500 index.

The empirical results of the paper lead to the following main conclusions. First, random jumps are indeed priced in the variance futures market. The prices of these contracts are found to be significantly bigger in magnitude than those of the composite options portfolio, which spans variance futures contracts under the assumption of no jumps. Second, a two-instruments hedging strategy, which also includes a variance futures contracts in the self-financing portfolio, is found to considerably improve upon the performance of the traditional delta hedging strategy often used in practice, which only includes the underlying asset and the zero-coupon bond in this portfolio. The improving performance of this strategy comes from the fact that variance futures contracts can efficiently hedge the exposure of a call, or a put, option to volatility risk. The inclusion of a finite (up to the $4^{th}$-order) number of higher-order moment futures contracts into the self-financing portfolio is found to further improve the performance of the above two-instruments hedging strategy, especially for short-term options which are more sensitive to jump risk.

The paper is organized as follows. Section 2 gives all necessary definitions of the variance and higher-order moment futures contracts, as well as the relation between them and the value of the composite portfolio of OTM options needed in the analysis of the paper. Section 3 shows how these contracts can be used to perfectly replicate the price of a contingent claim under the SV and SVJ models. Section 4 conducts a Monte Carlo simulation exercise to examine the performance of the suggested hedging strategies and to assess the magnitude of different sources of errors encountered when they are implemented in practice. Section 5 addresses the empirical questions of the paper. Section 6 concludes the paper. All derivations are given in a technical Appendix.
2 Variance and higher-order moment futures contracts

2.1 Variance futures contracts

Variance futures (or swaps) are derivative contracts in which one counterparty agrees to pay the other a notional amount times the difference between a fixed level and a realized level of variance. The fixed level is the variance futures price. Realized variance is determined by non-central second-order sample moment of the underlying asset over the life of the contract. More precisely, let $T - \tau = t_0 < t_1 < \ldots < t_n = T$ be a partition of time interval $[T - \tau, T]$ into $n$ equal segments of length $\Delta t$, where $\tau$ is accrual period of the contract, given as $n/252$. Then, the payoff of this variance futures contract is given as

$$V_{T-\tau,T}^{(n)} = \frac{252}{n} \sum_{i=1}^{n} \ln \left( \frac{S_i}{S_{i-1}} \right)^2,$$

where $S_i$ is the price of the underlying asset at time $t_i$. Since this contract worths zero on the inception date, no arbitrage dictates that the time-$t$ price of the above variance futures contract is given by the risk-neutral expected value of the contract payoff, i.e.,

$$FV_t^{(n)} = E_t^Q \left[ V_{T-\tau,T}^{(n)} \right],$$

where $Q$ denotes the risk-neutral measure. If time is continuous, i.e. $n \to \infty$, payoff $V_{T-\tau,T}^{(n)}$ converges in probability to the annualized quadratic variation of the log-price process of the underlying asset, denoted as $\frac{1}{\tau} \langle X, X \rangle_{T-\tau,T}$ (see Protter (1990)), where $X = \ln S$. Then, the continuous-time price of the variance futures contract, defined as $FV_t$, is given by the risk-neutral expected value of $\frac{1}{\tau} \langle X, X \rangle_{T-\tau,T}$, i.e.,

$$FV_t = E_t^Q \left[ \frac{1}{\tau} \langle X, X \rangle_{T-\tau,T} \right].$$

Under the assumption that log-price $X$ has no discontinuous component, variance futures contracts can be priced and perfectly hedged based on a composite portfolio of European call and put contracts (see, e.g. Carr and Lee (2009) and Gatheral (2006)), given

\footnote{As aptly shown by Broadie and Jain (2008), the effects of assuming continuous time on pricing variance futures contracts is negligible. This allows us to rely on continuous-time models in pricing and hedging contingent claims. The magnitude of the discrete-time approximation error of the variance futures contract payoff will be also examined in the simulation study presented in Section 4.}

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as
\[ V_{o,t} = \frac{2e^{\tau \rho}}{\tau} \left[ \int_{S_t}^{\infty} \frac{1}{K^2} C_t(\tau, K) dK + \int_{0}^{S_t} \frac{1}{K^2} P_t(\tau, K) dK \right]. \] (3)

The value of this portfolio, \( V_{o,t} \), depends on the values of two subportfolios of OTM European call and put options, with strike price \( K \) and expiration date \( T \), denoted as \( C_t(\tau, K) \) and \( P_t(\tau, K) \) respectively, where \( \tau = T - t \) denotes the maturity interval. However, this is not true if \( X \) contains a discontinuous component, which is found to characterize the price dynamics of many financial assets (see, e.g., Barndorff-Nielsen and Shepard (2006)). In this case, the price of variance futures contracts \( FV_t \), for \( T - \tau \leq t \leq T \), is given as
\[ FV_t = \frac{1}{\tau} \langle X, X \rangle_{T-\tau,t} + V_{o,t} + 2 \frac{e^{\tau \rho}}{\tau} E_t^{Q} \left[ \int_{t}^{T} \frac{dS_u}{S_u} - \frac{S_T - S_t}{S_t} \right] - 2 \sum_{j=3}^{\infty} \frac{1}{j!} E_t^{Q} \left[ \sum_{t<u<T} (\Delta X_u)^j \right]. \] (4)

See Appendix, for the proof. Term \( E_t^{Q} \left[ \int_{t+}^{T} \frac{dS_u}{S_u} - \frac{S_T - S_t}{S_t} \right] \) of formula (4) depends on the underlying asset’s price of the variance futures contract. If this asset is a stock (or a stock market index) that pays a constant dividend yield \( \delta \), then this term becomes
\[ E_t^{Q} \left[ \int_{t+}^{T} \frac{dS_u}{S_u} - \frac{S_T - S_t}{S_t} \right] = 1 + (r - \delta)T - e^{(r - \delta)T}. \]

If the underlying asset is a futures or swap contract, this term becomes zero.

Formula (4) implies that, if \( X \) has no discontinuous component, then a long position in a variance futures contract can be perfectly replicated by a portfolio constructed by holding: (a) a static long position in \( \frac{2}{\tau K^2} \) calls at strikes \( K > S_t \) and \( \frac{2}{\tau K^2} \) puts at strikes \( K < S_t \) with time to maturity \( \tau \), which form the composite options portfolio, (b) a dynamic position in \( \frac{2}{\tau} \left( \frac{1}{S_u} - \frac{1}{S_t} \right) \) shares at any time \( u \in [t, T] \), and (c) \( e^{-\tau \rho} \frac{1}{\tau} \langle X, X \rangle_{T-\tau,t} \) in cash (see, e.g., Carr and Lee (2009)).

However, formula (4) clearly indicates that the above portfolio can not perfectly replicate a long position in variance futures contracts if process \( X \) contains a discontinuous component, due to the sum of higher-order moment terms \( \sum_{j=3}^{\infty} \frac{1}{j!} E_t^{Q} \left[ \sum_{t<u<T} (\Delta X_u)^j \right] \). Ignoring these terms will lead to an imperfect hedge of a variance futures contract by means of traded option contracts. The magnitude of the hedging errors encountered will be investigated in Section 5, based on actual market data. To replicate these terms, in the
next section we will introduce futures contracts whose payoffs are defined by higher than second-order sample moments of $X$.

### 2.2 Higher-order moment futures contracts

Let the $j$th-order moment futures contract, for $j \geq 3$, be a contract in which one counterparty agrees to pay the other a notional amount times the difference between a fixed level (contract’s price) and a realized level of the $j$th-order non-central sample moment of process $X$. The payoff of this contract is given as

$$V^{(n)}_{(j), (T-\tau, T)} = \frac{252}{n} \sum_{i=1}^{n} \ln \left( \frac{S_i}{S_{i-1}} \right)^j, \text{ for } j \geq 3,$$

(5)

where $n$ are segments of time interval $[T-\tau, T]$ of length $\Delta t$. The price of this contract under the no arbitrage principle is given by the risk-neutral expected value of $V^{(n)}_{(j), (T-\tau, T)}$, i.e.

$$F_{(j), t} = E_t^Q \left[ V^{(n)}_{(j), (T-\tau, T)} \right].$$

In continuous time (i.e. $n \rightarrow \infty$), it can be shown that, under weak assumptions, payoff $V^{(n)}_{(j), (T-\tau, T)}$ converges in probability to $\frac{1}{j} \sum_{T-\tau < u \leq T} (\Delta X_u)^j$ (see Jacod (2008)). Thus, the continuous-time value at time $t$ of the above contract, denoted as $F_{(j), t}$, can be defined as

$$F_{(j), t} = E_t^Q \left[ \frac{1}{j} \sum_{T-\tau < u \leq T} (\Delta X_u)^j \right].$$

(6)

This definition of price $F_{(j), t}$ implies that the prices of odd-order moment futures will be negative, if large drops in $X$ occurring between $T - \tau$ and $T$ dominate its movements. In this case, positions in these futures contracts should be reversed. That is, a long position in a higher-order moment futures will be taken as a short position, and vice versa.

If we assume that a finite number of $j = 3, 4, ..., N$ higher-order moment futures are traded in the market, then formula (4) implies that long positions in variance futures contracts can be hedged against discontinuous movements in $X$, captured by term $\sum_{j=3}^{\infty} \frac{1}{j^2} E_t^Q \left[ \sum_{T-\tau < u \leq T} (\Delta X_u)^j \right]$. This will happen, if the replicating portfolio, defined in the previous section, include also a short position in $\frac{1}{\tau j^2}$ $j$th-order moment futures contracts, for $j = 3, 4, ..., N$. For sufficiently large $N$, it can be easily proved that this replicating
portfolio can approximately hedge long positions in variance futures contracts, even when
the underlying asset price contains discontinuous components. These results mean that
existence of higher-order moment futures can make the market approximately (or quasi)
complete (see Björk, Kabanov and Runggaldier (1997), or Jarrow and Madan (1999)).
Including these contracts in replicating portfolios of variance futures is thus expected to
improve their performance. This is first noted by Schoutens (2005), who has included a
third-order moment futures contract in these portfolios.

Note, at this point, that the above higher-order moment futures contracts can not
capture the variation of the discontinuous component of \( \mathbb{P} \), defined as
\[
\varepsilon \mathbb{P} \mathbb{P} - \varepsilon \mathbb{P} \mathbb{P} \leq 6 \Delta \mathbb{P},
\]
where
\[
\mathbb{P} \mathbb{P} \mathbb{P} = \sum_{u \leq T} (\Delta \mathbb{P})^2,
\]
and, hence,
\[
\frac{252}{n} \left( \sum_{i=1}^{n} \ln \left( \frac{S_i}{S_{i-1}} \right)^2 - \frac{\pi}{2} \frac{n}{n-1} \sum_{i=1}^{n-1} \ln \left( \frac{S_i}{S_{i-1}} \right) \ln \left( \frac{S_{i+1}}{S_i} \right) \right) \rightarrow \frac{1}{\tau} \sum_{T-\tau < u \leq T} (\Delta \mathbb{P})^2,
\]
As can be seen in Section 3, this contract can help us to separate the volatility from the

\footnote{Note also that formula (4) implies that the \( j \)-th order moment futures contract can be hedged by means of European calls and puts, the variance futures contract and a finite number of \( s \)-th order moment futures contracts, for \( s \neq j \). This can be done by solving (4) for \( F_{(j), \tau} \) using the definition of \( F_{(j), \tau} \) given by (6).}
jump risk. Both of these risks are encountered when investing in variance futures contracts. Furthermore, it is necessary to perfectly hedge contingent claims, like European options and options written on volatility swaps, against the above two sources of risk. Under the risk-neutral measure $Q$, the continuous-time price of a bipower variations futures contract at time $t$ is given as

$$F_{(2),t} = E_t^Q \left[ \frac{1}{\tau} \sum_{\tau < u \leq T} (\Delta X_u)^2 \right].$$

(9)

### 3 Pricing and hedging contingent claims using variance and higher-order moment futures contracts

In the previous section, we show how to price and perfectly hedge a variance future contracts in the presence of a discontinuous component in the underlying log-price process $X$. In this section, we generalize these results to the case of any contingent claim written on the price of a stock and/or its volatility. These contingent claims include plain vanilla European options, volatility swaps or options written on these swaps. The latter constitute more complex derivatives, which have non-linear payoffs with respect to volatility (see Carr and Lee (2009) and Friz and Gatheral (2005)). Most of these contingent claims are traded in the market. To price and hedge them, we will initially assume that stock prices follow the stochastic volatility (SV) model (see, e.g., Heston (1993)). Then, our analysis will be extended to the SV model allowing for jumps (SVJ) (see, e.g., Bates (1996)). This model has been found to better describe the dynamics of stock prices or their volatility and, thus, it is considered as a common specification in the literature. Our analysis starts with the SV model, as it is simpler and enables us to more clearly see the role of variance future contracts to perfectly hedge positions in contingent claims.

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4Note that subtracting $F_{(2),t}$ from $FV_t$ yields

$$FV_t - F_{(2),t} = E_t^Q \left[ \frac{1}{\tau} \langle X,X \rangle_{\tau,T} \right],$$

which shows that a long position in a variance futures contract and a short position in a bipower variation futures contract forms a portfolio whose payoff depends only on the continuous part of quadratic variation $\langle X,X \rangle^c$. The price of this portfolio can be thus used to determine the market price of volatility risk.
3.1 The SV model

Assume that the underlying asset is a stock which pays dividends at rate \( \delta \). Then, the SV model assumes that its price \( \Sigma \) and volatility \( \kappa \) obey the following processes:

\[
\frac{d\Sigma_t}{\Sigma_t} = \mu_t^S dt + \sqrt{\kappa} dW_t^{(1)}, \tag{10}
\]

\[
d\kappa_t = \kappa (\theta - \kappa_t) dt + \sigma \sqrt{\kappa_t} dW_t^{(2)}, \tag{11}
\]

where \( \mu_t^S = (r_t - \delta_t + \gamma S V_t) \) is the expected return of the underlying asset, where \( r \) is the rate of return of a zero-coupon bond and \( \gamma S V_t \) is the market price of risk, and processes \( W^{(1)} \) and \( W^{(2)} \) are two correlated Brownian motions, with correlation coefficient \( \rho \).\(^5\) The last assumption implies that there exists a Brownian motion process \( W^{(3)} \) which is independent of \( W^{(1)} \) and satisfies the following relationship: \( dW_t^{(2)} = \rho dW_t^{(1)} + \sqrt{1 - \rho^2} dW_t^{(3)} \). Under the SV model, the stochastic discount factor (SDF) process \( \Lambda_t \) used for asset pricing, is given as

\[
\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \gamma S \sqrt{\kappa_t} dW_t^{(1)} - \frac{\gamma V_t \sqrt{\kappa_t}}{\sigma \sqrt{1 - \rho^2}} dW_t^{(3)}, \tag{12}
\]

where parameter \( \gamma V_t \) accounts for the market price of volatility risk (see Chernov and Ghysels (2000)).

This market is incomplete in the sense that a position in a contingent claim can not be perfectly replicated by a self-financing portfolio, referred to as hedging portfolio, consisting of a position in the underlying stock and the zero-coupon bond. In this section, we will show that this market can be completed if the hedging portfolio contains, in addition to the above two assets, a position in a variance futures contract. To this end, our analysis starts with pricing variance futures contracts.

3.1.1 Pricing variance futures contracts

Consider that, apart from the stock and zero-coupon bond, a variance futures contract is traded in the market. This is written on stock price \( S \) and is defined over time interval \([0, T]\).

\(^5\)Note that our results do not depend on the specification of the volatility process \( V \). They hold under the following more general specification of \( V \):

\[
dV_t = \alpha(t, V_t) dt + \beta(t, V_t) dW_t^{(2)},
\]

where \( \alpha \) and \( \beta \) are appropriately defined functions.
The payoff of this contract at time $T$ is given as $\frac{1}{T} \int_0^T V_u du$. The following proposition gives analytic formulas of the price of this contract and its expected return, $E_t \left[ \frac{dFV_t}{FV_t} \right]$, under the SV model.

**Proposition 1** Assume that no-arbitrage opportunities exist in the market. Then, under the SV model, the price of a variance futures contract at time $t \in [0, T]$ is given as

$$FV_t = \frac{1}{T} \left( \int_0^t V_u du + F(t, V_t) \right),$$

where $F(t, V_t) = \tau \left( \psi_t V_t + (1 - \psi_t) \theta^Q \right)$ with $\psi_t = (1 - e^{-\kappa^Q \tau}) / (\kappa^Q \tau)$ and $\tau = T - t$, while its expected return, denoted $\mu_t^{FV}$, is given as

$$\mu_t^{FV} dt \equiv E_t \left[ \frac{dFV_t}{FV_t} \right] = \frac{\partial \ln FV_t}{\partial \ln V} \tilde{\gamma}_V dt,$$

where $\theta^Q$ and $\kappa^Q$ constitute risk-neutral counterparts of $\theta$ and $\kappa$, respectively, and $\tilde{\gamma}_V = \sigma \rho \gamma_S + \gamma_V$.

Proposition 1 shows that the expected return of a variance futures contract $\mu_t^{FV}$ depends on the market price of risk $\gamma_S$, due to the correlation between stock price $S$ and its volatility $V$, and the price of volatility risk $\gamma_V$. If $\gamma_V < 0$, as shown in many empirical studies (see, e.g., Carr and Wu (2010)), then $\mu_t^{FV}$ will be negative, given that $\gamma_S > 0$, $\rho < 0$ (due to the leverage effect) and $\partial \ln FV / \partial \ln V > 0$. The negative expected return of this contract is something to expect, since it pays when volatility unexpectedly increases. Due to their risk-averse behavior with respect to this event, investors wish to pay more to acquire variance futures contracts. Another interesting result of Proposition 1 is that, under the SV model, the price of volatility risk $\gamma_V$ can be uniquely determined by expected returns $\mu_t^{FV}$ and $\mu_t^S$, since $\gamma_V = \tilde{\gamma}_V - \sigma \rho \gamma_S$. This result implies that a position in any contingent claim can be perfectly replicated by a portfolio consisting of a number of stocks, bonds and a position in variance futures contract. It will be shown more rigorously in the next section.

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6 The negative sign of $\gamma_V$ can be also explained theoretically based on Lucas’ consumption asset pricing model in which the representative investor has a coefficient of risk aversion $\gamma$. Under these assumptions, discount factor $\Lambda_t$ becomes $\Lambda_t = C_t^{-\gamma}$, where $C_t$ denotes consumption at time $t$. Based on this model, it can be shown that $-\gamma E_t \left[ \frac{dC_t}{C_t} dV_t \right] = E_t \left[ \frac{d\Lambda_t}{\Lambda_t} dV_t \right] = -\gamma_V V_t dt$. The last relationship implies that, if $E_t \left[ \frac{d\Lambda_t}{\Lambda_t} dV_t \right] > 0$, meaning that the representative investor substitutes future with current consumption at a greater rate when volatility tend to increase, then $\gamma_V < 0$. 

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3.1.2 Pricing and hedging contingent claims

Consider now a contingent claim \( C \). Its price at time \( t \), denoted as \( C_t \), depends on both the stock price and the spot variance, i.e. \( C_t = C(t, S_t, V_t) \), where function \( C(\cdot) \) has continuous second order partial derivatives. As noted before, this claim can be a plain vanilla European option, a volatility swap or an option written on this swap. The self-financing portfolio which is required to hedge (replicate) a position in this claim consists of a position in the underlying stock, the zero-coupon bond and a variance futures contract. The numbers of these hedging instruments at time \( t \in [0, T] \), known as "deltas", will be defined by the following vector of real-value adapted processes \( \phi_t = (\phi_t^S, \phi_t^B, \phi_t^{FV}) \). At time \( t \), the value of this portfolio will be equal to the price of the contingent claim \( C_t \), i.e.

\[
C_t = \phi_t^S S_t + \phi_t^B B_t, \tag{15}
\]

where \( B \) denotes the price of zero-coupon bond. Note that the number of variance futures contacts \( \phi_t^{FV} \) is omitted from equation (15), since it costs nothing to take a position in them.

Next proposition derives the values of the vector of deltas \( \phi_t \). These can be employed to perfectly hedge changes in contingent claim price \( C_t \) driven by stochastic movements in stock price \( S \) and volatility \( V \). The proposition also derives the instantaneous expected return of contingent claim \( C_t \), defined as \( E_t \left[ \frac{dC_t}{C_t} \right] \).

**Proposition 2** Consider a contingent claim \( C \) written on the underlying stock and/or its volatility, with price function \( C_t = C(t, S_t, V_t) \), where \( C(\cdot) \) is a continuous function which has second order partial derivatives. Then, under the SV model and the no-arbitrage principle, the deltas of the self-financing portfolio replicating contingent claim price \( C_t \), at \( t \in [0, T] \), are given as follows:

\[
\phi_t^S = \frac{\partial C}{\partial S} \quad \text{and} \quad \phi_t^B = B_t^{-1} \left( C_t - \phi_t^S S_t \right)
\]

for the underlying stock and zero-coupon bond, respectively, and

\[
\phi_t^{FV} = \frac{\partial C}{\partial FV}
\]
for the variance futures contract. The instantaneous expected return of \( \mathbb{E} \) is given as

\[
E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln S} (\mu_t^S + \delta_t - r_t) dt + \frac{\partial \ln C}{\partial \ln FV} \mu_t^{FV} dt,
\]

(16)

where \( \mu_t^S \) and \( \mu_t^{FV} \) are the expected returns of the underlying stock and variance futures contract, respectively, defined in Proposition 1.

Proposition 2 proves that, under the SV model, a contingent claim \( C \) written on a stock and/or its volatility can be perfectly hedged by a self-financing portfolio consisting of a zero-coupon bond, the underlying stock and a variance futures contract. The price of this contingent claim, \( C_t \), is uniquely determined by the prices (or expected returns) of these three financial instruments. As shown in the appendix, this price can be obtained by solving Heston’s (1993) partial differential equation (PDE). In contrast to Heston’s approach, which obtains this PDE relying on equilibrium approach, we have derived it explicitly by eliminating all stochastic terms determining the contingent claim price. These results mean that, under the SV model, the market enlarged with a variance futures contract becomes complete. As mentioned before, contingent claim \( C \) can also include volatility swaps and options written on them. The results of Proposition 2 imply that these assets can be perfectly hedged by variance futures contracts, whose price equals the value of the composite portfolio of OTM options \( V_{o,t} \), under the SV model (see Proposition 1). Thus, these more complex derivatives, with non-linear payoffs, can be also perfectly hedged with respect to volatility by forming a portfolio of OTM options.

Formula (16) of Proposition 2 indicates that the expected return of contingent claim \( C \), \( E_t \left[ \frac{dC_t}{C_t} \right] \), in addition to the risk-free rate \( r \) and expected return of the stock (implied by the BS model), also depends on the volatility risk premium. The latter is reflected in the expected return of the variance futures contract \( \mu_t^{FV} \) (see equation (14)). This result means that, if the hedging portfolio of contingent claim \( C \) does not include the variance futures contract, then this will lead to non-zero delta-hedged gains, as it will not perfectly replicate claim \( C \). As shown by Bakshi and Kapadia (2003), these delta-hedged gains are expected to be negative under the physical measure, when \( \mu_t^{FV} < 0 \) and \( \frac{\partial \ln C}{\partial \ln FV} > 0 \). These results are consistent with the empirical findings of Coval and Shumway (2001), which show that BS model overestimate at-the-money (ATM) and long-term European options returns. This happens because \( \frac{\partial \ln C}{\partial \ln FV} \) is positive and larger in magnitude for these two categories of
The results of Proposition 2 have also a useful econometric implication. They can facilitate estimation of the parameters of the SV model under risk-neutral measure $\mathbb{Q}$. This can be done by exploiting information from options, variance futures and stock price data, jointly. In particular, the values of latent variable $V_t$, for all $t$, can be substituted by a linear function of variance futures prices $F V_t$, which are observable in the market, using formula (13). This will reduce the number of parameters required in the estimation of the model and will thus increase the efficiency of their estimates, given also that this estimation method exploits additional market information.

### 3.2 The SVJ model

Under the SVJ model, the processes driving stock price and volatility changes are given as follows:

$$
\frac{dS_t}{S_{t-}} = \mu_t^S dt + \sqrt{\nu_t} dW^{(1)}_t + J_t dN_t - \lambda \pi dt \tag{17}
$$

$$
dV_t = \kappa (\theta - V_t) dt + \sigma \sqrt{\nu_t} dW^{(2)}_t, \tag{18}
$$

where $\mu_t^S$ now is defined as $\mu_t^S = (\delta_t + \gamma_S V_t + \lambda (\pi - \pi^Q))$, $J_t$ is the random percentage jump conditional on a jump occurring distributed as $\ln(1+J_t) \sim N (\mu_J, \sigma_J^2)$, $N_t$ is a Poisson process with intensity parameter $\lambda$, i.e., $\text{prob}(dN_t = 1) = \lambda dt$, $\pi = \exp (\mu_J + \frac{1}{2} \sigma_J^2) - 1$ and $\pi^Q$ is the risk neutral counterpart of $\pi$ (see, e.g., Bates (1996)). As with the SV model, $W^{(1)}$ and $W^{(2)}$ are two Brownian motion processes which are correlated with each other.\footnote{As with the SV model, note that our results hold under a more general specification of $V$ (see fn 5) and the jump-size distribution $J$.}

For the SVJ model, the SDF process $\Lambda$ is given as

$$
\frac{d\Lambda_t}{\Lambda_{t-}} = -(\delta_t + \lambda \pi_A) dt - \gamma_S \sqrt{\nu_t} dW^{(1)}_t - \frac{\gamma \sqrt{\nu_t}}{\sigma \sqrt{1 - \rho^2}} dW^{(3)}_t + J_{\Lambda,t} dN_t,
$$

where $\ln(1+J_{\Lambda,t}) \sim N (\mu_{\Lambda,J}, \sigma_{\Lambda,J}^2)$ and $\pi_A$ denotes the mean jump size $\pi_A = \exp (\mu_J + \frac{1}{2} \sigma_J^2) - 1$. Following Pan (2002) and Broadie, Chernov and Johannes (2007), we set $\pi_A = 0.$

Under the SVJ model, a contingent claim $C$ written on a stock and/or the volatility can not be perfectly replicated by a portfolio consisting of the underlying stock and the zero-
coupon bond. To perfectly hedge $C$, in the next subsections we show that the self-financing portfolio must contain, in addition to the above two instruments, a variance futures contract, as happens with the SV model, and the higher-order moment futures contracts defined in Section 2. Before proving this result, we need to derive the price and expected return of these futures contracts under the SVJ model.

3.2.1 Pricing variance and higher-order moment futures contracts

The annualized quadratic variation of log-price process $X$ under the SVJ model is given as

$$
\frac{1}{T} \langle X, X \rangle_{0,T} = \frac{1}{T} \left( \int_0^T V_u du + \int_0^T \tilde{J}_u^2 dN_u \right),
$$

where $\tilde{J} = \ln (1 + J)$. The following proposition gives the price and instantaneous expected return of a variance futures contract.

**Proposition 3** Under the SVJ model and the no-arbitrage principle, the price of a variance futures contract at time $t \in [0, T]$ is given as

$$
FV_t = \frac{1}{T} \left( \int_0^t V_u du + \int_0^t \tilde{J}_u^2 dN_u + F(t, V_t) + G(t) \right),
$$

where $F(t, V_t)$ is defined in Proposition 1, $G(t) = \lambda(\mu_{(2)}^Q) \tau$, where $\mu_{(2)}^Q$ is the second-order non-central moment of $\tilde{J}$ under the risk-neutral measure $Q$. Its expected return is given as

$$
\mu_t^{FV} dt \equiv E_t \left[ \frac{dFV_t}{FV_t} \right] = \frac{\partial \ln FV}{\partial \ln V} \tilde{\gamma}_V dt + \frac{\lambda}{FV_t} \left( \mu_{(2)} - \mu_{(2)}^Q \right) dt,
$$

where $\tilde{\gamma}_V = \sigma \rho \gamma_S + \gamma_V$ and $\mu_{(2)}$ is the second-order non-central moment of $\tilde{J}$ under the physical probability measure $P$.

Proposition 3 indicates that the expected return of a variance futures contract $\mu_t^{FV}$ depends, in addition to the market and volatility risk premium, on a risk premium related to the jump component of the underlying stock price $S$. Given recent evidence that $\mu_{(2)}^Q < \mu_J < 0$ and $(\sigma_{(2)}^Q)^2 > \sigma_{(2)}^2$, where $\mu_{(2)}^Q$ and $\sigma_{(2)}^Q$ are respectively the risk-neutral counterparts of $\mu_J$ and $\sigma_J$ (see Broadie, Chernov and Johannes (2007)), this premium must be negative, implying $\mu_{(2)} - \mu_{(2)}^Q < 0$. This means that its presence will reduce the expected return of
a variance futures further than the volatility premium due to investors’ aversion towards jumps.

The prices and their expected changes of the bipower variation and higher-order moment futures contracts under the SVJ model are given in the next proposition. Note that, instead of the expected returns, the proposition gives the expected price changes of the above contracts as their expected returns depend on the sign of the current prices \( F_{(j),t} \), which can be negative for odd \( j \). The sign of the expected price changes of an investment in a bipower variation and higher-order moment futures contracts is only determined by that of the jump risk premium. The payoffs of these financial instruments under the SVJ model are respectively defined as 

\[
\begin{align*}
\Pi_j &= \mu^{(j)} \int_0^t \tilde{J}_u^2 dN_u + \lambda \mu^{(j)} \tau,
\end{align*}
\]

for \( j = 2 \) and \( j \geq 3 \), respectively. The expected changes of the prices of these derivatives are given as

\[
\mu_t^{(j)} dt \equiv E_t [dF_{(j),t}] = \lambda \left( \mu_{(j)} - \mu^{(j)} \right) dt,
\]

where \( \mu_{(j)} \) and \( \mu^{(j)} \) are the \( j \)-th order non-central moment of \( \tilde{J} \) under the physical and risk-neutral measure, respectively. These moments can be calculated respectively as \( \mu_{(j)} = \frac{\partial^{(j)} M(x)}{\partial x^{(j)}} \bigg|_{x=0} \) and \( \mu^{(j)} = \frac{\partial^{(j)} M^Q(x)}{\partial x^{(j)}} \bigg|_{x=0} \), where \( M(x) = \exp \left( \mu_j x + (1/2) (\sigma_j)^2 x^2 \right) \) and \( M^Q(x) = \exp \left( \mu^{(j)} x + (1/2) (\sigma^{(j)})^2 x^2 \right) \) are the moment-generating functions of random variable \( \tilde{J} \) under measures \( P \) and \( Q \), respectively.

Proposition 4 implies that the expected change of the price of the bipower variation futures, defined as \( \mu_t^{F(2)} \), is negative and lower in magnitude than that of the variance futures. This can be obviously attributed to the fact that, by definition, the bipower variation futures pays only when jumps occur in the market during time period \([0,T]\), whereas the variance futures contract accounts also for an increase in volatility. Using formulas (20) and (22), for \( j = 2 \), it can be easily seen that \( \tilde{\gamma}_V = \sigma \rho \gamma_S + \gamma_V \), which depends on the price of market
and volatility risk, can be calculated as

\[ \dot{\gamma}_V = \frac{F_{tV}}{\partial FV/\partial \ln V} \mu_{tV}^{FV} - \frac{1}{\partial FV/\partial \ln V} \mu_{tV}^{F(2)}. \]  

(23)

This relationship implies that the expected gains of a portfolio constructed by holding $1/(\partial FV/\partial \ln V)$ of the notional in a long position of a variance futures and $1/(\partial FV/\partial \ln V)$ of the notional in a short position of bipower variation futures, respectively, will be related to the volatility risk premium. This can be attributed to the fact that the return of this portfolio does not depend on the jump risk premium, as noted in Section 2 (see fn 4).

The results of Proposition 4 indicate that the sign of the expected changes of the prices of higher-order moment futures $\mu_t^{F(j)}$, for $j \geq 3$, depends on the sign of moments’ difference $\mu_{(j)} - \mu_{(j)}^Q$. Note at this point that moments $\mu_{(j)}$ and $\mu_{(j)}^Q$, for all $j$, exist due to the normality assumption of the log-jump size $\tilde{J}$. For $j = 3$, the sign of $\mu_{(3)} - \mu_{(3)}^Q$ is expected to be positive, which implies that the expected price change of the 3rd-order moment futures contract $\mu_t^{F(3)}$ will be positive. This compensates investors taking a long position in this futures contract for bearing the risk of possible negative jumps occurring during time period $[0, T]$.

Following analogous arguments, we can generalize the above results to the case of higher-order odd or even moments as follows:

\[ \mu_t^{F(j)} = \begin{cases} 
> 0, & \text{if } j \text{ is odd} \\
< 0, & \text{if } j \text{ is even} 
\end{cases} \]  

(24)

### 3.2.2 Pricing and hedging contingent claims

To price and hedge contingent claim $C$ under the SVJ model, we will assume that the self-financing portfolio, which replicates the price of contingent claim $C$, $C_t = C(t, S_t, V_t)$ for $t \in [0, T]$, consists, in addition to the financial instruments of the corresponding portfolio for the SV model, of higher-order moment futures contracts. That is, the vector of deltas $\phi_t$ now is defined as $\phi_t = (\phi_t^S, \phi_t^B, \phi_t^{FV}, \phi_t^{F(2)}, ..., \phi_t^{F(N)})^T$, where its elements denote the number

---

\[ \text{The positive sign of difference } \mu_{(3)} - \mu_{(3)}^Q \text{ can be more clearly seen by writing it as} \\
\mu_{(3)} - \mu_{(3)}^Q = \mu_j^3 - \left( \mu_j^Q \right)^3 + 3 \left( \mu_j \sigma_j - \mu_j^Q \sigma_j^Q \right)^2. \]

Given evidence that $\mu_j^Q < \mu_j < 0$ and $\left( \sigma_j^Q \right)^2 > \sigma_j^2$, the above equation implies $\mu_{(3)} - \mu_{(3)}^Q > 0$.  

---
of the underlying stock, zero-coupon bond, variance futures, bipower variation futures (for
\( j = 2 \)) and higher-order moment futures contracts (for \( j \geq 3 \)), at time \( t \), respectively.
Note that this replicating portfolio is assumed that consists of a finite, but sufficiently large
number of higher-order moment futures contracts \( N \), which is adequate to approximately
replicate \( C \) in the sense that
\[
\lim_{N \to \infty} dV_t(\phi_t) = dC_t \tag{25}
\]
for all \( t \in [0,T] \), where \( V_t(.) \) denotes the value of the self-financing portfolio. In the next
proposition, we derive analytic formulas of the elements of the vector of deltas \( \phi_t \) and
instantaneous expected return of contingent claim \( C \).

**Proposition 5** Consider a contingent claim \( C \) written on the underlying stock and/or its
volatility, with price function \( C_t = C(t, S_t, V_t) \), where \( C(.) \) is a continuous function which
has partial derivatives of any order. Then, under the SVJ model and the no-arbitrage
principle, the deltas of the self-financing portfolio replicating contingent claim price \( C_t \), at
\( t \in [0,T] \), are given as follows:

\[
\begin{align*}
\phi_t^S &= \frac{\partial C}{\partial S}, \quad \phi_t^B = B_t^{-1} (C_t - \phi_t^S S_t), \quad \phi_t^{FV} = \frac{\partial C}{\partial FV} \\
\phi_t^{F(2)} &= \frac{1}{2!} \left( \frac{\partial^{(2)} C}{\partial \ln S^{(2)}} - \frac{\partial C}{\partial \ln S} \right) - \frac{\partial C}{\partial FV}, \quad \text{and} \quad \phi_t^{F(j)} = \frac{1}{j!} \left( \frac{\partial^{(j)} C}{\partial \ln S^{(j)}} - \frac{\partial C}{\partial \ln S} \right), \quad \text{for} \ j \geq 3,
\end{align*}
\]

while the instantaneous expected return of \( C \) is given as

\[
E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln S} \left( \mu_t^S + \delta_t - r_t \right) dt + \frac{\partial \ln C}{\partial \ln FV} \mu_t^{FV} dt + \\
+ \left( \frac{1}{2} \left( \frac{\partial^{(2)} C}{\partial \ln S^{(2)}} - \frac{\partial \ln C}{\partial \ln S} \right) - \frac{\partial \ln C}{\partial FV} \right) \mu_t^{F(2)} dt + \\
+ \lim_{N \to \infty} \sum_{j=3}^N \frac{1}{j!} \left( \frac{\partial^{(j)} C}{\partial \ln S^{(j)}} - \frac{\partial \ln C}{\partial \ln S} \right) \mu_t^{F(j)} dt. \tag{27}
\]

The results of Proposition 5 imply that enlarging the market with variance, bipower
variation and higher-order moment futures contracts enables us to approximately replicate
contingent claim \( C \). For sufficiently large \( N \), this market becomes approximately (or quasi)
complete, thus implying that the delta-hedged gains of the self-financing portfolio given by
the proposition converge to zero. In such a market, there is a unique risk-neutral measure
\( Q \) (see Björk, Kabanov and Runggaldier (1997), or Jarrow and Madan (1999)). Under
this, we can derive the price of contingent claim \( C_t \) by solving Bates’s (1996) PDE (see Appendix). As with the SV model, we have derived this PDE by eliminating all stochastic terms determining stochastic movements in \( C_t \), and not based on equilibrium approach.

The formula of the expected return of the contingent claim \( E_t \left[ \frac{dC_t}{C_t} \right] \), given by Proposition 5, indicates that, in addition to the expected returns of the underlying asset and the variance futures contract, it also depends on those of the bipower variation and higher-order moment futures contracts. The bipower variation futures contract has two effects on \( E_t \left[ \frac{dC_t}{C_t} \right] \) which are complementary to each other. The first adjusts \( E_t \left[ \frac{dC_t}{C_t} \right] \) for its bias due to estimating the volatility premium based on the expected return of a variance futures contract, which also depends on the jump risk (see equation (20)). The second accounts for the exposure of contingent claim price \( C_t \) to the jump risk, especially, its component related to \( \tilde{J}^2 \) (see (22), for \( j = 2 \)). The total of the above two effects on \( E_t \left[ \frac{dC_t}{C_t} \right] \) depend on the sign and magnitude of delta coefficient \( \phi^{F(2)} \). Analogously, the effect of the higher-order moment futures on \( E_t \left[ \frac{dC_t}{C_t} \right] \) depends on the sign and magnitude of \( \phi^{F(j)} \) for \( j > 2 \). For European option contracts, which is the focus of our empirical analysis in Section 5, these coefficients depend on the maturity and moneyness of the option contract.

As the above definitions of \( \phi^{F(j)} \) are new in the literature, below we give a more detailed discussion about their sign and magnitude. This is done across different categories of moneyness of European calls and puts, and over short and long maturity intervals. To help this discussion, in Figure 1 we graphically present estimates of \( \phi^{F(j)} \), for \( j = 2, 3, 4 \), across different strike prices \( K \) and maturity intervals of one and six months. These rely on parameter estimates of the SVJ model, used in the simulation exercise of next section. Note that the put-call parity relation implies that \( \phi^{F(j)} \) will be the same for calls and puts of the same maturity interval and strike price, for all \( j \). This discussion is also very helpful in understanding sources of delta-hedged gains which are empirically observed (see, e.g., Section 5).

For deep-OTM puts and calls, delta coefficients \( \phi^{F(j)} \) will be almost zero, for all \( j = 2, 3, 4 \). In particular, \( \phi^{F(2)} \) will be close to zero, because both vega and gamma are very close to zero given that strike price \( K \) takes very small or large values.\(^9\) \( \phi^{F(3)} \) and \( \phi^{F(4)} \) will be also

\(^9\)This can be easily seen by writing \( \phi^{F(2)} \) as

\[
\phi^{F(2)} = \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial S^2} - \frac{\partial C}{\partial FV}
\]
close to zero, because deep-OTM puts and calls are very little influenced by the expectation of a jump occurrence, which is priced by the 3rd and 4th-order moment futures. The practical implication of these results is that a two-instruments hedging strategy, involving a short position in the underlying stock and a variance futures contract, will be sufficient to hedge the exposure of deep-OTM options to all sources of risk, as jump risk is negligible for this category of options. This is also true for long-term options of any moneyness. As long as volatility risk is hedged through variance futures, the exposure of these options to jump risk becomes also negligible.

In contrast to long-term, the sign of $\phi^{F(j)}$ for short-term options changes significantly across moneyness. In particular, for short-term OTM puts $\phi^{F(2)}$ will be negative. This can be attributed to the fact that a position held in variance futures contracts is in excess of that required to hedge the exposure of the option to a jump, independently of the sign of the jump. OTM puts benefit more by an increase in volatility, which can cause the price of the underlying stock to decrease due to leverage effect, compared to an equally possible positive or negative jump effect on the stock price, measured by $\tilde{J}^2$. This means that the option’s vega should be larger than the option’s gamma, implying $\phi^{F(2)} < 0$. For short-term OTM calls, the sign of $\phi^{F(2)}$ is expected to be positive. In this case, gamma should be larger than vega, given that a positive or negative jump is preferable than an increase in volatility.

Regarding the sign of higher-order deltas $\phi^{F(3)}$ and $\phi^{F(4)}$, this will be respectively negative and positive for short-term OTM puts. For $\phi^{F(3)}$, this happens because this category of options is considered by investors as a security instrument against severe adverse movements of the stock market. Thus, a long position on them embody a short position in a 3rd-order moment futures contract. The positive sign of $\phi^{F(4)}$ can be explained by the fact that OTM puts also embody a long position in the 4th-order moment futures contract, which pays when extreme negative or positive jumps occur. For short-term OTM calls, $\phi^{F(3)}$ and $\phi^{F(4)}$ are both positive. In this case, an investor who takes a long position in this category of options is averse to jumps, and thus embodies a long position in the 3rd and 4th-order moment futures contracts. The long position in the 3rd-order moment futures contract sells protection for a possible negative jump in the market. The long position in the 4th-order moment futures contract compensates the investor for an extreme positive or negative jump.
Finally, for short-term ATM options, the sign of $\phi^{F(j)}$ is expected to be negative, for all $j = 2, 3, 4$. More specifically, $\phi^{F(2)}$ will be negative, since a positive or negative jump is preferable than an increase in volatility causing leverage effects, as it happens with OTM puts. The negative value of $\phi^{F(3)}$ can be attributed to the fact that taking a long position in ATM options either leads to immediate gains in case of a put or to limited loses in case of a call. On the other hand, the negative sign of $\phi^{F(4)}$ can be explained by the fact that an extreme jump can render ATM options unprofitable.

4 Simulation exercise

In this section we perform a simulation study with the aim to evaluate the performance of hedging strategies for European call, or put, options under the SV and SVJ models, respectively, based on the theoretical results of the previous section. This study will enables us to assess the magnitude of two possible sources of errors encountered when implementing these results to discrete-time sets of data. The first is related to the discrete rebalancing of the hedging portfolios, followed by traders in practice. The second is due to the discrete sampling of realized higher-order moments defining payoffs $V^{(n)}_{T-\tau,T}$ and $V^{(n)}_{(j),T-\tau,T}$, for $j \geq 2$ (see (1) and (5)), in practice. For the SVJ model, our simulation study will also enables us to assess a third source of error related to the finite number of higher-order moment futures which can approximate the discontinuous component of stock price $S$.

Our simulation exercise is based on the Euler discretization scheme of the SV and SVJ continuous-time processes, with a sampling frequency of 5 minutes per day under the physical measure $P$, and a daily rebalancing of the alternative hedging portfolios considered. To generate empirically plausible values of stock price $S$, its volatility $V$ and a European call, or put, price $C$, we are based on values of the structural parameters of Heston’s (1993) SV model and Bates’s (1996) SVJ model reported in the empirical literature. In particular, the parameters of the models under measure $P$ are set to the following values: $\kappa = 5.04$, $\theta = 0.04$, $\sigma = 0.52$, $\rho = -0.66$, $\lambda = 4$, $\tau = -0.03$ and $\sigma_J = 0.04$, provided by Kaeck and Alexander (2011).\footnote{Note that Kaeck and Alexander (2011) reported an estimate of the jump intensity coefficient $\lambda = 1.8$. We have set this number to be equal to $4$ in order to increase the probability of a jump occurring during the simulated time period.} The values of the parameters of the models under risk-neutral measure $Q$ are set to our estimates of the models reported in next section (see Table 3). These are
given as $\kappa^Q = 0.72$, $\theta^Q = 0.31$, $\pi^Q = -0.07$ and $\sigma^Q_J = 0.05$. Note that the above values of the parameters of the SVJ model associated with the jump component of stock price $S$ implies a jump size of the log-return distribution which varies between -15\% and 9\% with probability 99.9\%. This interval encompasses daily price jumps which are empirically observed in the market. Finally, the initial values of stock price $S$ and its volatility are set to $S_t = 100$ and $\sqrt{\nu_t} = 0.3$, respectively. Interest rate $r$ and dividend yield $\delta$ are set to $r = 0.05$ and $\delta = 0.015$, respectively, and, without loss of generality, we have assumed that $\gamma_S = 0$.

For each of the above models, we conduct $M = 2,000$ iterations and we consider different moneyness levels and maturity intervals. In particular, we consider OTM puts, with $K/S_t = 0.9$, ATM calls, with $K/S_t = 1$, and OTM calls, with $K/S_t = 1.1$. We consider short-term options which are bought 30 days prior to maturity and long-term options which are bought 120 days prior to maturity. Both of these different maturity categories of options are delta-hedged for one day.

The variance futures prices are calculated based on formula (13), for the SV model, and formula (19), for SVJ model. For the last model, the bipower variation and higher-order moment futures prices are given by formula (21). For all the aforementioned futures contracts, their payoffs are calculated assuming a daily sampling of observations, often met in practice, based on formulas (1), (5) and (8). Next, we describe in more details the hedging strategies considered for each option pricing model in our simulation study and we assess their performance. The latter is done by calculating the mean absolute hedging error and the average delta-hedged gains.

4.1 SV model

First, we consider the traditional hedging strategy based on one instrument, i.e., the underlying asset, denoted as $HP_1$. In each iteration (denoted as $h$), a long position is taken on option $C$, with strike price $K$ and maturity interval $\tau$, at time $t$. This position is hedged by going short in $\phi^S_t = \partial C/\partial S$ shares of the stock and invest the residuals $U_t = C_t - \phi^S_t S_t$ in a zero-coupon bond. The combined position constitutes a self-financing portfolio. Next, at time $t + \Delta t$ (with $\Delta t = 1/252$), we calculate the hedging error of this strategy as

$$HP_h(t + \Delta t) = [C_{t+\Delta t} - C_t] - \left[\phi^S_t (S_{t+\Delta t} - S_t) + rU_t \Delta t + \delta \phi^S_t S_t \Delta t\right].$$ (28)
This procedure is repeated \( M = 2,000 \) times and, then, we calculate the mean absolute error, across all iterations \( h \), as

\[
\text{MAE} = \frac{1}{M} \sum_{h=1}^{M} |HP_h(t + \Delta t)|
\]  

(29)

and the average daily delta-hedged gains as

\[
\hat{\Sigma} = \frac{1}{M} \sum_{h=1}^{M} HP_h(t + \Delta t).
\]  

(30)

Second, we consider a two-instruments hedging strategy using the underlying stock and the variance futures contract with the same maturity interval \( \tau \). This strategy is denoted as \( HP_2 \). According to the theory, the long position in option \( C \) at time \( t \) is delta-hedged by going short in \( \phi_i^S = \partial C / \partial S \) shares of the stock, taking a short position in \( \phi_i^{FV} = \partial C / \partial FV \) variance futures and invest the residuals \( U_t = C_t - \phi_i^S S_t \) in a zero-coupon bond. The hedging error of this strategy at time \( t + \Delta t \) is calculated as

\[
HP_h(t + \Delta t) = [C_{t+\Delta t} - C_t] - [\phi_i^S (S_{t+\Delta t} - S_t) + \phi_i^{FV} (FV_{t+\Delta t} - FV_t) + rU_t \Delta t + \delta \phi_i^S S_t \Delta t],
\]  

for all \( h \). The MAE and the average daily delta-hedged gains for strategy \( HP_2 \) are calculated based on formulas (29) and (30), respectively.

The results of our simulation study for the SV model are presented in Table 1. These are in accordance with our theoretical predictions made in the previous section. They indicate that the two-instruments hedging strategy, \( HP_2 \), considerably improves upon the performance of \( HP_1 \), which uses only the underlying asset as hedging vehicle. This can be confirmed by the values of both the MAE and \( \hat{\Sigma} \) metrics reported in the table, which are found to be very close to zero. Note that the superiority of \( HP_2 \) strategy is more apparent for long-term ATM options, with time to maturity equal to \( \tau = 120 \) days and \( K/S_t = 1 \). This is something to expect, since the prices of this category of options are strongly affected by changes in volatility \( V \). The single-instrument strategy \( HP_1 \) yields negative values of \( \hat{\Sigma} \), for all different moneyness levels and maturities examined. This is also consistent with the theory. As discussed before, it can be attributed to the existence of a negative volatility
risk premium. Another interesting conclusion which can be drawn from the results of Table 1 is that the errors due to the discrete rebalancing of the portfolio and the calculations of the variance futures payoff over discrete-time sampling intervals are negligible.

4.2 SVJ model

For the SVJ model, in addition to strategies $HP_1$ and $HP_2$, our simulation study considers a multi-instruments hedging strategy, denoted as $HP_3$, which also includes the bipower variation and higher-order moment futures contracts in the replicating portfolio. The hedging errors of this strategy are calculated as follows:

$$
HP_n(t + \Delta t) = [C_{t+\Delta t} - C_t] - [\phi_t^S \left( S_{t+\Delta t} - S_t \right) + \phi_t^{FV} \left( FV_{t+\Delta t} - FV_t \right) + \sum_{n=2}^{N} \phi_t^{F(n)} (F_{(n),t+\Delta t} - F_{(n),t}) + rU_t \Delta t + \delta \phi_t^S S_t \Delta t],
$$

where $\phi_t^{F(2)} = \frac{1}{2} \left( \frac{\partial^{(2)} C}{\partial \ln S^2} - \frac{\partial C}{\partial \ln S} \right) - \frac{\partial C}{\partial FV}$, $\phi_t^{F(n)} = \frac{1}{n!} \left( \frac{\partial^{(n)} C}{\partial \ln S^n} - \frac{\partial C}{\partial \ln S} \right)$ and $N$ is the maximum number of higher-order moment futures contracts employed in the approximation of the discontinuous component of price $S$. As is expected by the theory, this approximation error will tend to zero, as $N$ increases. To determine the sufficient number $N$ which minimizes this approximation error, in our analysis we consider a sequence of different values of $N$, i.e., $N = 2, 3, ..., 10$.

The results of our simulation study for the SVJ model are presented in Table 2. The table presents values of the MAE and $\hat{\Sigma}$ metrics for the three alternative hedging strategies mentioned above. Panel A presents results based on the whole sample of iterations $M = 2,000$, while Panel B only for those which generate jumps. For space reasons, the table reports results of the above metrics only for the case of $N = 4$, which is found to sufficiently capture the effects of the discontinuous component of $S$ on hedging even when jumps occur. This can be confirmed by the inspection of both plots of Figure 2, which presents values of MAE for $N = 2, 3, ..., 10$. Plot A of the figure corresponds to the estimates of MAE obtained from the whole set of iterations, while Plot B for those which generate jumps.

Several interesting conclusions can be drawn from the results of Table 2 and the inspection of Figure 2. First, the values of the average delta-hedged gains $\hat{\Sigma}$ for single-instrument strategy $HP_1$ are negative and larger in absolute terms than those for the SV model. This
result is in accordance with theory. Under the SVJ model, hedging errors depend on both volatility and jump risk premia (see also Branger and Schlag (2008)). Second, strategy $HP_2$ improves upon the performance of strategy $HP_1$ for the SVJ model. This is true for all different moneyness levels and maturity categories examined. As with the SV model, this improvement is more profound for long-term options. But, it is not sufficient for short-term ones. As was expected by our theory (see Section 3.2.2), the benefits of strategy $HP_3$ are more profound for short-term options, which are more sensitive to random jumps in stock prices. The results of Table 2 and Figure 2 clearly indicate that a limited number of higher-order moment futures (i.e., $N = 4$) can adequately hedge option prices on daily basis, even when a jump in stock price $S$ occurs.

Summing up, the results of our simulation study indicate that strategies which include variance futures and/or a limited number of higher-order moment futures can be efficiently employed to hedge a long position in a contingent claim, under the SV and/or SVJ models. The approximation errors of these strategies due to the discrete rebalancing of the hedging portfolio and the discrete sampling of realized moments employed to calculate variance and higher-order moment payoffs are negligible.

5 Can volatility and jump risks be efficiently hedged by variance and higher-order moment futures?

Using data on variance futures contracts and European call and put prices, in this section we answer the following two questions. The first is if random jumps are priced in the variance futures market. As shown by formula (4), this must be reflected in the difference between the price of a variance futures contract and the value of the composite options portfolio, i.e. $FV_t - V_{0,t}$. This question will be examined without relying on any parametric model of the market. Answering this question is crucial for developing hedging strategies against random jumps in stock prices. The second question is if a long position in a European call or put can be efficiently hedged, in practice, based on the two-instruments hedging strategy $HP_2$, which also considers a variance futures contract as a hedging vehicle, or it requires multi-instruments hedging strategies like $HP_3$, which also employs higher-order moment futures. To address this question, we rely on estimates of the parameters of the SVJ model under the risk-neutral measure. These estimates are used to calculate the delta coefficients and to estimate the prices of higher-order moments futures. Note, at this point, that the purpose
of this study is not to test the SVJ model, but to employ it as a common specification in
order to implement multi-instruments hedging strategies in practice.

5.1 The data

The options price data used in our empirical analysis are taken from the OptionMetrics
Ivy data base. These are daily market closing prices of European ATM and OTM calls,
with $K \geq S_t$, and OTM puts, with $K < S_t$, written on the S&P 500 index. We use ATM
and OTM calls and puts because, as is well known in the literature, these options are more
actively traded compared to in-the-money (ITM). We have excluded option prices violating
the boundary conditions and options with maturity intervals less than 2 weeks. Finally,
following Driessen, Maenhout and Vilkov (2009), we have removed options with zero open
interest for liquidity reasons. Apart from option prices, we have also used the reported
implied volatility surfaces (IVS) of the above data base, with maturity intervals 1, 2, 3, 6
and 9 months. This is done in order to estimate the parameters of the SVJ model needed
in our analysis and calculate the values of the composite options portfolio. This data set
contains implied volatilities on both call and puts on a grid of 13 strike prices.

The variance futures price data used in our analysis are written on the S&P 500 index.
These contracts are traded over-the-counter. This data set consists of closing prices of
variance futures net of the accrued realized variance with maturity intervals of 1, 2, 3, 6
and 9 months. Thus, if we have a traded variance futures contract during period $[T - \tau, T]$,
then the recorded quote of $FV_t$ is adjusted as follows:

$$\overline{FV}_t \equiv E_t^Q \left[ \frac{1}{\tau} (X_t, X_{t+\tau}) \right] = FV_t - \frac{1}{\tau} \langle X_t, X_{t+\tau} \rangle_{T-\tau, t}.$$ 

Finally, the series of the risk-free interest rate and dividend yield are also
taken from the OptionMetrics data base. The interest rate is derived by British Banker’s
Association LIBOR rates and settlement prices of Chicago Mercantile Exchange Eurodollar
futures. The dividend yield is estimated by the put-call parity relation of ATM option
contracts.

Our data set covers the period from March 30, 2007 to October 29, 2010. This sample
interval consists of 900 trading days and it includes the period of the recent financial crisis,
the most serious at least since the 1930’s. Our analysis presents results for subsamples
before and after the date that this crisis seems to be intensified. This date is found to be
closely related to that of the collapse of Lehman Brothers. Our sample period is constrained
by the availability of over-the-counter variance futures price data.
5.2 Fitting the SVJ model into the data

To estimate the parameters of the SVJ model, we rely on daily values of IVS of put options written on the S&P 500 index, with constant maturity intervals 1, 2, 3 and 6 months. Fitting the SVJ model into constant-maturity IVS provides estimates of its parameters which are less sensitive to available maturity intervals every day. Since the following parameters of the SVJ model: $\sigma$, $\rho$, $\lambda$ and $\kappa \theta$ are the same under the physical and risk-neutral measures, in the estimation procedure we will set them to certain values provided in the literature, following Broadie, Chernov and Johannes (2007). In particular, these are set to $\kappa \theta = 5.04 \times 0.04 = 0.2016$, $\sigma = 0.52$, $\rho = -0.66$ and $\lambda = 1.8$, obtained by Kaeck and Alexander (2011) using a sample of data covering our sample interval 2007-2010. Given these values, we then estimate the vector of the risk-neutral parameters of the SVJ model, collected in vector $\Theta = \left( \kappa^0, \rho^0, \sigma^0 \right)$. This is done by solving the following least squares problem:

$$\min_\Theta \sum_{i=1}^{M_t} \sum_{j=1}^{N_i} \left( \ln \hat{P}_t \left( \tau_i, K_j, S_t, \tilde{F}V_t; \Theta \right) - \ln P_t(\tau_i, S_t, K_j) \right)^2,$$

for all $t$, where $\hat{P}_t \left( \tau_i, K_j, S_t, \tilde{F}V_t; \Theta \right)$ denotes estimates of put option prices implied by the SVJ model, with maturity intervals $\tau_i$ and strike prices $K_j$, and $P_t(\tau_i, K_j)$ are their corresponding market prices. $M_t$ denotes the number of different maturity intervals $\tau$, considered at any point in time $t$ (i.e. $M_t = 4$), and $N_i$ denotes the number of different put prices employed in the estimation procedure, for all $t$, i.e. $N_i = 13$. These values $M_t$ and $N_i$ imply that, for each time point (day) of our sample, a quite large cross-section set of 52 put option prices are used to fit the SVJ model into the data. Note that the estimates of put option prices $\hat{P}_t \left( \tau_i, K_j, S_t, \tilde{F}V_t; \Theta \right)$, derived by the above estimation procedure, rely on variance futures contract prices $\tilde{F}V_t$. As mentioned in the previous section, in this estimation procedure these values of $\tilde{F}V_t$ can substitute out those of volatility $V_t$, which is a latent variable. As noted before, this helps in obtaining more precise estimates of vector $\Theta$ from the data, as it exploits all available market information.

Table 3 presents the average values of the estimates of the elements of vector $\Theta$ over all

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11 Note that the choice between European calls and puts in estimating the SVJ model does not affect the results, due to put-call parity relation.
12 See, e.g., Bakshi, Cao and Chen (1997). Note that, in this estimation procedure, we use logarithms of put prices to avoid the problem of assigning more weight to relatively expensive options (e.g., ITM long-term options) and less weight to cheaper options (e.g., OTM options). See also Lin and Chang (2010).
time-points of the sample. Standard errors of these values are given in parentheses. Note
that table presents values of the elements of $\Theta$ for the whole sample and two subsamples
before and after September, 3 2008. This date is almost two weeks before the date of the
collapse of Lehman Brothers on September, 16 2008. It is chosen to split the sample into
two subsamples based on the daily estimates of $\Theta$, which indicate a clear cut shift in jump
parameters $\pi^Q, \sigma_j^Q$ after September 3, 2008 (see Figure 3). It reveals that the market was
expecting a significant negative jump two weeks before the collapse of this bank. To see
how well the SVJ model fits into the data, the table also presents values of the RMSE
(root mean square error) and the RMSPE (RMS pricing error) between the estimated and
actual put option prices $\hat{P}_t(\tau_i, K_j; S_t, FV_t; \Theta) \text{ and } P_t(\tau_i, K_j)$, respectively. These values
are found to be very small for the whole sample and the two subsamples considered, which
indicates that the SVJ fits satisfactorily into the data.

Regarding the elements of $\Theta$, the results of the table clearly indicate that the estimates
of parameter $\kappa^Q$ is significantly smaller than estimates of it under the physical measure
reported in the literature (i.e. $\kappa = 5.04$, see, e.g., Kaeck and Alexander (2011)). This is
ture for the whole sample and the two subsamples considered. For instance, for the whole
sample $\kappa^Q$ is found to be 0.72. This result was expected and it is consistent with recent
evidence provided in the literature (see, e.g., Egloff, Leippold and Wu (2010)). It implies
that the price of volatility risk $\gamma_V$ is negative. The estimates of risk-neutral parameters
$\pi^Q$ and $\sigma_j^Q$, reported in the table, are also consistent with the literature (see, e.g., Eraker
(2004)). As was expected, these estimates are found to be bigger in absolute terms for the
subsample after the date of the Lehman Brothers collapse. This can be also confirmed by
the inspection of Figure 3, which graphically presents the estimates of $\pi^Q$ and $\sigma_j^Q$, over the
whole sample. The plots of this figure reveal that the estimates of $\pi^Q$ and $\sigma_j^Q$ are virtually
zero until September 3, 2008. After that date, they vary substantially due to the subsequent
turmoils in the market triggered by the recent financial crisis sequence of events.

\footnote{Note that Egloff, Leippold and Wu (2010) estimate volatility premium using variance futures quotes.}
\footnote{Substantial time-variation in the values of $\pi^Q$ and $\sigma_j^Q$ during periods of market stress and turmoil are
also reported in the literature by Santa-Clara and Yan (2010).}
5.3 Are random jumps priced in the variance futures market?

According to formula (4), the difference between the price of a variance futures contract \( FV_t \) and value of the composite portfolio of European OTM calls and puts \( V_{o,t} \), given by formula (3), must be significantly different than zero, if random jumps are priced in the market. This difference, adjusted for the accrued realized variance effects \(-\frac{1}{\tau} \langle X, X \rangle_{T-\tau,t} \) and \( \frac{1}{\tau} \langle X, X \rangle_{T-t} \), and \( \frac{2}{\tau} (1 + (r - \delta) \tau - e^{(r-\delta)\tau}) \), will be henceforth written as \( \hat{FV}_t - \tilde{V}_{o,t} \), where \( \hat{FV}_t = FV_t - \frac{1}{\tau} \langle X, X \rangle_{T-\tau,t} \) and \( \tilde{V}_{o,t} = V_{o,t} + \frac{2}{\tau} (1 + (r - \delta) \tau - e^{(r-\delta)\tau}) \). It should reflect the sum of higher-order moment effects \( \sum_{j=3}^{\infty} \frac{1}{j!} E_t^Q \left[ \sum_{t<u<T} (\Delta X_u)^j \right] \), due to the discontinuous component of log-price \( X \). Note that, in our above definitions, we have set \( \tilde{\tau} = \tau \), since in our data set the accrued period of the variance futures contract coincides with the maturity interval of the European options, \( \tau \).

To empirically investigate if random jumps are priced in the market, we first need to retrieve values of \( V_{o,t} \), based on OTM call and put prices written on the S&P 500 index. To this end, we use weekly values of the IVS data of maturity intervals \( \tau = 1, 2, 3, 6, 9 \) months taken every Wednesday of the sample, and we rely on a numerical interpolation-extrapolation scheme suggested in Rompolis and Tzavalis (2013). In particular, we present results of two of these schemes, which differ with respect to the extrapolation procedure employed. The first sets the integral endpoints to their sample minimum and maximum values \( K_{\min} \) and \( K_{\max} \), respectively. That is, it chooses not to extrapolate the implied volatility functions, while the second assumes constant extrapolation. In Table 4, we present average values of the estimates of \( V_{o,t} \) over the whole sample and their corresponding approximation error bounds, obtained based on the error bound formula derived by Rompolis and Tzavalis (2013).\(^{15}\)

\[^{15}\text{Let } \hat{C}_t(\tau, y) \text{ and } \hat{P}_t(\tau, y) \text{ denote the approximated values of the true call and put prices } C_t(\tau, y) \text{ and } P_t(\tau, y), \text{ with respect to variable } y = \ln(K/S_t), \text{ respectively, based on an interpolation-extrapolation scheme. } \tilde{V}_{o,t} \text{ denotes the approximated value of the composite portfolio of options } V_{o,t}, \text{ given by formula (3), based on } \hat{C}_t(\tau, y) \text{ and } \hat{P}_t(\tau, y). \text{ Rompolis and Tzavalis (2013) have derived the following approximation error bound of } \tilde{V}_{o,t}: \]

\[
\left| \tilde{V}_{o,t} - V_{o,t} \right| \leq \frac{2e^{\tau \tau}}{\tau S_t} \left[ C_{\text{error}} \left( 1 - e^{-y_\infty} \right) + P_{\text{error}} \left( e^{-y_0} - 1 \right) + \varepsilon \right],
\]

where \( C_{\text{error}} = \max_{y \in (0, y_\infty)} \left| C_t(\tau, y) - \hat{C}_t(\tau, y) \right| \) and \( P_{\text{error}} = \max_{y \in (y_0, 0)} \left| P_t(\tau, y) - \hat{P}_t(\tau, y) \right| \) constitute upper bounds of the approximation errors of the European call and put option pricing function, respectively, \( y_0 = \ln(K_0/S_t), \) \( y_\infty = \ln(K_\infty/S_t), \) and \( \varepsilon \) is a truncation error of strike prices \( K \) at endpoints \( K_\infty \) and \( K_0, \) respectively, defined as

\[
\varepsilon = \int_{K_\infty}^{y_\infty} \frac{1}{K^2} C_t(\tau, K) dK + \int_{0}^{K_0} \frac{1}{K^2} P_t(\tau, K) dK.
\]
The results of Table 4 indicate that the values of $V_{o,t}$ obtained by the interpolation-extrapolation scheme of the implied volatility function which chooses not to extrapolate this function provide much smaller approximation error bounds, for all $\tau$. We will thus choose this scheme to calculate values of difference $\hat{FV}_t - \hat{V}_{o,t}$, employed in our empirical analysis. The choice of a non-extrapolation scheme is also economically intuitive. It can be justified by the fact that investing in the composite portfolio of OTM options in order to hedge variance futures does not imply trading on artificial data, but only on market available option data. The very small in size numerical approximation errors of this numerical scheme guarantee that the values of $\hat{FV}_t - \hat{V}_{o,t}$ will not be serially affected by numerical errors. In Table 5, we present average values of $\hat{FV}_t - \hat{V}_{o,t}$, based on this scheme of calculating $V_{o,t}$. This is done for the whole sample and the two subsamples considered in our analysis, and for maturity intervals $\tau = 1, 2, 3, 6, 9$ months. In parentheses, the table presents values of the $t$-statistic ($t$-stat) testing if the reported average values of $\hat{FV}_t - \hat{V}_{o,t}$ are different than zero, for all $\tau$.

A number of interesting conclusions emerge from the results of Table 5. First, the reported values of difference $\hat{FV}_t - \hat{V}_{o,t}$ are positive and significantly different than zero, for all $\tau$. This indicates that random jumps are indeed priced in the market of variance futures contracts, according to the predictions of formula (4). Second, the values of $\hat{FV}_t - \hat{V}_{o,t}$ increase both in terms of magnitude and volatility during the subsample which covers the recent financial crisis period (see also Figure 4). This result should be expected, since this period is one of market stress and fears of financial crisis accelerate. Finally, the results of Table 5 indicate that the mean values of $\hat{FV}_t - \hat{V}_{o,t}$ are very close to each other, across different maturity intervals. This can be confirmed by an ANOVA $F$-test, which can not reject the equality of these means. This result means that jumps in stock prices are time homogeneous.

Having established that difference $\hat{FV}_t - \hat{V}_{o,t}$ is significant, next we examine if its vari-

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16 The worse performance of the numerical approximation scheme which assumes constant extrapolation of implied volatility function can be attributed to the following three effects. First, it decreases the truncation error $\varepsilon$. Second, it increases $C_{error}$ and $P_{error}$ since the intervals $(y_0, 0)$ and $(0, y_\infty)$ increases, respectively. Finally, it increases the value of the function $e^{-\varphi} - 1$ that controls for the effect of $P_{error}$ on the bound. If the negative effect on $P_{error}$ dominates the positive effect on $\varepsilon$, then the extrapolation of the implied volatility function would increase the approximation error bounds, which explains the numbers of Table 4.

Since the true values of $V_{o,t}$ are unknown, the error bounds given by the above formula are calculated based on estimates of the whole set of parameters of the SVJ model. These are obtained by fitting the model into weekly option prices of our data. They are found to provide very small values of the root mean square errors ranging from 0.12 to 0.27, for all maturity intervals $\tau$ considered.

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ation can be explained by term $-\frac{2}{T} \sum_{j=3}^{\infty} \frac{1}{P_t^Q} \left[ \sum_{t<u<T} (\Delta X_u)^j \right]$, as the theory suggests. To this end, we regress logarithm values of $\tilde{FV}_t - \tilde{V}_{o,t}$ on estimates of this term, taken also in logarithms. In our analysis, we consider two different sets of estimates of this term. The first, denoted as $JT_t$, is obtained based on the estimates of the SVJ model, presented in Section 5.2, and it is calculated as

$$JT_t = 2\lambda \left( \mu^Q + \frac{1}{2} \left( \sigma^Q \right)^2 + \frac{1}{2} \left( \sigma^Q \right)^2 - e^{\mu^Q + (\sigma^Q)^2/2 + 1} \right),$$

(34)

for all $t$ (see Appendix, for a proof). Figure 4 graphically presents estimates of $JT_t$ against difference $\tilde{FV}_t - \tilde{V}_{o,t}$. The second method of estimating term $-\frac{2}{T} \sum_{j=3}^{\infty} \frac{1}{P_t^Q} \left[ \sum_{t<u<T} (\Delta X_u)^j \right]$ is model-free. It is based on estimates of the third-order risk-neutral moment of the underlying stock log-return implied from OTM option prices, given as

$$\mu_{3,t} = e^{\tau T} \left\{ \int_{S_t}^{t+\infty} G(K) C_t(\tau, K) dK + \int_0^{S_t} G(K) P_t(\tau, K) dK \right\},$$

where $G(K) = \frac{3}{K^2} \left[ \ln \left( \frac{K}{S_t} \right) \right] \left[ 2 - \ln \left( \frac{K}{S_t} \right) \right]$ (see Bakshi, Kapadia and Madan (2003)).

Least squares (LS) estimates of the coefficients of the above regressions are reported in Table 6. The variables (log-transformations) involved in these regressions are denoted as follows: $LN\text{FVO}_t$ stands for the logarithm of $\tilde{FV}_t - \tilde{V}_{o,t}$, $LNJ\text{T}_t$ for the logarithm of $JT_t$, and $LN\mu_{3,t}$ for the logarithm of $-\mu_{3,t}$, which is positive for all $t$. Panel A of the table presents estimation results of the regression of $LN\text{FVO}_t$ on $LNJ\text{T}_t$, while panel B presents those of the regression of $LN\text{FVO}_t$ on $LN\mu_{3,t}$. Note that, since the values of $JT_t$ are almost zero for the pre-crisis subsample (see Figure 4), the first of the above regressions was estimated only for the second subsample, covering the post-crisis period. In addition to LS estimates, the table presents fully-modified LS (FMLS) estimates of the regression coefficients, since all variables involved in these regressions are found to be integrated series

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Note that equation (34) holds also under a more general version of the SVJ model, which allows for jumps in volatility which are independent of those in the price level of the underlying asset often assumed in the literature (see, e.g., Broadie, Chernov and Johannes (2007)).

$\mu_{3,t}$ is an approximation of $-\frac{2}{T} \sum_{j=3}^{\infty} \frac{1}{P_t^Q} \left[ \sum_{t<u<T} (\Delta X_u)^j \right]$, which can be seen by writing

$$E_t^Q \left[ \sum_{t<u<T} (\Delta X_u)^3 \right] = \mu_{3,t} - 3E_t^Q \left[ \int_{t+}^{T} (X_u - X_t)^2 dX_u \right] - 3E_t^Q \left[ \int_{t+}^{T} (X_u - X_t) d\langle X, X \rangle_u \right].$$

This is derived by applying Ito’s lemma to function $f(x) = x^3$. To estimate $\mu_{3,t}$, we adopt the numerical procedure used to obtain values of $V_{o,t}$.
of order one, I(1). FMLS estimates correct the standard LS coefficient estimates for small-sample bias due to the omission of short-run dynamics of regression variables. To test if the estimated regressions are cointegrated, the table presents Phillips’ and Ouliaris (1990) $Z_t$ and $Z_a$ test statistics for cointegration. Rejection of cointegration between variables $LN_{FVO_t}$ and $LN_{JT_t}$, or $LN_{FVO_t}$ and $LN_{\mu_3,t}$, implies that variables $JT_t$ and $-\mu_3,t$ cannot explain long-run shifts in $\bar{FV}_t - \bar{V}_{o,t}$. This will constitute strong evidence against the theory.

The results of Table 6 indicate that variables $JT_t$ and $-\mu_3,t$, capturing random jumps, can explain long-run movements of difference $\bar{FV}_t - \bar{V}_{o,t}$, as the theory suggests. This is true for all maturity intervals $\tau$ considered. The $p$-values of test statistics $Z_t$ and $Z_a$, reported in the table, clearly indicate that the regressions of $LN_{FVO_t}$ on $LN_{JT_t}$, or $LN_{\mu_3,t}$, constitute cointegrating regressions. These results can be also confirmed pictorially by the inspection of the graph of Figure 4, which shows that $JT_t$ follows closely persistent shifts of $\bar{FV}_t - \bar{V}_{o,t}$, over the whole sample. The above results highlight the need to introduce higher-order moment futures contracts to hedge a position in variance futures contracts, as suggested in Section 2.

Regarding the regression coefficient estimates, the results of Table 6 (see Panel A) indicate that both the LS and FMLS estimates of $\beta_1$ are in the right direction, but they are less than unity which is their predicted value by the theory. These deviations of the estimates of $\beta_1$ from unity can be attributed to numerical approximation errors in the estimates of $V_{o,t}$, the existence of transaction costs and/or the inadequacy of the SVJ to fully capture the discontinuous component of stock price index dynamics. The last may be thought of as the most plausible explanation of them (see, e.g., Bates (2003)), given that the error bounds of $V_{o,t}$, reported in Table 4, and the effects of transaction costs on our results are found to very small.\footnote{To investigate if transaction costs constitute a significant source of the deviations of the estimates of $\beta_1$ from unity, we estimated $V_{o,t}$ based on ask option prices, instead of the mid bid-ask prices. These estimation results are not reported for reasons of space. They are very close to those reported in Table 6. This can be attributed to the fact that the average cost that should be paid to take a long position in the composite portfolio of options with $\tau = 1$ month maturity interval is $\$0.003$, which is very small. This value becomes smaller, as $\tau$ increases.}
5.4 Evaluation of dynamic hedging strategies accounting for volatility and jump risks

The findings of the previous section that random jumps are priced in the market of variance futures contracts implies that hedging strategies accounting for this source of risk are required in practice. Since this risk together with that of volatility are reflected in variance futures contracts, it will be interesting in examining if these assets can be employed to improve upon the performance of the traditional delta hedging strategy. The latter, denoted as $H_P^1$, hedge a long position in a European option by holding a self-financing portfolio consisting of $\phi_t^S = \partial C/\partial S$ number of shares of the underlying stock and invest the residuals, defined as $U_t = C_t - \phi_t^S S_t$, in a zero-coupon bond. In addition to this, it will be also interesting in investigating if there exist potential gains by adding higher-order moment futures in the hedging portfolio.

To empirically investigate the above questions, we will compare the following two hedging strategies with strategy $H_P^1$: The first is a two-instruments hedging strategy, denoted as $H_P^2$, which assumes that $H_P^1$ includes also a short position in $\phi_t^{FV} = \partial C/\partial FV$ number of a variance futures contracts. The second, denoted as $H_P^3$, is a strategy which extends $H_P^2$ to also include the bipower, 3rd and 4th-order moment futures contracts in the self-financing portfolio at numbers $\phi_t^{F(2)} = \frac{1}{2!} \left( \frac{\partial^{(2)} C}{\partial \ln S^2} - \frac{\partial C}{\partial \ln S} \right) - \frac{\partial C}{\partial FV}$, for $j = 2$, and $\phi_t^{F(j)} = \frac{1}{j!} \left( \frac{\partial^{(j)} C}{\partial \ln S^j} - \frac{\partial C}{\partial \ln S} \right)$, for $j = 3, 4$, respectively, as is implied by Proposition 5. The choice to include up to the 4th-order moment futures in the hedging portfolio is motivated by the results of the simulation exercise, which provide evidence that a limited number of higher-order moment futures can sufficiently hedge option price changes.

All these strategies are evaluated for OTM puts, with moneyness levels $K/S_t$ of 0.8 (deep-OTM) and 0.9, an ATM call, with $K/S_t = 1$, and OTM calls, with $K/S_t$ of 1.1 and 1.2 (deep-OTM). The expiration dates of these options lie within the maturity intervals of 15 to 35, 40 to 60 and 100 to 160 days. These intervals corresponds to options with time-to-maturity periods around one, two and a half and six months, respectively. In our analysis, we divide our sample in weeks and, for each week, we consider a daily rebalancing of the self-financing portfolio, implying that $\Delta t = 1/252$. The delta coefficients $\phi_t^S$, $\phi_t^{FV}$ and $\phi_t^{(j)}$, for $j = 2, 3, 4$, are calculated based on daily estimates of the parameters of the SVJ model, reported in Section 5.2. Since the bipower and higher-order moment futures
contracts are not traded, to evaluate strategy $HP_3$ we rely on estimates of their prices under the SVJ model (see equation (21)). The drift term of these prices are estimated based on the average of their daily estimates of parameters $\pi^Q$ and $\sigma^Q_j$, for all weeks of the sample. This guarantees that the prices of these contracts are consistent with the predictions of the SVJ model, for all weeks.\(^{20}\)

Tables 7A, 7B and 7C present values of the following evaluation metrics of the hedging errors of the three hedging strategies, given by formulas (28), (31) and (32), respectively, across different moneyness levels and maturity intervals:

$$\text{RMSE} = \sqrt{\frac{1}{D} \sum_{s=1}^{D} H(t + s\Delta t)^2},$$

(35)

where RMSE stands for root mean square error and $D$ is the number of days (denoted as $s$) of our sample,

$$\text{MAE} = \frac{1}{D} \sum_{s=1}^{D} |H(t + s\Delta t)|,$$

(36)

where MAE stands for the mean absolute error, and

$$\Sigma = \frac{1}{W} \sum_{j=1}^{W} \sum_{s=1}^{W_j} H(t + s\Delta t)$$

(37)

is the average cumulative delta-hedged gains across all weeks of the sample, where $W$ denotes the number of weeks of the sample and $W_j$ denotes the number of the trading days per week. Table 7A presents results for the whole sample, while Tables 7B and 7C for the two subsamples before and after the recent financial crisis, respectively. To test if there are significant hedging gains across the three strategies $HP_1$, $HP_2$ and $HP_3$, the table also reports $p$-values of an overidentified restrictions GMM based test statistic, denoted as $J$-stat (see Chernov and Ghysels (2000)). This is distributed as $\chi^2$ with one degree of freedom. This statistic can test if the following $(2 \times 1)$-dimension vector of second-order moment

\(^{20}\)To see if our results are sensitive to the estimator of the drift component of higher-order moment futures prices, we have also conducted our hedging evaluation exercise assuming that $\pi^Q$ and $\sigma^Q_j$ are given as the mean of the daily estimates of these parameters across the sample period. The results, which can be provided upon request, are very close to those reported in the current section. This is due to the fact that the performance of a hedging strategy mainly depends on the stochastic term of the price, which is approximated by the accrued realized non-central moment rather than the drift term, which is affected by the parameters of the jump component.
conditions are satisfied by the data:

\[
\left( \frac{1}{D} \sum_{s=1}^{D} H_k(t + s\Delta t)^2 - \mu \right) - \left( \frac{1}{D} \sum_{s=1}^{D} H_l(t + s\Delta t)^2 - \mu \right) = 0,
\]

for any pair of \( k \neq l \) hedging strategies, where \( H_i(t + s\Delta) \), for \( i = \{k,l\} \), is the hedging error of hedging strategy \( HP_i \), for \( i = 1,2,3 \), and \( \mu \) is a constant. This constant is assumed to be the same across any pair of strategies \( (k,l) \) examined. If this hypothesis is not true, then \( J \)-stat will reject the above moment conditions, which means that hedging strategy \( HP_k \) differs from \( HP_l \), for \( k \neq l \).

The results of Tables 7A-7C lead to the following conclusions. First, strategy \( HP_2 \), which, in addition to the underlying asset, includes also in the hedging portfolio short positions in a variance futures contract clearly outperforms traditional strategy \( HP_1 \), which considers only a short position in the underlying stock. This is true for all different maturity intervals and moneyness levels examined, as well as the two subsamples considered. It can be justified by the values of metrics RMSE and MAE, and the \( p \)-values of \( J \)-stat reported in the tables. The latter always reject the overidentified restrictions for the pair of strategies \( HP_1 \) and \( HP_2 \). The higher benefits of \( HP_2 \) than those of \( HP_1 \) are observed for ATM and long-term options. This can be attributed to the fact that these categories of options are more affected by a change in volatility, as is pointed out by the analysis of Section 3.1.2.

Second, the values of the RMSE and MAE metrics reported in the table indicate that \( HP_3 \) outperforms \( HP_2 \) for all options categories, with the exception of short-term OTM calls. The highest hedging benefits of \( HP_3 \) compared to \( HP_2 \) are observed in cases of the ATM call and the short-term OTM put with \( K/S_t = 0.9 \). This can be also justified by the \( p \)-values of the \( J \)-stat reported in the tables. The outperformance of \( HP_3 \) over \( HP_2 \) in these cases can be attributed to the fact that the above categories of options are exposed more to jump risk. These results are in accordance with our theoretical predictions of Section 3.2.2, which claim that ATM calls and short-term OTM puts are more sensitive to random jumps compared to deep-OTM calls and puts, as well as long-term OTM puts. Only for short-term OTM calls, we observe a worse performance of strategy \( HP_3 \) than \( HP_2 \). This result can be explained by the graphs of Figure 1 and the inadequacy of the SVJ model to fully capture the discontinuous component of stock price dynamics, as mentioned before.

As can be seen from these graphs, the delta coefficients of higher-order moment futures
$\phi^{F(j)}$, for $j = 2, 3, 4$, take extreme positive or negative values for short-term OTM calls. Thus, a small bias in the parameters estimates of the SVJ model leads to substantial deviations of $\phi^{F(j)}$ from their true values. A number of recent studies indicated that the observed returns of OTM calls can be explained by generalizing standard asset pricing theory assuming either a model that accommodates heterogeneity in beliefs about return outcomes and short-selling (see Bakshi, Madan and Panayotov (2010)) or a rank-dependent utility model with probability weighting function, which overweights tail events (see Polkova and Zhao (2012)). Finally, the results of the tables indicate that, for the post-crisis period, the benefits of $HP_3$ strategy compared to $HP_2$ become much higher for the deep-OTM put. This is consistent with the results of Section 5.3. It can be attributed to the significance of the jump risk premium for this period.

Third, the estimates of the average weekly delta-hedged gains $\Sigma$, reported in the tables, are negative for strategy $HP_1$, as they reflect risk premium effects due to volatility and random jumps. This is consistent with evidence provided in the literature (see Bakshi and Kapadia (2003)). These gains tend to increase with the maturity interval, which indicates that jump risk premium, which affects more short-term options rather than long-term, dominates volatility risk premium (see also Branger and Schlag (2008)). With respect to moneyness, the values of $\Sigma$ become closer to zero for OTM calls and puts rather than ATM calls, which means that the impact of volatility and jump premia is smaller for away-from-the-money options. For strategy $HP_2$, the values of $\Sigma$ tend to zero for most options categories examined, especially for deep-OTM and long-term options. This result means that the expected gains of strategy $HP_2$ are no longer affected by the volatility risk premium. The jump risk premium does affect the gains of strategy $HP_2$ for these two categories of options, since its effect is negligible (see our theoretical discussion in Section 3.2.2). This is not true for short-term OTM puts and ATM calls. For these categories of options, the value of $\Sigma$ become negative due to the significant influence of the jump risk premium effects. This sign of $\Sigma$ can be analytically explained by that of delta coefficients $\phi^{F(j)}$, for $j = 2, 3, 4$, discussed in Section 3.2.2 and the sign of the expected price changes of higher-order moment futures contracts $\mu^{F(j)}$, for $j = 2, 3, 4$, discussed in Section 3.2.1.

In particular, the negative sign of $\Sigma$ for short-term OTM puts means that the total effect of $\phi^{F(3)}$ and $\phi^{F(4)}$ on $E_t \left[ \frac{\partial C_t}{\partial \sigma} \right]$ (see formula (27)), which is negative, dominates that of $\phi^{F(2)}$, which is positive. The negative sign of $\Sigma$ for short-term ATM calls means that the
total effect of $\phi^{(3)}$ on $E_t \left[ \frac{dC_t}{C_t} \right]$, which is negative, dominates that of $\phi^{(2)}$ and $\phi^{(4)}$, which is positive. This negative sign of strategy $HP_2$ for short-term OTM puts and ATM calls can be explained by noticing that a long position in these contracts and a short position in the hedging portfolio of strategy $HP_2$ compensates investors for a negative jump, against which they are averse. For short-term OTM calls, the positive sign of $\Sigma$ can be attributed to the fact that these options are heavily exposed to a possible negative jump, which will tend to make them deep-OTM. When adopting strategy $HP_3$, the values $\Sigma_t$ tend to zero for OTM puts, for the whole sample.

The robustness of the above results has been investigated by carrying out two exercises. In the first, a monthly interval is chosen to construct the alternative hedging portfolios using daily rebalancing (see also Bollen and Whaley (2004)), while, in the second, the monthly interval of the hedging portfolios is combined with weekly rebalancing. The results of these two exercises do not change the above conclusions about the performance of the hedging strategies examined.

5.5 Delta-hedged gains, volatility and jump risk premia

To verify that the reported in the previous section delta-hedged gains depend on volatility and/or jump risk premia, we run the following regressions for ATM options:

$$\Delta \hat{\Sigma}_t^{HP_j}/S_t = \beta_0 + \beta_1 \Delta VOL_t + \beta_2 \Delta FVO_t + \varepsilon_{j,t}, \quad j = 1, 2, 3, \quad (38)$$

where $\Delta \hat{\Sigma}_t^{HP_j}/S_t$ are the first differences of the cumulative weekly delta-hedged gains of strategies $HP_j$, for $j = 1, 2, 3$, scaled by the current stock price $S_t$, $\Delta VOL_t$ is the first difference of historical volatility computed over the week prior to $t$, and $\Delta FVO_t$ is the first difference of variable $\hat{FV}_t - \hat{V}_{0,t}$, which captures the sum of higher-order moment effects $\sum_{j=3}^{\infty} \frac{1}{j!} \left[ \sum_{t<u\leq T} (\Delta X_u)^j \right]$ depending on jump premium. Regression (38) has been suggested by Bakshi and Kapadia (2003) to investigate if volatility risk is reflected in the expected delta-hedged gains of ATM options of strategy $HP_1$. We have extended this regression to two directions. First, we also consider hedging strategies $HP_2$ and $HP_3$, and, second, we employ a market based, direct measure of jump risk premium in its right hand.

21This weekly interval of the estimation of historical volatility ensures that our data series are not overlapping.
side, given by $\Delta FVO_t$. Testing whether volatility or jump risk are not priced are equivalent to testing if $\beta_1 = 0$ or $\beta_2 = 0$ can not be rejected by the data.

Since regressions (38), involve contemporaneous terms, to avoid any endogeneity bias estimation problems of slope coefficients $\beta_1$ and $\beta_2$ we have estimated them based on the GMM procedure, using the following instruments: the constant, and lagged values of the dependent variable $\Delta \hat{\Sigma}^H_{it}/S_t$ and the two independent variables $\Delta VOL_t$ and $\Delta FVO_t$ two periods back. The estimation results are presented in Table 8. This is done for the three maturity intervals considered in Tables 7A-7C. Newey-West standard errors corrected for heteroscedasticity and serial correlation are reported in parentheses. The order of serial correlation is chosen based on the Schwarz information criterion. The results of the table reveal the following.

First, the delta-hedged gains of strategy $HP_1$ reflect volatility and jump risk premia, given that the estimates of coefficients $\beta_1$ and $\beta_2$ are clearly different than zero at 5% significance level. The negative estimates of $\beta_1$ and $\beta_2$ indicate that the volatility and jump risk premia are negative, which is consistent with the findings of Bakshi and Kapadia (2003). Second, the delta-hedged gains of strategy $HP_2$ does not reflect volatility risk premium, since the estimates of $\beta_1$ are not significant. This result means that $HP_2$ can efficiently hedge against volatility risk. As was expected, the only source of risk that is reflected in the delta-hedged gains of strategy $HP_2$ is due to random jumps, as the estimates of $\beta_2$ are different than zero. The estimates of $\beta_2$ become insignificant for strategy $HP_3$, which also includes bipower and higher-order moment futures contracts in the hedging portfolio. This indicates that the delta-hedged gains of strategy $HP_3$ can also account for jump risk. The only exception is for long-term options. This may be attributed to the fact that delta-hedged gains of strategies $HP_2$ and $HP_3$ are very close to each other for this category of options (see the results of Table 7A). This can obviously attributed to the fact that the SVJ model can not fully capture the dynamics of the data, as argued in Section 5.3. As can be seen from Figure 1, $\phi^{F(i)}$ are very small for long-term options. Thus, a small difference in the estimates of $\phi^{F(i)}$, due to a slight mispecification of the SVJ model, can lead to significant biases in the delta-hedged gains.
6 Conclusions

This paper suggests perfect hedging strategies of contingent claims, including variance futures contracts, volatility swaps and plain vanilla European options, under the assumption that the underlying asset price follows the stochastic volatility (SV) and the stochastic volatility with jumps (SVJ) models. This is done by enlarging the market with appropriate futures contracts whose payoffs depend on higher-order sample moments of the underlying asset price. For the SV model, we demonstrate that enlarging the market with a variance futures (or swap) contract, which is nowadays traded in the market, makes it complete and, thus, perfectly hedge positions in contingent claims written on the underlying asset and/or its volatility.

For the SVJ model, the paper shows that, in addition to variance futures contracts, higher than second-order moment futures contracts should be included in the self-financing portfolio to replicate the price of a contingent claim. These futures contracts can hedge the exposure of a contingent claim price to random jumps. If this claim is a variance futures contract, this can be done together with the composite portfolio of OTM options. Since the size of jumps is random, the value of the self-financing hedging portfolio converges to the price of the contingent claim, as the number of higher-order moment futures contracts goes to infinity, thus making the market approximately (or quasi) complete. In practice, someone can satisfactorily approximate the value of this hedging portfolio with a finite number of higher-order moment futures contracts which can be traded in the market.

The results of the paper are used to empirically address two questions on pricing and hedging options. The first is if random jumps are priced in the market. This price of risk is reflected in the difference between the price of a variance futures contract and the value of the composite options portfolio. It can be estimated from market data without relying on any parametric model. The second question is if a position in a European option can be efficiently hedged based on a two-instruments hedging strategy, which considers a variance futures contract as a hedging vehicle, or it requires hedging strategies which rely on higher-order moment futures. To answer these questions, we rely on options and variance futures price data written on the S&P 500 index. The empirical results of the paper clearly indicate that, indeed, random jumps are priced in the variance futures market. The paper clearly shows that the prices of these contracts significantly exceed those of the composite options.
portfolio, thus implying that random jumps are priced in the variance futures market.

Regarding the second question, the results of both the empirical and simulation exercises clearly carry out by the paper indicate that a two-instruments hedging strategy, which also includes a variance futures contracts in the self-financing portfolio, is found to considerably improves upon the performance of the traditional delta hedging strategy often used in practice, which only includes the underlying asset and the zero-coupon bond in this portfolio. The improving performance of this strategy comes from the fact that variance futures contracts can efficiently hedge the exposure of a call, or a put, to volatility risk. These results also provide evidence that the inclusion of a finite (up to the 4th-order) number of higher-order moment futures contracts into the self-financing portfolio can further improve the performance of the above two-instruments hedging strategy, especially for short-term options which are more sensitive to jump risk. These results have important portfolio management implications. They indicate the need, first, to introduce variance futures contracts in the traditional delta hedging strategy in order to account for volatility risk, and second, to consider higher-order moment futures for hedging options and variance futures positions against their exposure to random jumps.

A Appendix

In this appendix, we provide proofs of equations (4) and (34) and the propositions presented in the main text.

A.1 Proof of equation (4)

To prove equation (4), we will rely on the conditional risk-neutral mean of future period log-return distribution given as

\begin{equation}
\mu_{1,t} = \left( E_t^Q \left[ \frac{S_T}{S_t} \right] - 1 \right) - e^{r_T} \left[ \int_{S_t}^{+\infty} \frac{1}{K^2} C_t(\tau, K) dK + \int_0^{S_t} \frac{1}{K^2} P_t(\tau, K) dK \right].
\end{equation}


By applying Ito’s lemma for semimartingales (see Protter (1990)) to function $S = e^X$, the

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we have
\[ dS_u = S_u - dX_u + \frac{1}{2} S_u - d \langle X, X \rangle_u + \left( S_u - S_u - S_u - \Delta X_u - \frac{1}{2} S_u - (\Delta X_u)^2 \right), \]
which implies
\[ \frac{1}{2} d \langle X, X \rangle_u = \frac{dS_u}{S_{u^-}} - dX_u + \left( \Delta X_u + \frac{1}{2} (\Delta X_u)^2 - \frac{S_u - S_{u^-}}{S_{u^-}} \right). \] (40)

By Taylor’s series expansion, the discontinuous component of the last relationship can be written as
\[ \Delta X_u + \frac{1}{2} (\Delta X_u)^2 - \frac{S_u - S_{u^-}}{S_{u^-}} = \Delta X_u + \frac{1}{2} (\Delta X_u)^2 + 1 - e^{\Delta X_u} = - \sum_{j=3}^{\infty} \frac{(\Delta X_u)^j}{j!}, \]
which implies that equation (40) can be written as
\[ \frac{1}{2} d \langle X, X \rangle_u = \frac{dS_u}{S_{u^-}} - dX_u - \sum_{j=3}^{\infty} \frac{(\Delta X_u)^j}{j!}. \]

Integrating the last formula yields
\[ \frac{1}{2} \langle X, X \rangle_{T - \tau, T} = \]
\[ = \int_{T - \tau}^{T} \frac{dS_u}{S_{u^-}} - \left( \ln \left( \frac{S_{T - \tau}}{S_{T - \tau}} \right) - \ln \left( \frac{S_T}{S_t} \right) \right) - \sum_{j=3}^{\infty} \frac{1}{j!} \left( \sum_{T - \tau < u \leq T} (\Delta X_u)^j \right), \] (41)

Taking the conditional risk-neutral expectations of equation (41) and, then, multiplying both sides of the resulting equation with \(2/\tau\) yields
\[ FV_t = \frac{1}{\tau} \langle X, X \rangle_{T - \tau, t} + \frac{2}{\tau} E_t^Q \left[ \int_t^T \frac{dS_u}{S_{u^-}} - \frac{2}{\tau} E_t^Q \left[ \ln \left( \frac{S_T}{S_t} \right) \right] - \frac{2}{\tau} \sum_{j=3}^{\infty} \frac{1}{j!} E_t^Q \left[ \sum_{t < u \leq T} (\Delta X_u)^j \right] \right]. \]

Substituting the first-order risk-neutral moment, given by formula (39), into the last formula
and rearranging terms yields
\[
FV_t = \frac{1}{\tau} (X, X)_{T-\tau, t} + 2e^{\frac{\tau}{\tau}} \left[ \int_0^{S_t} \frac{1}{K^2} C_t(\tau, K)dK + \int_0^S \frac{1}{K^2} P_t(\tau, K)dK \right] + \\
+ \frac{2}{\tau} E_t^Q \left[ \int_t^T \frac{dS_u}{S_u} - \frac{S_T - S_t}{S_t} \right] - \frac{2}{\tau} \sum_{j=3}^{\infty} \frac{1}{j!} E_t^Q \left[ \sum_{t<u<T} (\Delta X_u)^j \right],
\]
where \( \tau = T - t \), which proves formula equation (4).

A.2 Proof of Proposition 1

For the proof of formula (13), see Gatheral (2006). To derive formula (14), note that (13) implies that
\[
dFV_t = V_t dt + dF(t, V_t) = \left( V_t + \frac{\partial FV}{\partial t} \right) dt + \frac{\partial FV}{\partial V} dV_t,
\]
where \( \frac{\partial^2 FV}{\partial V^2} = 0 \). Substituting in the last relationship volatility process (11), we obtain
\[
dFV_t = \left( V_t + \frac{\partial FV}{\partial t} + \frac{\partial FV}{\partial V} \kappa (\theta - V_t) \right) dt + \frac{\partial FV}{\partial V} \sigma \sqrt{V_t} dW^{(2)}_t.
\]
Since the variance futures price must be a martingale under measure \( Q \), implying \( E_t^Q [dFV_t] = 0 \), the last relationship implies that
\[
V_t + \frac{\partial FV}{\partial t} + \frac{\partial FV}{\partial V} \kappa (\theta - V_t) = \frac{\partial FV}{\partial V} \tilde{V}_t
\]
or
\[
dFV_t = \frac{\partial FV}{\partial V} \tilde{V}_t dt + \frac{\partial FV}{\partial V} \sigma \sqrt{V_t} dW^{(2)}_t.
\]
The last equation implies that the instantaneous expected return of the variance futures price under the physical measure \( P \) is given as
\[
\mu^{FV}_t dt = E_t \left[ \frac{dFV_t}{FV_t} \right] = \frac{\partial \ln FV}{\partial V} \tilde{V}_t dt = \frac{\partial \ln FV}{\partial V} \tilde{V}_V dt,
\]
which proves formula (14) of Proposition 1.
A.3 Proof of Proposition 2

The portfolio is self-financing implying that

$$dC_t = \phi_t^S dS_t + \phi_t^B dB_t + \phi_t^{FV} dFV_t + \delta_t \phi_t^S S_t dt.$$ 

By applying Ito’s lemma to contingent claim price process $C_t$ we obtain

$$dC_t = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial V} dV_t + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 dt + \frac{1}{2} \frac{\partial^2 C}{\partial V^2} \sigma^2 V_t dt + \frac{\partial C}{\partial S} \rho \sigma S_t dt$$

Equating the last two equations and using equation (42) yields

$$\phi_t^S dS_t + \phi_t^B r_t dt + \phi_t^{FV} \left( \frac{\partial FV}{\partial t} dt + V_t dt + \frac{\partial FV}{\partial V} dV_t \right) + \delta_t \phi_t^S S_t dt$$

$$= \left( \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 V_t + \frac{1}{2} \frac{\partial^2 C}{\partial V^2} \sigma^2 V_t + \frac{\partial C}{\partial S} \rho \sigma S_t V_t \right) dt + \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial V} dV_t$$

Set $\phi_t^S = \partial C/\partial S$ and $\phi_t^{FV} = \frac{\partial C/\partial V}{\partial FV/\partial V} = \partial C/\partial FV$, and substitute equations (15) and (43) into the last formula. This yields

$$C_t r_t - \frac{\partial C}{\partial S} S_t (r_t - \delta_t) + \frac{\partial C}{\partial V} (V_t \gamma_V - \kappa (\theta - V_t))$$

$$= \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 V_t + \frac{1}{2} \frac{\partial^2 C}{\partial V^2} \sigma^2 V_t + \frac{\partial C}{\partial S} \rho \sigma S_t V_t.$$ 

Replacing in the last relationship the parameters of volatility process $V_t$ with their risk-neutral counterparts and rearranging terms yields

$$\frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 V_t + \frac{\partial C}{\partial S} S_t (r_t - \delta_t) - C_t r_t + \frac{\partial C}{\partial S} \rho \sigma S_t V_t + \frac{1}{2} \frac{\partial^2 C}{\partial V^2} \sigma^2 V_t + \frac{\partial C}{\partial V} \kappa Q (\theta^Q - V_t) = 0,$$

which is the partial differential equation (PDE) of contingent claim price $C_t$, derived by Heston (1993) under equilibrium approach. This PDE implies that the drift term of $dC_t$ is equal to

$$r_t C_t - \frac{\partial C}{\partial S} S_t (r_t - \delta_t) + \frac{\partial C}{\partial V} (\gamma_V V_t - \kappa (\theta - V_t)).$$
Given this, the stochastic process driving $C_t$ can be written as

$$dC_t = \left( r_t C_t - \frac{\partial C}{\partial S} S_t (r_t - \delta_t) + \frac{\partial C}{\partial V} (\gamma_t V_t - \kappa (\theta_t - V_t)) \right) dt + \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial V} dV_t$$

$$= r_t C_t dt + \frac{\partial C}{\partial S} S_t V_t \gamma_s dt + \frac{\partial C}{\partial V} \gamma_t V_t dt + \left( \frac{\partial C}{\partial S} S_t \sqrt{\gamma_t} V_t + \frac{\partial C}{\partial V} \sigma \sqrt{\gamma_t} \right) dW^{(1)}_t + \frac{\partial C}{\partial V} \sigma \sqrt{1 - \rho^2} \sqrt{V_t} dW^{(3)}_t,$$

which implies

$$E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln S} \gamma_t V_t dt + \frac{\partial \ln C}{\partial \ln V} \gamma_t V_t dt.$$

Using $\gamma_t V_t = \delta \mu_t^S + \delta_t - r_t$ and formula (14), the last equation yields

$$E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln S} (\mu_t^S + \delta_t - r_t) dt + \frac{\partial \ln C}{\partial \ln FV} \mu_t^{FV} dt,$$

which proves formula (16) of Proposition 2.

### A.4 Proof of Proposition 3

Under the no-arbitrage principle, we have

$$FV_t = \frac{1}{T} E_t^Q \left[ \langle X, X \rangle_{0,T} \right] = \frac{1}{T} \left( \int_0^T V_u du + \int_0^T \tilde{J}_u^2 dN_u + E_t^Q \left[ \int_0^T \tilde{J}_u^2 dN_u \right] \right)$$

$$= \frac{1}{T} \left( \int_0^T V_u du + \int_0^T \tilde{J}_u^2 dN_u + F(t, V_t) + G(t) \right),$$

where $F(t, V_t) = \tau (\psi_t V_t + (1 - \psi_t) \theta^Q)$ and $\psi_t = (1 - e^{-\kappa^Q t}) / (\kappa^Q \tau)$ (see Proposition 3), and $G(t) = \lambda \left( \mu^Q_{(2)} \right)^{\tau}$, where $\mu^Q_{(2)}$ is the second-order non-central moment of $\tilde{J}$ under measure $Q$. The last relationship proves formula (19) of Proposition 3. This relationship implies that

$$dFV_t = V_t dt + \tilde{J}_t^2 dN_t + dF(t, V_t) + dG(t)$$

$$= \left( V_t + \frac{\partial FV}{\partial t} + \frac{\partial G}{\partial t} \right) dt + \frac{\partial FV}{\partial V} dV_t + \tilde{J}_t^2 dN_t. \quad (44)$$
Substituting into the last relationship volatility process \( V_t \) and equation (43), which also holds under the SVJ model, we obtain

\[
dFV_t = \left( \frac{\partial FV}{\partial V} \tilde{\gamma}_V V_t + \frac{\partial G}{\partial t} \right) dt + \frac{\partial FV}{\partial V} \sigma \sqrt{V_t} dW_t^{(2)} + \tilde{J}_t^2 dN_t.
\]

The last relationship implies that the instantaneous expected return of the variance futures contract is given as

\[
\mu^F_t dt \equiv E_t \left[ \frac{dFV_t}{FV_t} \right] = \frac{\partial \ln FV}{\partial \ln V} \tilde{\gamma}_V dt + \frac{\lambda}{FV_t} (\mu_{(2)} - \mu_{(2)}^Q) dt,
\]

where \( \mu_{(2)} \) is the second-order non-central moment of \( \tilde{J} \) under \( P \). This proves formula (20) of Proposition 3.

A.5 Proof of Proposition 4

Under the no-arbitrage principle, we have

\[
F_{(j),t} = \frac{1}{T} E_t^Q \left[ \int_0^T \tilde{J}_u^j dN_u \right] = \frac{1}{T} \left( \int_0^t \tilde{J}_u^j dN_u + E_t^Q \left[ \int_t^T \tilde{J}_u^j dN_u \right] \right)
\]

\[
= \frac{1}{T} \left( \int_0^t \tilde{J}_u^j dN_u + \lambda \mu_{(j)}^Q \right),
\]

for \( j \geq 2 \), where \( \mu_{(j)}^Q \) is the \( j \)th-order non-central moment of \( \tilde{J} \) under \( Q \). The last relationship proves formula (21). To prove formula (22), note that futures contracts price \( F_{(j),t} \) follows the data-generating process:

\[
dF_{(j),t} = \tilde{J}_t^j dN_t - \lambda \mu^Q_{(j)} dt.
\]

Taking the conditional expectation of the last relationship yields

\[
\mu^F_{(j)} dt \equiv E_t \left[ dF_{(j),t} \right] = \lambda \left( \mu_{(j)} - \mu^Q_{(j)} \right) dt,
\]

where \( \mu_{(j)} \) is the \( j \)th-order non-central moment of \( \tilde{J} \) under the physical measure. The last relationship proves formula (22) of Proposition 4.
A.6 Proof of Proposition 5

Before proving Proposition 5, we need to prove the following lemma.

**Lemma 1** Consider contingent claim price function \( C_t = C(t, S_t, V_t) \) which is infinitely differentiable with respect to \( S \) and assume that

\[
\sup_{t < t^*, S < S^*, V < V^*} \sum_{n=2}^{\infty} \left| \frac{\partial^{(n)} C}{\partial \ln S^{(n)}} - \frac{\partial C}{\partial \ln S} \right| R^n < \infty
\]  

(46)

for all \( t^*, S^*, V^* > 0 \) and \( R > 0 \). Then, the following result holds:

\[
\Delta C_t - \frac{\partial C}{\partial S} \Delta S_t = \Delta C_t - \frac{\partial C}{\partial \ln S} \frac{\Delta S_t}{S_t} = \Delta C_t - \frac{\partial C}{\partial \ln S} \frac{J_t}{n!}.
\]  

(47)

where \( J = \ln(1 + J) = \ln \left( 1 + \frac{dS}{S} \right) \).

**Proof.** By taking a Taylor’s series expansion of the discontinuous component of the contingent claim price with respect to the stock price, we have that

\[
\Delta C_t - \frac{\partial C}{\partial S} \Delta S_t = \Delta C_t - \frac{\partial C}{\partial \ln S} \frac{\Delta S_t}{S_t} = \Delta C_t - \frac{\partial C}{\partial \ln S} \frac{J_t}{n!}.
\]  

(48)

The last equation comes from the fact that

\[
J_t = e^{\ln(1 + J_t)} - 1 = \lim_{N \to \infty} \left( \sum_{n=1}^{N} \frac{J^n_t}{n!} \right).
\]

Note that formula (48) is equivalent to

\[
\Delta C_t - \frac{\partial C}{\partial \ln S} \frac{J_t}{n!} \lim_{N \to \infty} \left( \sum_{n=2}^{N} \frac{\tilde{J}^n_t}{n!} \right) = \lim_{N \to \infty} \sum_{n=2}^{N} \frac{\partial^{(n)} C}{\partial \ln S^{(n)}} \frac{\tilde{J}^n_t}{n!} - \frac{\partial C}{\partial \ln S} \lim_{N \to \infty} \left( \sum_{n=2}^{N} \frac{J^n_t}{n!} \right) =
\]

where the sum converges because of (46), which proves equation (47).

Given Lemma 1, next we prove Proposition 5.
Proof. The portfolio is self-financing portfolio which implies that

\[ dV_t(\phi) = \phi_i^S dS_t + \phi_i^B dB_t + \phi_i^{FV} dFV_t + \sum_{n=2}^{N} \phi_i^{F(n)} dF_{(n),t} + \delta_t \phi_i^S S_t dt. \]

This portfolio replicates the contingent claim price in the sense that, as \( N \to \infty \), we have

\[ \lim_{N \to \infty} dV_t(\phi) = dC_t, \]

which implies that

\[ dC_t = \phi_i^S dS_t + \phi_i^B dB_t + \phi_i^{FV} dFV_t + \lim_{N \to \infty} \sum_{n=2}^{N} \phi_i^{F(n)} dF_{(n),t} + \delta_t \phi_i^S S_t dt. \]

Applying Ito’s lemma to contingent price function \( C_t = C(t, S_t, V_t) \) and substituting equations (44) and (45) into the last relationship yields

\[
\begin{align*}
\frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial V} dV_t + \frac{\partial^2 C}{2 \partial S^2} S_t^2 V_t dt + \frac{1}{2} \frac{\partial^2 C}{\partial V^2} \sigma^2 V_t dt + \frac{\partial C}{\partial S \partial V} \rho \sigma S_t V_t dt + & \\
+ \left( \Delta C_t - \frac{\partial C}{\partial S} \Delta S_t \right) dN_t & \\
= & \phi_i^S dS_t + \phi_i^B dB_t + \phi_i^{FV} \left( V_t + \frac{\partial FV}{\partial t} + \frac{\partial G}{\partial t} \right) dt + \phi_i^{FV} \frac{\partial FV}{\partial V} dV_t + \phi_i^{FV} \tilde{\Gamma}_t^2 dN_t + \\
+ \lim_{N \to \infty} \sum_{n=2}^{N} \phi_i^{F(n)} \left( \tilde{J}_t^n dN_t - \lambda \mu_{(0)} dt \right) + \delta_t \phi_i^S S_t dt.
\end{align*}
\]

Assume now that price function \( C(t, S_t, V_t) \) is infinitely differentiable with respect to \( S \). Substituting equation (47) into the last relationship and taking a Taylor’s series expansion of the discontinuous component of the contingent claim price with respect to stock price \( S \) yields

\[
\begin{align*}
\frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial V} dV_t + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 V_t dt + \frac{1}{2} \frac{\partial^2 C}{\partial V^2} \sigma^2 V_t dt + \frac{\partial C}{\partial S \partial V} \rho \sigma S_t V_t dt + & \\
+ \lim_{N \to \infty} \left( \sum_{n=2}^{N} \frac{\partial^{(n)} C}{\partial \ln S^{(n)}} - \frac{\partial C}{\partial \ln S} \right) \tilde{J}_t^n dN_t & \\
= & \phi_i^S dS_t + \phi_i^B dB_t + \phi_i^{FV} \left( V_t + \frac{\partial FV}{\partial t} + \frac{\partial G}{\partial t} \right) dt + \phi_i^{FV} \frac{\partial FV}{\partial V} dV_t + \phi_i^{FV} \tilde{\Gamma}_t^2 dN_t + \\
+ \lim_{N \to \infty} \sum_{n=2}^{N} \phi_i^{F(n)} \left( \tilde{J}_t^n dN_t - \lambda \mu_{(0)} dt \right) + \delta_t \phi_i^S S_t dt.
\end{align*}
\]
Setting $\phi_t^S = \partial C / \partial S$, $\phi_t^{FV} = \partial C / \partial FV = \partial C / \partial FV$, $\phi_t^{F(2)} = \frac{1}{2} \left( \frac{\partial^2 C}{\partial \ln S^2} - \frac{\partial^2 C}{\partial \ln S} \right) - \partial C / \partial FV$ and $\phi_t^{F(j)} = \frac{1}{n} \left( \frac{\partial^{(n)} C}{\partial \ln S^{(n)}} - \frac{\partial C}{\partial \ln S} \right)$, for $j > 2$, into the last relationship and using equations (15) and (43) yields

$$\lim_{N \to \infty} \left[ \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S^2_t V_t + \frac{\partial C}{\partial S} S_t (r_t - \delta_t) - C_t r_t + \frac{\partial C}{\partial S} \rho \sigma S_t V_t + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 V_t \\
+ \frac{\partial C}{\partial V} \kappa_t Q (\theta^Q - V_t) + \sum_{n=2}^{N} \frac{1}{n!} \left( \frac{\partial^{(n)} C}{\partial \ln S^{(n)}} - \frac{\partial C}{\partial \ln S} \right) \lambda^{Q(n)} \right] = 0. \quad (49)$$

The last formula can be simplified by taking the risk-neutral conditional expectation of equation (47), which implies

$$\lim_{N \to \infty} \sum_{n=2}^{N} \frac{1}{n!} \left( \frac{\partial^{(n)} C}{\partial \ln S^{(n)}} - \frac{\partial C}{\partial \ln S} \right) \lambda^{Q(n)} = \lambda E_t^Q [\Delta C_t] - \frac{\partial C}{\partial S} S_t \lambda^Q. \quad (50)$$

Then, (49) can be written as

$$\frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 V_t + \frac{\partial C}{\partial S} S_t (r_t - \delta_t - \lambda^Q) - C_t r_t + \frac{\partial C}{\partial S} \rho \sigma S_t V_t + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 V_t \\
+ \frac{\partial C}{\partial V} \kappa_t Q (\theta^Q - V_t) + \lambda E_t^Q [\Delta C_t] = 0, \quad (51)$$

which is the PDE derived Bates (1996) under equilibrium approach. As with the SV model, this PDE now is derived by forming a portfolio that approximately replicates contingent claim price $C_t$. Equation (51) implies that the drift term of $dC_t$ is given as

$$r_t C_t - \frac{\partial C}{\partial S} S_t (r_t - \delta_t - \lambda^Q) - \frac{\partial C}{\partial V} \kappa_t Q (\theta^Q - V_t) - \lambda E_t^Q [\Delta C_t],$$

which implies that the stochastic process of $dC_t$ can be written as

$$dC_t = \left( r_t C_t + \frac{\partial C}{\partial S} S_t \gamma S V_t + \frac{\partial C}{\partial V} \gamma V_t - \lambda E_t^Q [\Delta C_t] \right) dt + \frac{\partial C}{\partial S} S_t \sqrt{V_t} dW^{(1)}_t + \\
+ \frac{\partial C}{\partial V} \sigma \sqrt{V_t} dW^{(2)}_t + \Delta C_t dN_t. \quad (52)$$

The last equation gives

$$E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln \gamma S} V_t dt + \frac{\partial \ln C}{\partial \ln \gamma V} dt + \lambda \left( E_t \left[ \frac{\Delta C_t}{C_t} \right] - E_t^Q \left[ \frac{\Delta C_t}{C_t} \right] \right) dt.$$
Using \( \gamma_S V_t = \mu_t^S + \delta_t - r_t + \lambda (\pi_t^Q - \pi) \), where \( \mu_t^S \) is the expected return of the stock at \( t \) and equation (23), we can write the last relationship as

\[
E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln S} (\mu_t^S + \delta_t - r_t) dt + \\
+ \frac{\partial \ln C}{\partial \ln V} \left( \frac{1}{\partial \ln FV/\partial \ln V} \mu_t^{FV} - \frac{1}{\partial \ln V/\partial \ln V} \mu_t^{F(2)} \right) dt + \\
+ \lambda \left( E_t \left[ \frac{\Delta C_t}{C_t} \right] - E_t^Q \left[ \frac{\Delta C_t}{C_t} \right] + \frac{\partial \ln C}{\partial \ln S} (\pi_t^Q - \pi) \right) dt.
\]

Using (50), the last term of the right-hand-side of the last formula can be written as

\[
\lambda \left( E_t \left[ \frac{\Delta C_t}{C_t} \right] - E_t^Q \left[ \frac{\Delta C_t}{C_t} \right] + \frac{\partial \ln C}{\partial \ln S} (\pi_t^Q - \pi) \right) = \frac{1}{2} \left( \frac{\partial^{(2)} C/C}{\partial \ln S^{(2)}} - \frac{\partial \ln C}{\partial \ln S} \right) \mu_t^{F(2)} + \lim_{N \to \infty} \sum_{j=3}^{N} \frac{1}{j!} \left( \frac{\partial^{(j)} C/C}{\partial \ln S^{(j)}} - \frac{\partial \ln C}{\partial \ln S} \right) \mu_t^{F(j)}
\]

Given this, equation (53) can be written as

\[
E_t \left[ \frac{dC_t}{C_t} \right] = r_t dt + \frac{\partial \ln C}{\partial \ln S} (\mu_t^S + \delta_t - r_t) dt + \frac{\partial \ln C}{\partial \ln FV} \mu_t^{FV} dt + \\
+ \left( \frac{1}{2} \left( \frac{\partial^{(2)} C/C}{\partial \ln S^{(2)}} - \frac{\partial \ln C}{\partial \ln S} \right) \right) \mu_t^{F(2)} dt + \\
+ \lim_{N \to \infty} \sum_{j=3}^{N} \frac{1}{j!} \left( \frac{\partial^{(j)} C/C}{\partial \ln S^{(j)}} - \frac{\partial \ln C}{\partial \ln S} \right) \mu_t^{F(j)} dt
\]

which proves formula (27) of Proposition 5. ■

**A.7 Proof of equation (34)**

From the proof of formula (4), we have

\[
- \frac{2}{\tau} \sum_{j=3}^{\infty} \frac{1}{j!} E_t^{Q} \left[ \sum_{t<u<T} (\Delta X_u)^j \right] \]

\[
= \frac{2}{\tau} \left( E_t^{Q} \left[ \sum_{t<u<T} \Delta X_u \right] + \frac{1}{2} E_t^{Q} \left[ \sum_{t<u<T} (\Delta X_u)^2 \right] - E_t^{Q} \left[ \sum_{t<u<T} \frac{S_u - S_{u-}}{S_u} \right] \right),
\]

where \( \tau = T - t \). Under the SVJ model, the discontinuous component of process \((S_u)_{u \in [t,T]}\) is given as \( J_t dN_t \), where \( J_t \) is the random percentage jump conditional on a jump occurring
with probability $\ln(1 + J_t) \sim N\left(\mu_j^Q, \left(\sigma_j^Q\right)^2\right)$ and $N_t$ is a Poisson process with intensity $\lambda$ under measure $Q$. Given these definitions the above relationship can be written as

$$\frac{2}{\tau} E_t^Q \left[ \int_t^T \ln(1 + J_u) dN_u + \frac{1}{2} \int_t^T \ln(1 + J_u)^2 dN_u - \int_t^T J_u dN_u \right]$$

$$= 2\lambda \left( \mu_j^Q + \frac{1}{2} \left(\sigma_j^Q\right)^2 + \frac{1}{2} \left(\mu_j^Q\right)^2 - e^{\mu_j^Q + (\sigma_j^Q)^2/2 + 1} \right),$$

which proves equation (34).
References


Schoutens, W., 2005, Moment swaps, Quantitative Finance 5, 525-530.

<table>
<thead>
<tr>
<th>Moneyness (K/S_t)</th>
<th>MAE (in $) (\Sigma) (in $)</th>
<th>30</th>
<th>120</th>
<th>30</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9 (put) (\Sigma)</td>
<td>(HP_1)</td>
<td>0.0762</td>
<td>0.1994</td>
<td>-0.0112</td>
<td>-0.0308</td>
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</table>

Notes: This table presents values of the mean absolute error (MAE) and average delta-hedged gains (\(\Sigma\)) of hedging strategies \(HP_1\) and \(HP_2\) for the SV option pricing model of Heston (1993). \(HP_1\) constitutes the traditional hedging strategy, which holds \(\partial C/\partial S\) stocks and the remaining cash in zero-coupon bonds to hedge a long position in a European option. \(HP_2\) is a two-instruments strategy, which holds \(\partial C/\partial S\) units of stocks, \(\partial C/\partial FV\) units of variance futures and the remaining cash is invested in zero-coupon bonds. The results of the table are based on \(M = 2,000\) iterations. The parameter values of the SV model used to generate the data are given as follows: \(\kappa = 5.04\), \(\theta = 0.04\), \(\sigma = 0.52\) and \(\rho = -0.66\), and \(\kappa^Q = 0.72\) and \(\theta^Q = 0.31\) under risk-neutral measure \(Q\). The prices of variance futures contracts values are calculated based on formula (13), assuming that its payoff is calculated on a discrete-time (daily) basis. Interest rate and dividend yield are set to \(\tau = 0.05\) and \(\delta = 0.015\), respectively, while \(\gamma_k\) is set to zero. We assume two different maturity intervals of \(\tau = \{30, 120\}\) days. The first considers a short-term option, which is bought 30 days prior to maturity and it is delta-hedged for one day. The second considers a long-term option, which is bought 120 days prior to maturity and it is also delta-hedged for one day.
Table 2: Simulation results for the SVJ model

<table>
<thead>
<tr>
<th>$K/S_0$</th>
<th>Panel A: Total number of iterations</th>
<th>Panel B: Iterations generating jumps</th>
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<tbody>
<tr>
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<td>MAE (in $)</td>
<td>$\Sigma$ (in $)$</td>
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<tr>
<td>0.9 (put)</td>
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<td>$HP_1$ 0.0840 0.1741</td>
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<td>$HP_2$ 0.0132 0.0060</td>
<td>0.0039 -0.0001</td>
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<td></td>
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<tr>
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<td>$HP_1$ 0.1262 0.1769</td>
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<td></td>
<td>$HP_2$ 0.0032 0.0023</td>
<td>0.0004 -0.0002</td>
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<td></td>
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<td>$-5.7 \times 10^{-5}$ -0.0001</td>
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<td></td>
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<td>$HP_2$ 0.0062 0.0023</td>
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<td>$HP_3$ 0.0068 0.0029</td>
<td>-0.0002 -0.0007</td>
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Notes: This table presents values of the mean absolute error (MAE) and average delta-hedged gains ($\Sigma$) of hedging strategies $HP_1$, $HP_2$ and $HP_3$ for the SVJ option pricing model of Bates (1996). $HP_1$ constitutes the traditional hedging strategy, which holds $\partial C/\partial S$ stocks and the remaining cash in zero-coupon bonds to hedge a long position in a European call. $HP_2$ is a two-instruments strategy, which holds $\partial C/\partial S$ units of stocks, $\partial C/\partial FV$ units of variance futures and the remaining cash is invested in zero-coupon bonds. $HP_3$ holds, in addition to the two instruments of strategy $HP_2$, a position in bipower variation futures, given by $\phi^{F(2)}_t$, and a position (defined as $\phi^{F(n)}_t$, for $n = 3, 4$) in the third and fourth-order moment futures contracts, respectively. Panel A of the table reports results of the MAE and $\Sigma$ across the total number of $M = 2,000$ iterations, while Panel B reports results only for the iterations generating jumps. The parameter values of the SVJ model used to generate the data are taken as follows: $\kappa = 5.04$, $\theta = 0.04$, $\sigma = 0.52$, $\rho = -0.06$, $\lambda = 4$, $\varpi = -0.03$ and $\sigma_J = 0.04$, and $\kappa_Q = 0.72$, $\theta_Q = 0.31$, $\varpi_Q = -0.07$ and $\sigma_Q = 0.05$ under the risk-neutral measure $Q$. The initial values for the underlying stock price and its volatility are set to $S_0 = 100$ and $\sqrt{v_0} = 0.3$, respectively. The prices of variance futures values are calculated using formula (19), while those of the bipower variation futures and higher-order moment futures by formula (21). These are based on discrete-time (daily) intervals of payoffs. Interest rate and dividend yield are set to $r = 0.05$ and $\delta = 0.015$, respectively, while $\gamma_s$ is set to zero. We assume two different maturity intervals of $\tau = \{30, 120\}$ days. The first considers a short-term option, which is bought.
30 days prior to maturity and it is delta-hedged for one day. The second considers a long-term option, which is bought 120 days prior to maturity and it is also delta-hedged for one day.
Table 3: Estimates of the SVJ model parameters

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<tr>
<th>Estimation periods</th>
<th>$\kappa^Q$</th>
<th>$\pi^Q$</th>
<th>$\sigma_j^Q$</th>
<th>RMSE</th>
<th>RMSPE (in $)</th>
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<td>(0.052)</td>
<td>(2.85 $\times 10^{-7}$)</td>
<td>(1.58 $\times 10^{-5}$)</td>
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<td>0.83</td>
<td>-0.12</td>
<td>0.08</td>
<td>0.04</td>
<td>3.05</td>
</tr>
<tr>
<td></td>
<td>(0.062)</td>
<td>(0.003)</td>
<td>(0.002)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: This table presents average values of the cross-section estimates of the elements of vector $\Theta = (\kappa^Q, \pi^Q, \sigma_j^Q)'$, for all time points (days) of the sample. Standard errors of these values are given in parentheses. RMSE and RMSPE denote the root mean square and the root mean square pricing errors of the put option prices implied by the SVJ model, respectively. The table reports values of vector $\Theta$ for the whole sample period (March 30, 2007 to October 29, 2010), and for the following subsamples: March 30, 2007 to September 3, 2008, and September 4, 2008 to October 29, 2010.
**Table 4: Average estimates of $V_{o,t}$ with their error bounds**

<table>
<thead>
<tr>
<th>Maturity intervals</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>No extrapolation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_{o,t}$ (in $$)</td>
<td>0.0608</td>
<td>0.0591</td>
<td>0.0578</td>
<td>0.0551</td>
<td>0.0535</td>
</tr>
<tr>
<td>Error bounds</td>
<td>0.0004</td>
<td>0.0003</td>
<td>0.0003</td>
<td>0.0004</td>
<td>0.0003</td>
</tr>
<tr>
<td><strong>Constant extrapolation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_{o,t}$ (in $$)</td>
<td>0.0803</td>
<td>0.0791</td>
<td>0.0781</td>
<td>0.0758</td>
<td>0.0748</td>
</tr>
<tr>
<td>Error bounds</td>
<td>0.0242</td>
<td>0.0377</td>
<td>0.0353</td>
<td>0.0458</td>
<td>0.0473</td>
</tr>
</tbody>
</table>

**Notes:** The table presents average values of the estimates of $V_{o,t}$ over the whole sample and their approximation error bounds. The sample estimates of $V_{o,t}$ are calculated on weekly basis (every Wednesday). This is done for 5 different maturity intervals, i.e., for $\tau = 1, 2, 3, 6, 9$ months. The table presents two sets of results. The first assumes no extrapolation while the second it assumes constant extrapolation.
Table 5: Descriptive Statistics of $\widetilde{FV}_t - \widetilde{V}_{o,t}$

<table>
<thead>
<tr>
<th>Maturity intervals</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole sample 30/3/2007 - 29/10/2010</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.0258</td>
<td>0.0273</td>
<td>0.0288</td>
<td>0.0303</td>
<td>0.0311</td>
</tr>
<tr>
<td>$t$-stat</td>
<td>(13.11)</td>
<td>(14.16)</td>
<td>(15.24)</td>
<td>(17.82)</td>
<td>(19.86)</td>
</tr>
<tr>
<td>Mean</td>
<td>0.011</td>
<td>0.0121</td>
<td>0.0129</td>
<td>0.0144</td>
<td>0.0153</td>
</tr>
<tr>
<td>$t$-stat</td>
<td>(20.27)</td>
<td>(22.07)</td>
<td>(22.72)</td>
<td>(23.23)</td>
<td>(24.46)</td>
</tr>
<tr>
<td>Subsample 4/9/2008 - 29/10/2010</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.0356</td>
<td>0.0373</td>
<td>0.0393</td>
<td>0.0409</td>
<td>0.0416</td>
</tr>
</tbody>
</table>

Notes: The table presents average values of difference $\widetilde{FV}_t - \widetilde{V}_{o,t}$. The latter are based on weekly (every Wednesday) estimates of $V_{o,t}$. These are obtained based on the interpolation-extrapolation scheme that chooses not to extrapolate the implied volatility function. In parentheses, we present values of the test statistic (denoted $t$-stat) if these values are different than zero.
Table 6: Regression analysis of $LNFVO_t$

<table>
<thead>
<tr>
<th>Maturity intervals</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A:</strong> Regression model $LNFVO_t = \beta_0 + \beta_1 LNJT_t + \varepsilon_t$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>LS estimates</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>0.40</td>
<td>0.18</td>
<td>-0.14</td>
<td>-0.50</td>
<td>-0.80</td>
</tr>
<tr>
<td></td>
<td>(0.08)</td>
<td>(0.09)</td>
<td>(0.10)</td>
<td>(0.10)</td>
<td>(0.11)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.69</td>
<td>0.63</td>
<td>0.56</td>
<td>0.48</td>
<td>0.42</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.02)</td>
<td>(0.02)</td>
<td>(0.02)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.94</td>
<td>0.93</td>
<td>0.90</td>
<td>0.87</td>
<td>0.83</td>
</tr>
<tr>
<td></td>
<td>FMLS estimates</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>0.37</td>
<td>0.13</td>
<td>-0.28</td>
<td>-0.68</td>
<td>-1.04</td>
</tr>
<tr>
<td></td>
<td>(0.11)</td>
<td>(0.18)</td>
<td>(0.18)</td>
<td>(0.20)</td>
<td>(0.22)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.68</td>
<td>0.63</td>
<td>0.54</td>
<td>0.45</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.03)</td>
<td>(0.03)</td>
<td>(0.03)</td>
<td>(0.04)</td>
</tr>
<tr>
<td>$Z_t$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.02</td>
<td>0.08</td>
</tr>
<tr>
<td>$Z_\alpha$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.02</td>
<td>0.07</td>
</tr>
<tr>
<td><strong>Panel B:</strong> Regression model $LNFVO_t = \beta_0 + \beta_1 LN\mu_{3,t} + \varepsilon_t$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>LS estimates</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>2.24</td>
<td>1.22</td>
<td>0.35</td>
<td>-0.74</td>
<td>-1.18</td>
</tr>
<tr>
<td></td>
<td>(0.15)</td>
<td>(0.12)</td>
<td>(0.11)</td>
<td>(0.10)</td>
<td>(0.12)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.68</td>
<td>0.64</td>
<td>0.56</td>
<td>0.45</td>
<td>0.41</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.01)</td>
<td>(0.01)</td>
<td>(0.02)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.90</td>
<td>0.91</td>
<td>0.89</td>
<td>0.82</td>
<td>0.70</td>
</tr>
<tr>
<td></td>
<td>FMLS estimates</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>2.29</td>
<td>1.30</td>
<td>0.45</td>
<td>-0.17</td>
<td>-0.78</td>
</tr>
<tr>
<td></td>
<td>(0.18)</td>
<td>(0.21)</td>
<td>(0.23)</td>
<td>(0.30)</td>
<td>(0.58)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.69</td>
<td>0.65</td>
<td>0.57</td>
<td>0.55</td>
<td>0.49</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.03)</td>
<td>(0.03)</td>
<td>(0.05)</td>
<td>(0.09)</td>
</tr>
<tr>
<td>$Z_t$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$Z_\alpha$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>
Notes: The table presents LS and FMLS estimates of the regression coefficients of $LN FVO_t$ (log-transform of $FV_t - \tilde{V}_{0,t}$) on $LN JT_t$ (log-transform of $JT_t$) (see Panel A), and of $LN FVO_t$ on $LN \mu_{3,t}$ (log-transform of $-\mu_{3,t}$) (see Panel B). The estimation period of the first regression is from 4/9/2008 to 29/10/2010, while of the second is the whole sample. Standard errors are in parentheses. $\bar{R}^2$ is the coefficient of determination of the LS regression. $Z_t$ and $Z_a$ are Phillips-Ouliaris’ test statistics for cointegration. Their $p$-values are reported in the table.
Table 7A: Evaluation of alternative hedging strategies (30/3/2007 - 29/10/2010)

<table>
<thead>
<tr>
<th>K/Si</th>
<th>RMSE (in $)</th>
<th>MAE (in $)</th>
<th>( \sum ) (in $)</th>
<th>J-stat (p-values)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>15-35</td>
<td>40-60</td>
<td>100-160</td>
<td>15-35</td>
</tr>
<tr>
<td>0.8</td>
<td>(put)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>HP1</td>
<td>1.02</td>
<td>1.29</td>
<td>1.78</td>
</tr>
<tr>
<td></td>
<td>HP2</td>
<td>0.98</td>
<td>1.11</td>
<td>1.18</td>
</tr>
<tr>
<td></td>
<td>HP3</td>
<td>0.98</td>
<td>1.08</td>
<td>1.17</td>
</tr>
<tr>
<td>0.9</td>
<td>(put)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>HP1</td>
<td>1.95</td>
<td>2.14</td>
<td>2.37</td>
</tr>
<tr>
<td></td>
<td>HP2</td>
<td>1.83</td>
<td>1.80</td>
<td>1.56</td>
</tr>
<tr>
<td></td>
<td>HP3</td>
<td>1.73</td>
<td>1.71</td>
<td>1.54</td>
</tr>
<tr>
<td>1</td>
<td>(call)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>HP1</td>
<td>3.72</td>
<td>3.60</td>
<td>3.40</td>
</tr>
<tr>
<td></td>
<td>HP2</td>
<td>3.44</td>
<td>3.05</td>
<td>2.46</td>
</tr>
<tr>
<td></td>
<td>HP3</td>
<td>2.95</td>
<td>2.77</td>
<td>2.35</td>
</tr>
<tr>
<td>1.1</td>
<td>(call)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>HP1</td>
<td>1.99</td>
<td>2.85</td>
<td>3.49</td>
</tr>
<tr>
<td></td>
<td>HP2</td>
<td>1.80</td>
<td>2.35</td>
<td>2.51</td>
</tr>
<tr>
<td></td>
<td>HP3</td>
<td>1.99</td>
<td>2.22</td>
<td>2.42</td>
</tr>
<tr>
<td>1.2</td>
<td>(call)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>HP1</td>
<td>1.05</td>
<td>1.93</td>
<td>3.18</td>
</tr>
<tr>
<td></td>
<td>HP2</td>
<td>0.96</td>
<td>1.64</td>
<td>2.37</td>
</tr>
<tr>
<td></td>
<td>HP3</td>
<td>1.15</td>
<td>1.75</td>
<td>2.34</td>
</tr>
</tbody>
</table>

Notes: The table presents values (in US dollars) of the RMSE and MAE metrics, as well as average over the whole sample values of delta-hedged gains (denoted \( \sum \)) of the following hedging strategies: \( HP_1 \), \( HP_2 \) and \( HP_3 \). \( HP_1 \) considers only a short position in the underlying stock. \( HP_2 \) is a two-instruments strategy, which considers short positions in the underlying stock and a variance futures contract, and \( HP_3 \), in addition to these two
instruments, includes the bipower and higher-order moment futures in the hedging portfolio. The number (deltas) of the above hedging instruments are based on the daily estimates of the SVJ model, presented in Section 5.2. We consider a daily rebalancing of the hedging portfolios, calculating hedging errors at each time-point (day) of the sample. The $p$-values reported in the table are the probabilities of error type I of statistic $J$-stat. This test the null hypothesis that there are no hedging benefits between a pair of hedging strategies $HP_k$ and $HP_l$, for $k \neq l$. It is distributed as $\chi^2$ with one degree of freedom.
<table>
<thead>
<tr>
<th>(K/S_i)</th>
<th>(15-35)</th>
<th>(40-60)</th>
<th>(100-160)</th>
<th>(15-35)</th>
<th>(40-60)</th>
<th>(100-160)</th>
<th>(15-35)</th>
<th>(40-60)</th>
<th>(100-160)</th>
<th>(15-35)</th>
<th>(40-60)</th>
<th>(100-160)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8 (put)</td>
<td>(HP_1)</td>
<td>0.46</td>
<td>0.90</td>
<td>1.58</td>
<td>0.24</td>
<td>0.59</td>
<td>1.11</td>
<td>-0.15</td>
<td>-0.21</td>
<td>0.02</td>
<td>(HP_1-HP_2)</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>(HP_2)</td>
<td>0.44</td>
<td>0.81</td>
<td>1.13</td>
<td>0.23</td>
<td>0.53</td>
<td>0.79</td>
<td>-0.14</td>
<td>-0.19</td>
<td>0.06</td>
<td>(HP_2-HP_3)</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>(HP_3)</td>
<td>0.43</td>
<td>0.80</td>
<td>1.12</td>
<td>0.23</td>
<td>0.52</td>
<td>0.78</td>
<td>-0.17</td>
<td>-0.25</td>
<td>0.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.9 (put)</td>
<td>(HP_1)</td>
<td>1.38</td>
<td>1.84</td>
<td>2.26</td>
<td>0.86</td>
<td>1.28</td>
<td>1.62</td>
<td>-0.57</td>
<td>-0.35</td>
<td>0.08</td>
<td>(HP_1-HP_2)</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>(HP_2)</td>
<td>1.30</td>
<td>1.60</td>
<td>1.60</td>
<td>0.80</td>
<td>1.10</td>
<td>1.13</td>
<td>-0.51</td>
<td>-0.32</td>
<td>0.09</td>
<td>(HP_2-HP_3)</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>(HP_3)</td>
<td>1.23</td>
<td>1.54</td>
<td>1.58</td>
<td>0.76</td>
<td>1.08</td>
<td>1.12</td>
<td>-0.76</td>
<td>-0.57</td>
<td>-0.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 (call)</td>
<td>(HP_1)</td>
<td>2.99</td>
<td>3.03</td>
<td>3.06</td>
<td>2.21</td>
<td>2.25</td>
<td>2.24</td>
<td>-0.54</td>
<td>-0.32</td>
<td>0.10</td>
<td>(HP_1-HP_2)</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>(HP_2)</td>
<td>2.78</td>
<td>2.63</td>
<td>2.33</td>
<td>2.07</td>
<td>1.97</td>
<td>1.70</td>
<td>-0.37</td>
<td>-0.31</td>
<td>0.08</td>
<td>(HP_2-HP_3)</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
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<td>2.27</td>
<td>1.88</td>
<td>1.89</td>
<td>1.67</td>
<td>-1.51</td>
<td>-0.93</td>
<td>-0.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1 (call)</td>
<td>(HP_1)</td>
<td>1.30</td>
<td>1.91</td>
<td>2.62</td>
<td>0.76</td>
<td>1.28</td>
<td>1.83</td>
<td>0.07</td>
<td>-0.04</td>
<td>0.06</td>
<td>(HP_1-HP_2)</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>(HP_2)</td>
<td>1.21</td>
<td>1.61</td>
<td>1.85</td>
<td>0.71</td>
<td>1.07</td>
<td>1.27</td>
<td>0.12</td>
<td>-0.03</td>
<td>0.04</td>
<td>(HP_2-HP_3)</td>
<td>0.54</td>
</tr>
<tr>
<td></td>
<td>(HP_3)</td>
<td>1.23</td>
<td>1.57</td>
<td>1.79</td>
<td>0.70</td>
<td>1.03</td>
<td>1.24</td>
<td>-0.32</td>
<td>-0.60</td>
<td>-0.28</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.2 (call)</td>
<td>(HP_1)</td>
<td>0.52</td>
<td>0.86</td>
<td>1.79</td>
<td>0.20</td>
<td>0.48</td>
<td>1.14</td>
<td>-0.01</td>
<td>0.13</td>
<td>0.01</td>
<td>(HP_1-HP_2)</td>
<td>0.02</td>
</tr>
<tr>
<td></td>
<td>(HP_2)</td>
<td>0.50</td>
<td>0.71</td>
<td>1.21</td>
<td>0.19</td>
<td>0.41</td>
<td>0.73</td>
<td>0.01</td>
<td>0.11</td>
<td>-0.01</td>
<td>(HP_2-HP_3)</td>
<td>0.54</td>
</tr>
<tr>
<td></td>
<td>(HP_3)</td>
<td>0.48</td>
<td>0.71</td>
<td>1.22</td>
<td>0.19</td>
<td>0.40</td>
<td>0.72</td>
<td>-0.10</td>
<td>-0.16</td>
<td>-0.30</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: See Table 7A. The sample interval is 30/3/2007 - 3/9/2008.
Table 7C: Evaluation of alternative hedging strategies (4/9/2008-29/10/2010)

<table>
<thead>
<tr>
<th>K/S_i</th>
<th>RMSE (in $)</th>
<th>MAE (in $)</th>
<th>Σ (in $)</th>
<th>J-stat (p-values)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>15-35</td>
<td>40-60</td>
<td>100-160</td>
<td>15-35</td>
</tr>
<tr>
<td>0.8 (put)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HP_1</td>
<td>1.26</td>
<td>1.49</td>
<td>1.90</td>
<td>0.56</td>
</tr>
<tr>
<td>HP_2</td>
<td>1.20</td>
<td>1.27</td>
<td>1.20</td>
<td>0.54</td>
</tr>
<tr>
<td>HP_3</td>
<td>1.21</td>
<td>1.22</td>
<td>1.19</td>
<td>0.50</td>
</tr>
<tr>
<td>0.9 (put)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HP_1</td>
<td>2.24</td>
<td>2.31</td>
<td>2.44</td>
<td>1.21</td>
</tr>
<tr>
<td>HP_2</td>
<td>2.10</td>
<td>1.91</td>
<td>1.53</td>
<td>1.12</td>
</tr>
<tr>
<td>HP_3</td>
<td>1.99</td>
<td>1.81</td>
<td>1.52</td>
<td>1.01</td>
</tr>
<tr>
<td>1 (call)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HP_1</td>
<td>4.12</td>
<td>3.92</td>
<td>3.60</td>
<td>2.46</td>
</tr>
<tr>
<td>HP_2</td>
<td>3.80</td>
<td>3.29</td>
<td>2.54</td>
<td>2.24</td>
</tr>
<tr>
<td>HP_3</td>
<td>3.24</td>
<td>2.95</td>
<td>2.40</td>
<td>2.18</td>
</tr>
<tr>
<td>1.1 (call)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HP_1</td>
<td>2.32</td>
<td>3.31</td>
<td>3.95</td>
<td>1.00</td>
</tr>
<tr>
<td>HP_2</td>
<td>2.09</td>
<td>2.72</td>
<td>2.86</td>
<td>0.89</td>
</tr>
<tr>
<td>HP_3</td>
<td>2.34</td>
<td>2.55</td>
<td>2.74</td>
<td>1.20</td>
</tr>
<tr>
<td>1.2 (call)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HP_1</td>
<td>1.28</td>
<td>2.38</td>
<td>3.81</td>
<td>0.37</td>
</tr>
<tr>
<td>HP_2</td>
<td>1.16</td>
<td>2.03</td>
<td>2.87</td>
<td>0.34</td>
</tr>
<tr>
<td>HP_3</td>
<td>1.42</td>
<td>2.16</td>
<td>2.84</td>
<td>0.41</td>
</tr>
</tbody>
</table>

Notes: See Table 7A. The sample period is 4/9/2008 - 29/10/2010.
Table 8: Delta-hedged gains, volatility and jump risk premia

<table>
<thead>
<tr>
<th>Maturity</th>
<th>$\Delta \hat{\Sigma}^{HP_1}_t / S_t$</th>
<th>$\Delta \hat{\Sigma}^{HP_2}_t / S_t$</th>
<th>$\Delta \hat{\Sigma}^{HP_3}_t / S_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15-35</td>
<td>$\beta_0$</td>
<td>$\beta_1$</td>
<td>$\beta_2$</td>
</tr>
<tr>
<td></td>
<td>$-0.0001$</td>
<td>$-0.026$</td>
<td>$-0.71$</td>
</tr>
<tr>
<td></td>
<td>$(0.0002)$</td>
<td>$(0.008)$</td>
<td>$(0.12)$</td>
</tr>
<tr>
<td>40-60</td>
<td>$5.6 \times 10^{-5}$</td>
<td>$-0.029$</td>
<td>$-0.75$</td>
</tr>
<tr>
<td></td>
<td>$(0.0005)$</td>
<td>$(0.01)$</td>
<td>$(0.14)$</td>
</tr>
<tr>
<td>100-160</td>
<td>$-2.2 \times 10^{-5}$</td>
<td>$-0.015$</td>
<td>$-0.8$</td>
</tr>
<tr>
<td></td>
<td>$(0.0003)$</td>
<td>$(0.007)$</td>
<td>$(0.17)$</td>
</tr>
<tr>
<td></td>
<td>$-3 \times 10^{-5}$</td>
<td>$0.0001$</td>
<td>$-0.57$</td>
</tr>
<tr>
<td></td>
<td>$(0.0001)$</td>
<td>$(0.008)$</td>
<td>$(0.13)$</td>
</tr>
<tr>
<td></td>
<td>$-9.6 \times 10^{-5}$</td>
<td>$0.002$</td>
<td>$-0.71$</td>
</tr>
<tr>
<td></td>
<td>$(0.0001)$</td>
<td>$(0.006)$</td>
<td>$(0.15)$</td>
</tr>
</tbody>
</table>

Notes: The table presents GMM estimates of regression models (38), for $j = 1, 2, 3$, based on ATM options. This is done for the following maturity intervals: 15-35, 40-60 and 100-160 days, using the whole sample period 30/3/2007 - 29/10/2010. Newey-West standard errors correcting for heteroscedasticity and serial correlation are in parentheses. The instruments used in the GMM estimation procedure are: the constant, and two-periods back lagged values of the dependent variable and the two independent variables.
Figure 1: This figure presents values of $\phi^{(2)}$, $\phi^{(3)}$ and $\phi^{(4)}$ with respect to strike price $K$, for maturity intervals of 1 and 6 months. The vertical line in the middle of graphs indicates current stock price.
Figure 2: This figure presents values of the mean absolute error (MAE) for strategy $HP_3$ across the number of higher-order moment futures $N$ used in the hedging portfolio.
Figure 3: This figure presents estimates of $\pi^Q$ (top graph) and $\sigma_J^Q$ (bottom graph) from March 3, 2007 to October 29, 2010. The vertical line corresponds to September 4, 2008.
Figure 4: Weekly estimates of $\bar{V}_t - \tilde{V}_{o,t}$ (solid line) and $JT_t$ (dashed line), for $\tau = 1$ month.