Optimal dynamic pricing with inventories

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Abstract

We characterize optimal dynamic pricing by a monopolist with a finite stock of goods to sell. We show that the monopolist finds it optimal to charge a price that increases monotonically as his stock depletes. © 2001 Elsevier Science B.V. All rights reserved.

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JEL classification: D4; D8

1. Introduction

Consider a seller who has a finite number of units of a good to sell. Such trading environments are abundant in everyday life. Examples include firms selling their inventories while going out of business, stores selling their stock of a discontinued product line (or one for which it would take a long time to produce additional units), individuals or estates selling their inheritance, or just a breeder of a rare species of an animal selling a new-born litter. In each of these cases, the seller has to decide the price at which to sell its stock, as the latter evolves as a result of past trades. What is often observed in such situations is a tendency for the posted price to increase over time. Reasons for this pattern may include the seller updating favorably his prior about the demand for his product, the consumers updating favorably their opinion about the product quality, or decreased competition. In this note we focus on the effect of the seller’s inventory on his pricing strategy.

We use a simple model to study the price dynamics that results from optimal pricing behavior on the part of a seller. We show that as the inventory depletes as a result of past sales, the seller raises the...

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posted price for the good even though all units of the good and (expected) demand in every period are identical. In other words, the seller chooses a strictly increasing price profile. The intuition is as follows. With discounting of inter-temporal profits, and an infinite sequence of buyers, the seller faces a trade-off between posting a low price and thus raising the probability of a sale today, and posting a high price which yields high current profits, conditional on a sale taking place, at the cost of a reduced probability of sale today (but with the option to sell tomorrow). With a large stock left to be sold, the option value to the seller of holding a unit of the good for sale in the future is relatively low because of the discounting of the entire stream of future profits. This makes it more attractive for the seller to sell the unit today, even if at a relatively lower price, in order to realize the stream of future profits sooner. On the other hand, when the stock left to be sold is relatively small, the seller incurs less of a cost by way of delayed future profits as a result of being unable to sell a unit today. As a result, his optimal strategy consists of posting a relatively high price today, even though that raises the probability of being unable to sell today.

After we derive and characterize the optimal solution, we present some numerical calculations to illustrate the price dynamics. In accordance with the intuition underlying our result, holding the discount factor constant, the price of any given unit is found to be decreasing in the size of the total stock. Moreover, we find that for a fixed starting stock, the price of any given unit is increasing in the discount factor.

2. The model: payoffs and information

Consider a seller (monopolist) who has \( k \) units of a (non-perishable) good. The seller’s opportunity cost for each unit is \( c = 0 \). Time is discrete and is indexed by \( t = 0, 1, 2, \ldots \) in an infinite horizon. The seller encounters one new buyer every period. The buyer who arrives in period \( t \) desires one unit of the good and has value \( v_t \) for that one unit. Each \( v_t \) is an independent and random draw from the probability distribution function \( F \), with support \([0, \bar{v}]\) and corresponding density function \( f \). Assume that \( F \) is strictly increasing everywhere on its support.

In each period \( t \), the seller posts a price \( p_t \) for each unit of the good in stock. The seller does not observe \( v_t \) before posting the price. The arriving buyer decides whether or not to buy one unit of the good at the posted price. If the buyer arriving in period \( t \) chooses to buy, the seller receives an instantaneous payoff of \( p_t \) and the buyer receives a payoff of \( (v_t - p_t) \). Otherwise, each receives an instantaneous payoff of zero. The seller seeks to maximize the present discounted value of expected future profits using a discount factor \( \delta \in (0,1) \).

\footnote{To keep things simple, the valuations of buyers is assumed independent of their arrival date.}

\footnote{This result may be surprising in view of the documentation of the ‘declining price (anomaly)’ in the empirical auctions literature. For example, Ashenfelter (1989) and Beggs and Grady (1997) have demonstrated empirically that prices of identical objects may decline when auctioned sequentially. On the other hand, our result accords well with other observations including the commonly observed practice in which department stores advertise ‘early bird specials’ in which prices are discounted even below sale price levels during the first few hours of a sale event.}
3. The optimal price path

Clearly, since buyers do not get a second chance to buy, the buyer in period $t$ will follow the simple strategy of buying at the posted price if and only if $v \geq p_t$. Let us therefore focus on the seller’s strategy. The seller’s objective function then becomes $\sum_{t=1}^{\infty} \delta^t p_t \{1 - F(p_t)\}$ where $1 - F(p_t)$ is the probability of a sale in period $t$. To start with, denote by $(x - 1)$ the number of units of the good that have been sold up until the period in question. The seller chooses $p_t$ as a map from the history of the game up until the end of period $t$ into a number in $\mathbb{R}_+$. Given the independence in buyer valuations across periods and the instantaneous realization of payoffs every period, it easily follows that $p_t$ depends only on the value of $x$ (and, in particular, not on the sequence of prices and buyer decisions that constitute the history of the game). In other words, the seller does not change the posted price in any period $t$ unless a sale occurred in period $t - 1$.

Let us define some more notation that would allow us to set up the dynamic pricing decision that faces our seller. We can restrict attention to the sequence of $k$ prices $\{p_i\}_{i=1,\ldots,k}$ by way of the seller’s strategy. Define $V(x)$ to be the optimal continuation payoff to the seller when he has already sold $(x - 1)$ units of the good (i.e., the next unit to be sold is the $x$th unit). We can then write the seller’s problem in state $x$ as

$$V(x) = \max_{p_x} \{1 - F(p) \} [p + \delta V(x + 1)] + \delta F(p) V(x).$$

(1)

The first-order condition for the seller’s problem is

$$1 - F(p_x) - f(p_x) [p_x + \delta V(x + 1)] + \delta f(p_x) V(x) = 0$$

(2)

which can be simplified to yield

$$p_x = \frac{1 - F(p_x)}{f(p_x)} + \delta [V(x) - V(x + 1)].$$

(3)

Also, if $p_x$ is indeed the optimal price in state $x$, then from (1), it must be that

$$V(x) = \{1 - F(p_x) \} [p_x + \delta V(x + 1)] + \delta F(p_x) V(x)$$

(4)

which simplifies to

$$V(x) = \{1 - F(p_x) \} [p_x + \delta V(x + 1)].$$

(5)

Equations (3) and (5) give us a two-equation system relating $p_x$ (the optimal decision rule) with $V(x)$ and $V(x + 1)$ (the value functions). But we also know that

$$V(k + 1) = 0$$

(6)

Note that if the seller does not discount the future ($\delta = 1$), he finds it optimal to charge the highest price ($v$) at which a sale could possibly take place.
since in state \( k \) there are no further units in stock. Therefore, starting with \( x = k \), we can use (6), (5) and (3) to solve backwards for the price sequence \( \{p_x\}_{x=k,k-1,...,1} \).

Note that it is straightforward to show that the optimal price sequence is unique. To do so formally, we can define the transformation \( G \) as

\[
G = \left( \frac{1 - F(p_x)}{f(p_x)} + \delta \frac{1 - F(p_x)}{1 - \delta F(p_x)} \right) \{p_x + \delta V(x + 1) - V(x + 1)\}. \tag{7}
\]

Simplifying and rearranging, we get

\[
\delta V(x + 1) = A(p_x) - p_x, \tag{8}
\]

where we define

\[
A(p_x) = \frac{(1 - F(p_x))(1 - \delta F(p_x))}{(1 - \delta)f(p_x)}. \tag{9}
\]

Using Eq. (8) and an analogous expression for \( \delta V(x) \) to substitute the term within square brackets in (3), we get

\[
 p_x = \frac{1 - F(p_x)}{f(p_x)} + [(p_x - A(p_x)) - (p_{x-1} - A(p_{x-1}))] \tag{10}
\]

which can be rearranged to yield

\[
p_{x-1} - A(p_{x-1}) = - [p_x - \frac{1 - F(p_x)}{f(p_x)}] + p_x - A(p_x). \tag{11}
\]

Equation (11) allows the following characterization of the price path:

**Proposition 1.** If \( F(.) \) has a non-decreasing hazard rate, then \( \{p_x\}_{x=1,...,k} \) is an increasing sequence of prices.

**Proof.** First, consider the expression \( p_x - A(p_x) \). If \( F(.) \) has a non-decreasing hazard rate, then \( A(p_x) \) is strictly decreasing (since \( F(.) \) is assumed to be strictly increasing everywhere on its support — see Eq. (9)). Therefore, \( p_x - A(p_x) \) is strictly increasing in \( p_x \).

Now, \( V(x) - V(x + 1) > 0 \). To see this observe that at state \( x \), the seller can always post the price sequence \( \{p_{x+1}\} \) till he reaches the state where only one unit remains to be sold. But since future profits are discounted at a rate strictly greater than zero, the resulting discounted expected profits must be greater than \( V(x + 1) \). Then (3) implies that

\[\text{A formal proof of this claim is available from the authors upon request.}\]
Equation (11) then implies that
\[ p_{x-1} - A(p_{x-1}) \leq p_x - A(p_x) \]  
(13)

But we have already established that \( p_x - A(p_x) \) is strictly increasing in \( p_x \). This proves the Proposition. \( \square \)

The assumption of non-decreasing hazard rate of the distribution of valuations is quite common in the information economics literature. Examples of distributions with a non-decreasing hazard rate include, among others, the uniform, exponential, normal and Laplace distributions, and any distribution with an increasing density function.

The intuition for this result, as alluded to in Section 1, is straightforward. When a large number of units remain to be sold, it is relatively costly to delay the sale of each unit due to the discounting of future profits. In other words, the option value of waiting to sell the early units is low. As fewer and fewer units remain to be sold, the option value of waiting increases, resulting in a higher posted price by the seller.

4. Examples

We now illustrate our solution by presenting explicit calculations for the cases of the uniform and exponential distributions. Let us first consider the case of uniformly distributed buyer valuations, in other words \( F \) is uniform on \([0, \bar{v}]\). The numerical results are provided in Table 1. In addition to prices increasing with the number of units sold, they also illustrate that the prices are increasing in the discount factor.

<table>
<thead>
<tr>
<th>( \delta = 0.8, \bar{v} = 100, k = 10 )</th>
<th>( \delta = 0.9, \bar{v} = 100, k = 10 )</th>
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<tr>
<td>( x )</td>
<td>Price</td>
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<td>9</td>
<td>61.125</td>
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<tr>
<td>10</td>
<td>69.098</td>
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</tbody>
</table>
Consider now the case where the distribution function is exponential: $f(x) = \lambda e^{-\lambda x}$, so that $F(x) = 1 - e^{-\lambda x}$ and the hazard rate is constant, $f(x)/[1 - F(x)] = \lambda$. Then,

$$A(p) = \frac{1 - \delta(1 - e^{-\lambda p})}{(1 - \delta)\lambda}$$

(14)

and it is straightforward to show (by induction) that the optimal price path satisfies

$$p_{k-n} + p_{k-n+1} + \ldots + p_k = \frac{n}{\lambda} + A(p_{k-n}), \text{ for } n = 0, 1, \ldots, k - 1.$$  

(15)

This expression can be used to calculate the optimal path recursively, starting from $n = 0$ and proceeding towards $n = k - 1$. Table 2 provides some numerical calculations to illustrate the solution.

### 5. Conclusion

In this note we have used a simple model to capture a trading situation that is quite common in everyday economic transactions, but to the best of our knowledge, has not been formally modelled. Our analysis provides some intuition for the optimal price dynamics of a seller with a finite inventory. Regarding extensions of this work, the introduction of competition in this situation appears of interest since the tendency of prices to increase as inventories deplete will interact with the strategic behavior among sellers.

### References

