Dynamic Pricing of Differentiated Products

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Abstract

We examine the dynamic pricing decision of a firm facing random demand while selling a fixed stock of two differentiated products over an infinite horizon. Prices in each period depend on the available stock of both products (varieties). In addition to the standard trade-off between a higher revenue and the probability of selling the product, a higher price for one product also affects the probability of a sale for the other product. We characterize the optimal price paths of the two varieties and find that the market may be optimally ‘covered’ or ‘not covered’. With a positive stock of only one variety, the price path is nondecreasing. The same holds for the price paths with two varieties, with respect to the own stock level and as long as the probability of no sale is positive. When this probability equals one, the price of a given variety is decreasing in its own stock and increasing in the stock of the other variety. The optimal paths reflect the property that the stream of expected future profits is higher when the inventory levels of the two products tend to be closer to each other.

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1 Introduction

How should a seller price different varieties of a product when he has different inventories of each? This problem appears important for many firms: situations where firms are multi-product, selling different varieties of a product, with these varieties viewed by consumers as imperfect substitutes to one another, appear to be the norm in markets, rather than an exception. At the same time, the ability of firms to sell each particular variety is restricted by the current inventory (stock) that each firm holds. How available inventories evolve over time depends on the sales that take place, which in turn depend at least partly on the prices set for each variety. Importantly, since the varieties are viewed as substitutes by the consumers, the prices of all varieties influence the evolution of the available stock of each variety. Decreasing, say, the price of one variety while keeping constant the price of another will tend to affect the sales of the second variety (and not only of the first one) and, thus, also affect the prices the seller would like to set in the future for both varieties. Motivated by the above-described problems, in this paper, we set up a simple dynamic model to study optimal dynamic pricing of differentiated products when there are capacity constraints (or inventories) for each product. Specifically, our objective is to examine how the available stock of two products, differentiated horizontally along a single dimension, and its reduction over time influence the optimal intertemporal price paths of these products.

The analysis of the behaviour of multiproduct firms is both of theoretical interest and of practical importance. There are several ways in which the decisions of a seller who offers more than one product about one particular product may influence the profit from selling the rest of the products. In general, such an interdependency may arise either through demand or through costs.\footnote{For a discussion of the basic issues related to the multiproduct firm see e.g. Tirole (1986) and the references therein.} In the presence of demand dependencies, the posted price for one of the products affects the demand for the rest.\footnote{In the marketing literature, the negative effect of one product in the product line on the rest is known as ‘cannibalization’.} With cost dependencies, the issue of economies of scope arises. These two general considerations may be viewed either in a static or in a dynamic setting. In the present paper we focus on a distinct mechanism that creates an important intertemporal link between the pricing decisions of a multiproduct monopolist, that of inventory reduction.

To clarify the focus of our analysis, consider the following simple example. An automobile retailer has a given number of cars in stock. All the cars are the same model of a certain brand and differ only with respect to their color. Suppose that there are only two varieties, blue and red cars. Every time that a consumer enters the store, the retailer announces the price of the
car, possibly depending on its color, not knowing the customer’s preferences about the color of the car. A number of questions arise in this set-up. First, how do optimal prices depend on the total stock of cars? When would the seller serve a buyer irrespective of his preferences and when would he let a buyer with certain preferences leave empty-handed? Second, how do optimal prices depend on the relative availability of the two varieties? In other words, would the firm set a higher or a lower price for the variety of which it has less stock? Third, independent of the relative stock of red and blue cars, would the seller increase, decrease or leave unchanged the next-period price of the variety he has just sold? How would the price of the other variety change? These questions have remained relatively unexplored in the literature, although, casual evidence suggests that they are relevant in practice.\(^3\)

While the above example refers to differentiation in the product characteristics space, the model applies to spatial differentiation too. Consider, for example, a firm that sells the same product at two stores located opposite to each other in the outskirts of some city. The nature of the product is such that it is very difficult (costly) to transfer stock between the two locations.\(^4\) In each period a consumer living in the city decides which store to purchase from, depending on the prices posted and the transportation cost he incurs and the goal of the firm is to maximize its total revenue.

In an attempt to offer answers to the above questions, we examine the joint effect of three factors on the dynamic pricing decisions of the firm: nonreplenishable stock (or capacity constraints), product differentiation and demand uncertainty. We use as a base of our model a horizontal-differentiation framework, roughly along the lines of Hotelling. A monopoly firm holds fixed stock of two distinct varieties of some (non-perishable) product. Consumers have unit demands and arrive sequentially, one in each period, with preferences that are independently and identically distributed and are represented by their location on the Hotelling line. At the beginning of the period the firm sets two prices, one for each variety, not knowing the consumers’ preferences over varieties. The objective of the firm is to maximize the discounted value of expected future profits over an infinite horizon. The role of prices is threefold and, as a result, the monopolist faces a trade-off between posting low and high prices. Firstly, the two prices taken together determine the probability of a sale within a given period. The lower the

\(^3\) For instance, in a recent article about the car industry in China, it is mentioned that “...oddly, the finished cars then sit in parking lots for up to 90 days before they are sold, usually at a discount because they are not the colour or do not have the optional extras that the buyer wants.” (“Ripe for revolution”, The Economist, September 4th 2004). Also, Bitran and Caldentey (2003) discuss the possibility of a firm price discriminating when customers’ preferences differ and state that “Retailers, on the other hand, are much more active in this way, charging different prices for a blue shirt and a red shirt (same model, brand and size)”.\(^4\) Otherwise only total capacity would matter and not its initial allocation or subsequent fluctuations across the two locations.
prices are, the higher the probability the customer will buy one of the two varieties. Secondly, conditional on a sale taking place, the variety with the lower price has a higher probability to be sold. Finally, the higher is the price of the product sold, the higher is the per-period revenue obtained. As we will see, it is important in that setting that the optimal price of each variety depends not only on its available stock but also on the stock of the other variety.

Within this set-up we characterize the optimal price paths. Our main results are as follows. In a given period, two possible states of the market arise in equilibrium: either it is ‘covered’ so that there is a purchase with probability one, or it is ‘not covered’, so that the probability that the buyer leaves empty-handed is positive. Depending on the initial stock of the two varieties and on the parameters’ values, the market may be covered or not covered in all periods, or it may be covered initially and not covered in later periods (when less units are left in stock). We find that when the market is covered, the price of the variety which has been sold in the immediately previous period increases, while that of the other variety decreases. We can show this parametrically, for small and symmetric initial stocks, and numerically, for a larger range of initial stock levels. We also prove that expected profits increase when total capacity is distributed more evenly between the two varieties. When the market is not covered, the price of the variety sold in the previous period increases, while the price of the other variety remains unchanged.

When the stock of one of the varieties has been sold out, the price path of the other variety is nondecreasing over time. More specifically, as the available stock of the product with positive stock decreases, its price increases, if the market is not covered, and remains constant, as long as the monopolist wants to ensure that a sale is made with probability one.

Our paper contributes to the multi-product firm literature by proposing a dynamic model of pricing differentiated products under capacity constraints. Regarding product differentiation, an important literature has characterized the optimal pricing and product line decisions of a multi-product firm, as well as some issues that emerge in this setting, like the possibility of entry deterrence. This literature includes Mussa and Rosen (1978), Schmalensee (1978), Eaton and Lipsey (1979) and some important work following on such contributions. However, the behavior of a firm selling multiple products has been studied in a static setting. To the best of our knowledge, the literature studying the optimal pricing of substitutable products in a dynamic context is very limited. Our paper also contributes to the literature studying the relation between capacity constraints or inventories and optimal pricing. Somewhat surprising, this issue has not received the attention it deserves in the core of economic theory; however, it is at the heart of many studies in operation research.\(^5\)

\(^5\) The strand of operations research literature that examines the problem of a firm maximizing the revenue from selling a given quantity over time, is known as “revenue, yield or demand management”.

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revenue management literature. The problems examined by this literature fall into two broad
categories depending on whether firms manage directly the available capacity or set prices over
time. Our work is related to the second group of models, where prices are the control used to
manage sales over time. Most of these models analyze single-product finite-horizon problems.
Depending on the assumptions, the optimal prices are either decreasing over the entire time
horizon or fluctuate up and down. Nonmonotonic price paths are obtained, for example, by
Gallego and van Ryzin (1994) who study the dynamic pricing of a fixed inventory over a finite
selling horizon. In their model, the optimal price is a function both of the number of units
in stock and of the length of the horizon. They show that, at a given point in time the price
decreases as the stock increases, and that, for a given level of stock the price is higher if there
is more ‘selling’ time left. As a result, the price path exhibits ups and downs, being decreasing
between sales (i.e. in the intervals of time where there is no purchase) and jumping upwards
immediately after a sale. In contrast, Das Varma and Vettas (2001) allow for an infinite selling
horizon, so optimal prices are a function of the available stock only. Time is discrete and buyers,
with unit demand and random valuations (unknown to the monopolist at the time of the price
setting), arrive sequentially. In that model, because of discounting of future profits, the option
value of a unit of the product increases as the available stock decreases, which leads to a price
path increasing in the number of units sold. The present paper builds on that work, with the
main important difference that we extend the analysis to a differentiated-product context.

In contrast to the single-product problem, the multi-product dynamic pricing problem has
remained relatively unexplored. Most of the revenue management literature related to multi-
product problems studies the allocation of a fixed capacity of a single resource to multiple
products, where it is assumed that customers’ demand for a given product is independent of
the availability of other products (and the prices associated to them). Talluri and van Ryzin
(2004) extend this framework by considering a single-resource, multi-product set-up where the

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6 The monotonicity of the price paths may also depend on learning about the demand, which is not considered
in our model. For instance, in Lazear (1986), prices are declining over time because consumers are identical and
there is demand learning.

7 Our assumption of an infinite horizon may be interpreted as capturing existing uncertainty with respect to
the exact end date of a finite selling horizon. Then the discount factor embodies the probability with which the
market closes.

8 In these problems the prices of the different products are predetermined. Customers arrive sequentially and
request a product. The seller observes the request and decides whether to accept or reject it. Since this kind of a
problem has been historically first analyzed in the context of the airline industry, the term usually employed to
denote a product and its associated price is “fare product”. For instance, a business class seat and an economy
class seat (with the corresponding prices) constitute two different fare products.
probability of a consumer choosing a given product depends on the other products available.\textsuperscript{9} The objective of the seller then becomes to choose which fare products to offer in each period. They find that, the greater the available capacity or the more time left for the sale, a larger subset (characterized by a higher probability of a purchase and a higher revenue) will be chosen. The study of stochastic multi-product pricing problems have been very limited due to the complexity of such problems.\textsuperscript{10} One approach has been to approximate these problems by their deterministic counterparts. For instance, Gallego and van Ryzin (1997) study the joint problem of allocating many resources over many products and their optimal pricing over a finite horizon. They show that the solutions of the deterministic problem are asymptotically optimal, suggest heuristic policies and present applications to networks. An alternative approach has been proposed by Maglaras and Meissner (2003), who reduce the multiproduct dynamic pricing problem to one of choosing, first, the optimal aggregate rate at which capacity is used and, subsequently, determining the optimal per-period prices subject to the constraint that the optimal capacity utilization rate is satisfied. We differ from the papers mentioned above in that we assume an infinite selling horizon and focus explicitly on horizontally differentiated products.

The structure of the paper is the following. Section 2 presents the model. Section 3 discusses the case where one of the varieties has been sold out and we characterize the optimal dynamic pricing of the one remaining variety. In Section 4 we examine our main case, where there is positive stock of both varieties. We conclude in Section 5. Some proofs and numerical examples are presented in the Appendix.

2 The Model

We consider a (single) seller who holds a fixed stock of two varieties of some product.\textsuperscript{11} In each period the seller determines the per-unit price of each variety, not knowing the buyers’ preferences for one variety or the other. Then a single buyer arrives who either buys one unit (of one of the two varieties) or leaves empty-handed. A buyer that does not buy never comes back. The same sequence of actions is repeated in each period until the stock of both varieties is sold out. Time is discrete, denoted by $t = 1, 2, 3...$ and the horizon is infinite.

In order to analyze the above problem we set up a dynamic model, building on a properly

\textsuperscript{9}See Anderson, de Palma and Thisse (1996) for a presentation of discrete choice models and their applications (their ch. 4 discusses the relationship between the discrete choice and the address approach).

\textsuperscript{10}The multiple-resource problem is sometimes referred to as ‘network revenue management’.

\textsuperscript{11}The analysis in the remainder of the paper will be made with explicit reference to differentiation in a characteristics space. However, it applies equally well to a spatial framework, if one considers location instead of preference for variety.
modified horizontal-product-differentiation framework. More specifically, normalizing the degree of differentiation, we assume that the two varieties are situated at the two ends of an interval with length equal to unity. We denote each buyer’s preference for variety (location) by a parameter \( v^t \in [0,1] \). Then, the utility of a buyer characterized by a preference parameter \( v^t \) is

\[
U^t = \begin{cases} 
  s - p^t_1 - cv^t, & \text{if he buys variety 1} \\
  s - p^t_2 - c(1 - v^t), & \text{if he buys variety 2}.
\end{cases}
\]

(1)

The parameter \( s \) captures the surplus from buying a unit (no matter of which variety) of the product and is common to all buyers. For instance, returning to our cars example in the Introduction, \( s \) is the gross utility a buyer obtains from buying a certain car irrespective of its color.\(^{12}\) Importantly, and in contrast to the main body of the horizontal-product-differentiation literature, we do not impose the assumption of a high enough gross surplus, which ensures that the market is covered.\(^{13}\) The parameter \( c \) can be thought as capturing both the transportation cost per unit of distance, say \( \bar{c} \), and the degree of product differentiation, say \( \hat{v} \), that is \( c \equiv \bar{c}\hat{v} \).\(^{14}\) Consequently, the generalized cost that a buyer incurs if he buys a unit of product one, equals \( p^t_1 + cv^t \), while, if he buys a unit of product two, this cost is \( p^t_2 + c(1 - v^t) \), where \( p^t_1 \) and \( p^t_2 \) are the prices of the two products posted by the seller in period \( t \).\(^{15}\)

Having described the buyers’ side of the model, we now turn to the seller. We denote by \( K_i \) the initial stock of product \( i \) (\( i = 1,2 \)) and by \( k^t_i \) the available stock (i.e. the units left unsold) of variety \( i \) in period \( t \), where \( k^t_i = 0, 1, ..., K_i, \ i = 1,2 \). As we have mentioned above, in each period the seller sets prices \( p^t_1 \) and \( p^t_2 \), not knowing the value of \( v^t \), the reservation price of the buyer in the period. Thus, from the seller’s point of view, the preference parameter \( v \) is a random variable, which, we assume, is distributed uniformly on \([0,1]\).\(^{16}\) Given the uncertainty that the seller faces, the role of price determination in each period is threefold. By determining

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\(^{12}\)The assumption of a common \( s \) represents a situation where there is no differentiation along some other, non-horizontal, dimension. For instance, quality differentiation could be incorporated into the model by assuming a different gross surplus \( s_i \) for each variety.

\(^{13}\)Our following analysis shows that the price dynamics are different when the market is covered and when it is not.

\(^{14}\)As in a standard Hotelling model, \( \bar{c} \) denotes the disutility a buyer obtains from not purchasing his most preferred variety and it is common to all buyers. The degree of differentiation is captured by the distance between the products, which equals \( \hat{v} \). Because of the linearity of the transportation cost and without loss of generality, we have normalized the distance between the products to be equal to a unit.

\(^{15}\)As we have mentioned above, the model may be equally well interpreted in terms of spatial differentiation. In this case, \( v^t \) would denote the address (location) of the buyer who arrives in period \( t \), \( \bar{c} \) would capture the transportation cost per unit of distance and \( \hat{v} \) would be the distance between the two stores.

\(^{16}\)Respectively, the realizations of the preference parameter in each period, \( v^t \), are independent and random draws from this distribution.
prices, the seller controls: first, the probability that there is a sale in the current period (the lower the prices of both products, the higher this probability); second, given that there is a sale, the relative probability of each variety being sold (given the price of one variety, the lower the price of the other variety is, the higher the probability that it will be bought) and third, the current-period payoff (the higher the price of the variety sold, the higher the current revenue conditional on a sale). Clearly there are opposite forces in work, which would jointly determine the optimal prices in each period.

The buyer’s problem

In order to determine the demand for each variety in every period we need to consider two possibilities depending on the prices posted by the seller: either the optimal prices are such that they ensure a sale of one of the two varieties or they are such that a buyer, with a valuation in a certain range, would possibly leave empty-handed.

Let us first assume that prices in period \( t \) are low enough that the market is ‘covered’, in the sense that certainly there will be a sale in this period. Whether the buyer will purchase one unit of variety 1 or of variety 2 depends on prices and on his preferences, i.e. on the value of \( v^t \). Therefore, as in a standard spatial model, we have to determine first the preference \( \tilde{v}^t \) of the buyer indifferent between the two varieties. By equalizing \( U^t_1 \) and \( U^t_2 \) from expression 1, we find that

\[
\tilde{v}^t = (p^t_2 - p^t_1 + c)/2c.
\]

(2)

Note that, since \( \tilde{v}^t \in [0, 1] \), optimal prices should satisfy

\[
|p^t_1 - p^t_2| \leq c.
\]

(3)

Moreover, there will be a sale with probability one only if the net utility of the buyer with a preference parameter \( \tilde{v}^t \) (the indifferent buyer) is nonnegative. By substituting \( \tilde{v}^t \) in either one of the equations in (1) we obtain the following condition:

\[
p^t_1 + p^t_2 \leq 2s - c.
\]

(4)

Consequently, when the seller posts prices that ensure a sale of one unit, a buyer will purchase variety 1, if he is characterized by a preference parameter \( v^t \in [0, \tilde{v}^t] \), and will purchase variety 2, if \( v^t \in (\tilde{v}^t, 1] \).\(^{17}\) If we let \( p^t_i > p^t_j \), it follows from (3) and (4) that \( p^t_i \leq s, i, j = 1, 2 \). We, therefore, have that when the market is covered the higher of the two prices should not exceed \( s \).\(^{18}\) This condition ensures that the probabilities of a sale we employ in our analysis are well

\(^{17}\)As a convention, we assume that, given prices, if a buyer is indifferent between the two varieties he buys product 1.

\(^{18}\)This constraint is without loss of generality. Both a price equal to \( s \) and a price higher than \( s \) imply a zero probability of a sale for the product priced at this level and, thus, have exactly the same implication for the analysis.
defined.

The other possibility to be considered is that the seller has announced prices such that buyers with certain preferences would choose not to buy at all. This is the case when, given prices, the indifferent buyer (and, by continuity, some buyers in his neighborhood) receives strictly negative net utility, which will happen only if prices satisfy

\[ p_1 + p_2 > 2s - c. \]

Then, only a buyer, characterized by a preference parameter such that his utility from a purchase is positive, will buy a unit of the product. More specifically, a buyer will buy variety 1 only if \( U_1^t \geq 0 \), or equivalently if \( v^t \leq (s - p_1^t)/c \), and will buy variety 2 only if \( U_2^t \geq 0 \), or equivalently if \( v^t \geq 1 - (s - p_2^t)/c \). Again, since \( v^t \in [0,1] \), we obtain \( p_1^t, p_2^t \leq s \), which ensures that the probability of a sale when the market is not covered is well defined.

**The seller’s problem**

By choosing the prices he will set, the seller aims at maximizing the present expected value of future profits with an one-period discount factor equal to \( 0 < \delta \in (0,1) \). We next determine the probabilities of each variety being bought, conditional on a sale taking place.

Assume first that, in some period, the seller optimally wants to ensure a sale from either one of the varieties. The probability of variety 1 being sold equals the probability that the current-period valuation, \( v^t \), is not higher than the valuation of the indifferent buyer, \( \tilde{v}^t \). Given that \( v^t \sim U[0,1] \), this probability equals \( F(\tilde{v}^t) = \tilde{v}^t \). After substituting for \( \tilde{v}^t \), from expression (2) we obtain

\[ F(\tilde{v}^t) = 1/2 + (p_2^t - p_1^t)/2c. \]

The respective probability of variety 2 being sold is

\[ 1 - F(\tilde{v}^t) = 1/2 - (p_2^t - p_1^t)/2c. \]

We can see from the above expressions that a higher price of a given variety reduces the probability of this variety being bought, while a higher price of the other variety increases this probability. Further, we know that there will be a sale with probability one whenever prices are such that the indifferent buyer receives non-negative utility. However, leaving the indifferent buyer with strictly positive utility is never optimal for the seller. The reason is the following. Imagine that prices are such that \( U(\tilde{v}^t) > 0 \). Then the seller can increase both prices by the same amount, keeping in this way \( \tilde{v}^t \) unchanged and, consequently, the probability \( F(\tilde{v}^t) \) unchanged, but increasing his current period expected payoff. Therefore, the optimal prices that

\footnotetext{19}{From (2), the location of the indifferent buyer, \( \tilde{v}^t \), depends only on the difference between the two prices and therefore changing them by the same amount does not alter the value of \( \tilde{v}^t \). Moreover, given that one of the products will be sold for sure, by increasing both prices, the seller increases his current-period payoff.}
ensure a sale with probability one should satisfy \( U(\bar{v}) = 0 \), i.e. should leave zero utility to the indifferent buyer, which, after substituting for \( \bar{v} \), leads to the following optimality condition

\[
p_1^t + p_2^t = 2s - c. \quad (5)
\]

Clearly, the above reasoning would not hold if the two varieties were offered by two separate firms. However, in the case of a single seller that we consider here, it captures the fact that in determining optimal prices the monopolist internalizes the effect of the price of the one variety on the other, being able in this way to extract more of the surplus.

What if, now, in the presence of positive stock of both varieties, the Seller wanted to ensure that a particular variety, any of the two, is sold with probability one? He would set the price of this variety equal to \( p_i^t = s - c \). Again, posting a lower price cannot be optimal, since the probability of a sale remains the same, but the profit decreases.

Further, we consider the possibility that the market is not covered, i.e optimal prices satisfy \( p_1^t + p_2^t > 2s - c \), so there is a positive probability that the buyer will not purchase at all in that particular period. If prices satisfy this condition, from the analysis of the buyer’s problem it follows that the probability of a sale of variety 1 is equal to

\[
F\left(\frac{s - p_1^t}{c}\right) = \frac{s - p_1^t}{c},
\]

while that of a sale of variety 2 is equal to

\[
1 - F\left(1 - \frac{s - p_2^t}{c}\right) = \frac{s - p_2^t}{c}.
\]

Direct calculations give us the probability of no sale, which equals

\[
1 - \left(1 - F\left(1 - \frac{s - p_2^t}{c}\right)\right) - F\left(\frac{s - p_1^t}{c}\right) = \left(1 - \frac{2s - p_1^t - p_2^t}{c}\right).
\]

Note that this probability equals zero (that is, the probability of a sale equals one) exactly when \( p_1^t + p_2^t = 2s - c \). In addition, our requirement that the higher of the two prices should not exceed the valuation \( s \) and the fact that the seller will never charge a price lower than \( s - c \), ensure that the above probability functions are well defined. Equivalently, one may define the probabilities of a sale as above for \( p_1^t, p_2^t \leq s \), and equal to zero for \( p_1^t, p_2^t > s \). We summarize below:

**Lemma 1** Optimal monopoly prices in period \( t \) satisfy \( p_1^t + p_2^t \geq 2s - c \). The probability of no sale is positive, if \( p_1^t + p_2^t > 2s - c \), and equals zero, otherwise.

Having derived the probabilities of a sale of the two products in each period, we can now formulate the seller’s problem. Let \( V(t)(k_1, k_2) \) denote the continuation expected payoff (discounted sum of future profits), at time \( t \), when there are \( k_1^t \) and \( k_2^t \) units in stock from the respective varieties (equivalently, when \((K_1 - k_1)\) and \((K_2 - k_2)\) units of each variety have been sold). Given
that the buyers’ preferences are independent across time, the only link between the periods is the stock of unsold goods \((k_1, k_2)\), which represents the ‘state’ of the problem. This allows us to simplify the notation by dropping the time indexes. Following standard arguments, we can write the Bellman’s equation of the value function, when there is positive stock of both varieties, as follows:

\[
V(k_1, k_2) = \max_{p_1 + p_2 \geq s - c} \left\{ \frac{(s - p_1)}{c} \left( p_1 + \delta V(k_1 - 1, k_2) \right) + \frac{(s - p_2)}{c} \left( p_2 + \delta V(k_1 - 1, k_2 - 1) \right) + (1 - \frac{2s - p_1 - p_2}{c}) \delta V(k_1, k_2) \right\}
\]

(6)

The first term in the maximand constitutes the profit from selling a unit of the first variety multiplied by the respective probability. The profit itself consists of the current period profit, which equals the price of that variety and the continuation payoff with one unit less in stock, \(V(k_1 - 1, k_2)\). The second term represents the profit from selling a unit of the second variety and can be interpreted in a similar way. The last term represents the profit if there is no sale in the current period, which equals the continuation payoff with the same number of units in stock, times the probability of no sale. In the following Sections we characterize the solution of the above problem. In the following Section, we examine first the case where there is a positive stock of only one variety left. After this is done, we will turn to the case of two varieties.

3 The case of only one variety

Reasoning backwards, we first analyze the case where the stock of the one variety has been sold out and there is only one variety offered. To simplify the notation and since the problem is symmetric, we drop the indexes denoting variety. Then, modifying (6) accordingly, the seller’s problem can be written as:

\[
V(k) = \max_{s \geq p \geq s - c} \left\{ \frac{s - p}{c} (p + \delta V(k - 1)) + \delta \left( 1 - \frac{s - p}{c} \right) V(k) \right\},
\]

(7)

where \(V(k)\) is the value function with \(k\) units of one of the varieties (or \(V(k) = V(k, 0) = V(0, k)\)). In addition, since in our model capacity is limited while the selling horizon is infinite, we have the following boundary condition: \(V(0) = 0\). Clearly, the probability of a sale equals one, if \(p = s - c\), and zero, if \(p(k) = s\). Taking the relevant first-order condition with respect to \(p\), we obtain:\(^{20}\)

\[
p(k) = \frac{s + \delta (V(k) - V(k - 1))}{2}.
\]

\(^{20}\)In corresponding one-product models, where no substitute product exists, the difference \(V(k) - V(k - 1)\) represents the opportunity cost of selling the unit today, or equivalently, the ‘bid price’ (see Bitran and Caldentey, 2003).
and, after solving the system of equations (7) and (8), we obtain the optimal price in state k:

\[ p(k) = \frac{c(1 - \delta) + \delta s - \sqrt{c(1 - \delta)(c(1 - \delta) + \delta s + \delta^2 V(k - 1))}}{\delta}. \] (9)

Given that \( V(0) = 0 \) one could solve recursively for the optimal price path using expression (9). From Proposition 1 in Das Varma and Vettas (2001), we know that \( V(k) > V(k - 1) \) and that the price sequence \( \{p_k\}_{k=K}^1 \) is increasing. However, the analysis in Das Varma and Vettas (2001) refers to one variety only, where buyers’ valuations are distributed on some interval \([0, \pi]\). In this case, only a price equal to zero would ensure a sale with probability one, which is equivalent to throwing away a unit and cannot be optimal given that the number of units in stock is finite.

In the present set-up, where the buyers are aware of the (potential) existence of more than one varieties and have certain preferences among them, the seller could ensure a sale of one unit of a given variety by setting a price for that variety equal to \( s - c \). Hence, optimal prices cannot be lower than \( s - c \) and it follows, from (9), that \( p(k) \geq (s - c) \) only if the following condition holds:\(^{21}\)

\[ c \geq \frac{(1 - \delta)s}{2 - \delta} + \frac{\delta(1 - \delta)V(k - 1)}{2 - \delta} \equiv g_k(c). \] (10)

Given that \( p(k) > 0 \), for every \( k \), and by the reasoning in Das Varma and Vettas (2001), we have that \( V(k) > V(k - 1). \)^{22} It follows that \( g_k(c) > g_{k-1}(c) \). Furthermore, \( V(k) \) is decreasing in \( c \), therefore \( g_k(c) \) is also decreasing in \( c \), with \( g_k(0) > 0 \). Consequently, \( c = g_k(c) \) will have a unique solution \( \tilde{c}_k \), for every \( k = 1, 2, ..., K \), such that, if \( c \leq \tilde{c}_k \), the price that the seller posts when he has a stock \( k \) is \( p(k) = s - c \), which ensures that, irrespective of the preferences of the current-period buyer, a unit of the product will be sold for sure (i.e. the market is covered). If \( c > \tilde{c}_k \), the price is \( p(k) > s - c \) and there is positive probability that the buyer leaves empty-handed. Moreover, it is easy to check that the sequence of the threshold values \( \{\tilde{c}_k\}_{k=K}^1 \) is decreasing. Hence, we extend the result in Das Varma and Vettas (2001) to take into account the (potential) existence of a substitute product, the stock of which has already been sold out.

**Proposition 1** When there is only one variety left to be sold, the optimal price path \( \{p_k\}_{k=K}^1 \) is non-decreasing. Specifically, if \( \tilde{c}_k \) is the solution of \( c = g_k(c) \) (as given by 10) and

i) if \( c > \tilde{c}_K \), then \( p(k) > s - c \) and the market will not be covered for every \( k = 1, ..., K \)

ii) if \( c \leq \tilde{c}_1 \), then \( p(k) = s - c \) and the market will be covered for all \( k = 1, ..., K \)

iii) if \( \tilde{c}_j \geq c > \tilde{c}_{j-1} \), then \( p(k) > s - c \), for \( k < j \), and \( p(k) = s - c \), for \( k \geq j \), where \( j = 1, 2, ..., K \).

\(^{21}\)In addition, the optimal prices given by (9) are always smaller than \( s \), which ensures a positive probability of a sale.

\(^{22}\)The argument is as follows. When the firm has \( k \) units in stock, that is, in state \( k \) it could always mimic the price sequence \( \{p_{k-1}\} \) till the first \( k - 1 \) units are sold. But, then the Seller will have one more unit for a sale and, since \( \delta > 0 \), the discounted profits, \( V(k) \), should be strictly higher than \( V(k - 1) \).
The above Proposition states that the price sequence of the variety, from which there is positive stock left, is nondecreasing. The intuition for this result is as follows. As mentioned above, the price posted by the seller in a certain period determines, on the one hand, the probability of a sale and, on the other hand, the current-period profit. So, given that buyers’ valuations are unknown and that the stock is finite while the number of buyers is infinite, there is an ‘option value’ of keeping a unit and selling it in the future. If the stock is large, this option value is small and the firm is better off by setting a relatively low price, thus, selling today with high probability and, therefore, realizing positive profits sooner.\textsuperscript{23} The opposite holds, if the number of units in stock is relatively small. The difference from Proposition 1 in Das Varma and Vettas (2001) is that in our set-up we have to distinguish between three cases, depending on the value of the parameter $c$, i.e., depending on the degree of differentiation and/or the level of the transportation cost. First, if $c$ is relatively small ($c \leq \bar{c}_1$), there will be a purchase with probability one in every period until the stock is sold out. Prices will be constant at $(s-c)$ and after substituting into (7) we obtain the discounted expected profit from selling a stock $k$ as

$$V(k) = \frac{(s-c)(1-\delta^k)}{1-\delta}. \quad (11)$$

Second, if there is a high degree of differentiation (or a high transportation cost), namely, if $c > \bar{c}_K$, then the seller will always post prices such that there is some positive probability of not selling. In this case, the price in each period is given by (9), the price sequence is strictly increasing and the value function satisfies:

$$V(k) = \left(\sqrt{c(1-\delta) + \delta s + \delta^2 V(k-1) - \sqrt{c(1-\delta)}}\right)^2 \quad (12)$$

Finally, the most interesting case is the one where the market is covered for some levels of the stock and not covered for other. Since $\{\bar{c}_k\}_{k=1}^{K}$ is decreasing, the only possibility is that, when the stock is large, prices are relatively low ($p(k) = s-c$), so that the market is covered, but they start to increase ($p(k-1) > p(k) > s-c$) as the stock decreases. In this case, we can establish a simple threshold value for the continuation payoff, such that, when solving backwards for the optimum prices, we can determine the stock size when the optimal price falls and stays at $(s-c)$ so that the market will be covered for any larger stock size. More specifically, when the value function for some stock level $(k-1)$ becomes larger than $\tilde{V}(k-1) = (2-\delta)c/\delta(1-\delta) - s/\delta$, then we know that for a stock of $k$ units and for any larger stock the price will equal $(s-c)$ and a sale will take place for sure. Further, if we denote by $\hat{k}$ the stock level, which denotes the period when a switch from a covered to a not-covered market occurs, it follows that, for $k \leq \hat{k}$,\textsuperscript{23} Because of discounting, the present value of units sold farther in the future is small and, therefore, it is not profitable to forego a sale today and delay the realization of the stream of future profits.
Table 1: Optimal price paths and expected profits for $s = 8$ and
(a) $c = 6$, $\delta = 0.9$; (b) $c = 3$, $\delta = 0.9$; (c) $c = 6$, $\delta = 0.8$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$p(k)$</th>
<th>$V(k)$</th>
<th>$\tilde{c}_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.003</td>
<td>0.028</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.501</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>5.411</td>
<td>17.029</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5.862</td>
<td>14.129</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4.585</td>
<td>16.161</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>4.773</td>
<td>17.700</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4.611</td>
<td>16.147</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>4.511</td>
<td>20.283</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>4.430</td>
<td>21.239</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>4.263</td>
<td>22.045</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>4.077</td>
<td>22.723</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>4.026</td>
<td>23.307</td>
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</tr>
<tr>
<td>13</td>
<td>3.921</td>
<td>23.799</td>
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</tr>
<tr>
<td>14</td>
<td>3.909</td>
<td>24.217</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>3.900</td>
<td>24.673</td>
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</tr>
<tr>
<td>16</td>
<td>3.877</td>
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</tr>
<tr>
<td>17</td>
<td>3.817</td>
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<td>18</td>
<td>4.090</td>
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<td>19</td>
<td>4.865</td>
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</tr>
<tr>
<td>20</td>
<td>4.273</td>
<td>27.707</td>
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</tr>
</tbody>
</table>

The value function is given by equation (12), while direct computations show that for $K \geq k > \hat{k}$ it satisfies:

$$V(\hat{k} + n) = \frac{(s-c)(1-\delta^n)}{1-\delta} + \delta^n V(\hat{k}),$$

(13)

where $\hat{k} + n = k$ and $n = 1, 2, ..., K - \hat{k}$ denotes the units sold, while the market is covered. We can check that, if the market is covered for all available stock levels, i.e. if $\hat{k} = 0$, then $V(\hat{k}) = 0$ and (13) equals (11), i.e. we are back to the case where in every period optimal prices ensure that there is a purchase with probability one.

We conclude this Section with some numerical examples that illustrate the properties of the solution. In Table (1) we present the optimal prices and the expected profits for stock levels $k = 1, 2, ..., 20$. We have set $s = 8$, $\delta = 0.9$, while $c$ varies across Tables (1a) and (1b). Specifically, we have set $c = 6$ for calculating the results presented in Table (1a), which ensures that $p(k) > s - c$, for all $k$. Hence, the price sequence increases as the available stock decreases (by Proposition 1). The optimal prices and profits presented in Table (1b) are calculated for $c = 3$. The latter results show that, when the transportation cost or the degree differentiation between the two varieties are decreased substantially (here, $c$ declines from 6 to 3), for any stock level $k \geq 9$, $p(k) = s - c = 5$ and a unit will be sold for sure.

We can see from the last column of Table (1b) that this holds as soon as $\tilde{c}_k$ becomes higher than $c$ (which happens for $k = 9$). When there are only 8 units left in stock, the seller starts to increase the price, hence, leaving some positive probability that a customer with not so strong preference for the particular variety does not buy. We present diagrammatically the increasing profile of optimal prices as the available stock decreases (see Figure 1). Further, we examine
how the optimal price path changes in response to a decrease in the discount factor, $\delta$. Table (1c) shows that (with a lower discount factor) for every stock level the price decreases. This is very intuitive since, as the seller becomes more ‘impatient’ he is expected to post lower prices in order to realize the stream of profits earlier.

From the above analysis we conclude that, depending on the parameters’ values, the seller may initially post prices that ensure that a unit of the variety left unsold is purchased even by a buyer with a strong preference for the other variety, while after a certain number of periods the seller starts increasing steadily the price after each period when a sale takes place. We next continue the analysis by examining our main case of interest, where the seller holds positive stock of both varieties.

4 Positive stock of both varieties

In case there is a positive stock $(k_1, k_2)$ of both varieties, optimal prices should satisfy the Bellman’s equation given by (6). Following some slight modifications this can be written as

$$ V(k_1, k_2) = \max_{p_1 + p_2 \geq 2s - c, p_1, p_2 \leq s} \left\{ \left( \frac{s - p_1}{c} p_1 + \frac{s - p_2}{c} p_2 \right) + \delta \left( \frac{s - p_1}{c} V(k_1 - 1, k_2) + \frac{s - p_2}{c} V(k_1, k_2 - 1) \right) ight\}, $$

(14)

where the terms in the first bracket denote the current-period expected profit and the terms in the second bracket denote the expected continuation payoff.

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24In the cases where the market is covered the optimal price remains $p = s - c$. 
4.1 Positive probability of no sale: the market is not covered.

First, let the optimal prices satisfy $p_1 + p_2 > 2s - c$, meaning that there is a positive probability of no sale. We obtain the optimal prices, from the relevant first-order conditions, as follows:

$$p_1(k_1, k_2) = \frac{s + \delta(V(k_1, k_2) - V(k_1 - 1, k_2))}{2},$$  \hspace{1cm} (15)

$$p_2(k_1, k_2) = \frac{s + \delta(V(k_1, k_2) - V(k_1, k_2 - 1))}{2},$$  \hspace{1cm} (16)

from which it follows that $p_2(k_1, k_2) = p_1(k_1, k_2) + \delta(V(k_1 - 1, k_2) - V(k_1, k_2 - 1))/2$. Substituting for $V(k_1, k_2)$ in (15) and (16) we obtain the following expressions for the optimal pair of prices:

$$p_1(k_1, k_2) = \frac{c(1 - \delta) + 2\delta s + \delta^2(V(k_1 - 1, k_2) - V(k_1, k_2 - 1))}{2\delta} - Z \frac{2\delta}{4\delta},$$  \hspace{1cm} (17)

$$p_2(k_1, k_2) = \frac{c(1 - \delta) + 2\delta s + \delta^2(V(k_1 - 1, k_2) - V(k_1, k_2 - 1))}{2\delta} - Z \frac{2\delta}{4\delta},$$  \hspace{1cm} (18)

where

$$Z \equiv \sqrt{\frac{4c(1 - \delta)[c(1 - \delta) + 2\delta s + \delta^2(V(k_1 - 1, k_2) + V(k_1, k_2 - 1))] - \delta^4(V(k_1 - 1, k_2) - V(k_1, k_2 - 1))^2}{-\delta^4(V(k_1 - 1, k_2) - V(k_1, k_2 - 1))^2}}.$$  \hspace{1cm} (19)

Using expressions (17) and (18) we obtain the value function in state $(k_1, k_2)$,

$$V(k_1, k_2) = \frac{c(1 - \delta) + 2\delta s + V(k_1 - 1, k_2) + V(k_1, k_2 - 1)}{2\delta} - Z \frac{2\delta}{2\delta^2}.$$  \hspace{1cm} (20)

The optimal price paths of the two varieties can be computed recursively using the relations $V(k_1, 0) = V(k_1), V(0, k_2) = V(k_2)$ and $V(0) = 0$.

Furthermore, note that when the probability of no sale is positive given the prices that have been optimally set, the two pricing optimality conditions are independent from each other, since the probability of a sale of each of the products is independent of the probability of a sale of the other product (see expression 14). This means that the seller’s pricing decision regarding one variety is not constrained by the existence of the other variety. In this case, clearly the optimal price of product $i$, $p_i(k_1, k_2)$, has to be equal to $p_i(k_1)$, the optimal price defined above for the case when only one of the varieties remains to be sold, $i = 1, 2$.\(^{25}\) We, therefore, obtain the following result

**Proposition 2** When the probability of no sale is positive, the two pricing problems are independent and optimal the prices satisfy $p_1(k_1, k_2) = p(k_1)$ and $p_2(k_1, k_2) = p(k_2)$.

\(^{25}\)Remember that $p(k)$ is the price that maximizes the seller’s profit from the sale of the $k^{th}$ unit of one of the products when the stock of the other product has been sold out.
From the above analysis we obtain that the properties of the price paths, in the cases where there is positive stock of the two varieties and the market is not covered, follow directly from the properties of the price sequence \( \{p_k\}_{k=1}^{K} \) defined in Section 3. In Proposition 1 we have established that the latter is nondecreasing. We now show that the sequence of prices \( p_i(k_1, k_2) \) strictly increases as \( k_i \) decreases, for each \( i = 1, 2 \). This is because the optimal prices \( p(k) \), when only one variety is available, remain unchanged as capacity decreases in time only if \( p(k) = s - c \), that is, only if the market is covered in every period. But this cannot hold in the present case, since optimal prices are such that the market is not covered with positive probability. Consequently, \( p_i(k_1, k_2) \neq s - c \), so \( p_i(k_1, k_2) \) increases after a unit of variety \( i \) is sold, while the price of the other variety remains unchanged. Hence, the following holds.

**Corollary 1** If the optimal prices are set so that the probability of no sale is positive, capacities are \((k_1, k_2)\) and one unit of variety 1 is sold, then \( p_1(k_1 - 1, k_2) > p_1(k_1, k_2) \) and \( p_2(k_1 - 1, k_2) = p_2(k_1, k_2) \). Respectively, if one unit of variety 2 is sold, \( p_1(k_1, k_2 - 1) = p_1(k_1, k_2) \) and \( p_2(k_1, k_2 - 1) > p_2(k_1, k_2) \).

A direct consequence of the above result is that, if it is optimal to have a positive probability of no sale for certain stock levels, then this probability should increase as the stock decreases. This is because the price of the variety from which a unit has been sold increases and consequently the probability of selling a unit of that variety decreases. At the same time, the price of the other variety remains unchanged and so does the probability of selling a unit of it. Hence, we obtain that

**Corollary 2** If optimal prices at some \( t \) are such that the probability of no sale is positive, then this probability will be positive in all subsequent periods and nondecreasing.

The above result implies that, if the market is not covered in some period, it will not be covered in the subsequent periods. Consequently, it also holds that, if optimal prices with stock \((k_1, k_2)\) are such that the market is not covered, then it will not be covered when there is positive stock of only one variety left.

Proposition 2 and Corollary 2 also imply that

**Proposition 3** \( V(k_1, k_2) = V(k_1) + V(k_2) \) when the probability of no sale is positive.

**Sketch of the proof.** We have established that, if the market is not covered in some period, it will not be covered in any subsequent period and that for any stock level \( p_i(k_1, k_2) = p_i(k_i) \). It follows that \( V(k_1, k_2) = V(k_1) + V(k_2) \) (for a detailed proof see the Appendix).

Finally, we show that the seller is better off when his remaining capacity is more symmetric. Given a total number of units available, the seller’s expected profit is increasing if these units become more evenly distributed across the two varieties. Let \( k_1 > k_2 \). Then, since \( p_1(k_1) \leq p_2(k_2) \)
and by Proposition 2, we have that \( p_1(k_1, k_2) = p_1(k_1) < p_2(k_2) = p_2(k_1, k_2) \) (see Corollary 1 for why the inequality is strict). Substituting with the respective expressions given by (15) and (16), we obtain that

**Corollary 3** If the probability of no sale is positive and the number of units in the stock of each variety satisfy \( k_1 > k_2 \), then \( V(k_1 - 1, k_2) > V(k_1, k_2 - 1) \).

So far, we have analyzed the case where the seller optimally leaves some positive probability that there is no purchase. Then the pricing decisions regarding one of the products are independent of the pricing decisions regarding the other product. In addition, if for some stock levels the market is not covered, it will not be covered for any lower stock levels. However, the market may be optimally covered for higher stock levels. In the next Section we examine this possibility.

### 4.2 Zero probability of no sale: the market is covered.

From Lemma 1, we know that when the seller ensures that a buyer purchases one of the varieties with probability one, optimal prices should satisfy \( p_1(k_1, k_2) + p_2(k_1, k_2) = 2s - c \). Therefore, the problem is reduced to that of determining only one of the prices, while the other follows directly from the above equation. Given that, when the constraint \( p_1(.) + p_2(.) \geq 2s - c \) is binding, the probability of no sale is zero, we can write (6) for this case as follows

\[
V(k_1, k_2) = \max_{s - c \leq p_1 \leq s} \left\{ \frac{(s - p_1)}{c} (p_1 + \delta V(k_1 - 1, k_2)) + \left( 1 - \frac{(s - p_1)}{c} \right) (2s - c - p_1 + \delta V(k_1, k_2 - 1)) \right\}. \tag{21}
\]

From the first order conditions with respect to \( p_1 \) we obtain

\[
p_1(k_1, k_2) = \frac{2s - c}{2} - \frac{\delta (V(k_1 - 1, k_2) - V(k_1, k_2 - 1))}{4}. \tag{22}
\]

From \( p_2(k_1, k_2) = 2s - c - p_1(k_1, k_2) \) it follows that

\[
p_2(k_1, k_2) = \frac{2s - c}{2} + \frac{\delta (V(k_1 - 1, k_2) - V(k_1, k_2 - 1))}{4}. \tag{23}
\]

Substituting the prices back into (21), we obtain

\[
V(k_1, k_2) = \frac{2s - c + \delta (V(k_1 - 1, k_2) + V(k_1, k_2 - 1))}{2} + \frac{\delta^2 (V(k_1, k_2 - 1) - V(k_1 - 1, k_2))^2}{8c}. \tag{24}
\]

As before, the optimal price path for each variety can be computed recursively using equations (22), (23) and (24). Since the sum of the two prices within a period (that is, for given levels of inventories) is constant, it is clear that, when the price of one variety increases, the price
of the other variety should decrease and vice versa (as long as the market is covered in the next period). We examine the properties of the optimal price paths for small symmetric initial stock levels and then (due to the complication of the implied relations) we present numerical examples for a wider range of stock levels. First, we establish a result, which shows that having a wider available variety of products is profitable. This is expected since, when two differentiated products are offered for sale instead of one, the transportation cost for some of the buyers is reduced, which allows the monopolist to charge higher prices.

**Lemma 2** \( V(k_1, k_2) \geq V(k_1 + k_2) \).

**Proof.** It cannot be that \( V(k_1, k_2) < V(k_1 + k_2) \) since, with \( k_1 \) units of product 1 and \( k_2 \) units of product 2, the seller could always mimic the optimal price path of a single product the stock of which is \((k_1 + k_2)\). He could do this by setting \( p_1(k_1, k_2) = p(1 + k_2) \) for, say, the first variety and a price \( p_2(k_1, k_2) = s \), for the second variety (which ensures that no buyer would buy the latter), as long as there is positive stock of the first variety. Once this stock has been depleted, the seller sets \( p_2(k_1, k_2) = p(k_2) \). Therefore, the seller cannot be worse-off when he has a positive stock of both varieties than when he has the same (total) number of units of one variety only.\(^{26}\)

Now, suppose that the seller has two units of each variety in stock and parameter values are such that it is optimal to sell a unit in the first period with probability one. Hence, the optimal first-period prices equal \( p_1(2, 2) = p_2(2, 2) = (2s - c)/2 \) (by equation (22)). Since the market is covered, the first-period buyer will buy a unit of his preferred variety (also taking prices into account) for sure. Assume that the buyer’s preferences end up being such that he purchases one unit of variety 1. Therefore, the second-period stock is \((1, 2)\) and the price of the first variety in that period equals \( p_1(1, 2) = (2s - c)/2 - \delta(V(2) - V(1)) / 4 \). By Lemma 2, \( V(1, 1) \geq V(2) \) and, consequently, \( p_1(1, 2) \geq p_1(2, 2) \) and \( p_2(1, 2) \leq p_2(2, 2) \). We, therefore, obtain that, after a sale of one unit of a certain variety, the next-period price of that variety increases, while the price of the other variety decreases (as long as the market is covered). We can show by direct calculations that this is also the case if the firm holds initially 3 units of each variety. The first period prices are \( p_1(3, 3) = p_2(3, 3) = (2s - c)/2 \). Assume again that a unit of the first variety is sold in each of the following periods. Then the prices of this variety will become \( p_1(2, 3) = (2s - c)/2 - \delta(V(1, 3) - V(2, 2))/4 \) and \( p_1(1, 3) = (2s - c)/2 - \delta(V(3) - V(1, 2))/4 \), respectively. In the Appendix we show that \( p_1(3, 3) \leq p_1(2, 3) \leq p_1(1, 3) \) and therefore the price of variety 2, the stock of which remains unchanged, is nonincreasing, \( p_2(3, 3) \geq p_2(2, 3) \geq p_2(1, 3) \).

\(^{26}\)Note that for the case where the market is not always covered (for the optimal prices) then \( V(k_1, k_2) = V(k_1) + V(k_2) \) and thus this property implies that \( V(k_1) + V(k_2) \geq V(k_1 + k_2) \).
Remark 1 When the initial stock is \((2,2)\) or \((3,3)\), and the probability of a sale is one, the price of the variety which is sold does not decrease in the next period, while the price of the other variety does not increase.

The intuition from the cases considered regarding the monotonicity of prices can be extended to cases of higher and arbitrary capacity levels. However, the formal derivations appear overly complicated (as the number of possible states for each future period is increasing very quickly). Thus, in the remainder of this Section we turn to numerical calculations which illustrate that the above findings hold for a wider range of stock levels.

4.3 Numerical Examples

First, we compute the optimal price paths assuming that \(c = 6\), \(s = 8\), \(\delta = 0.9\) and letting \(k_1 = 1, \ldots, 20\) and \(k_2 = 1, \ldots, 10\). Details about the procedure we follow and the Tables with the respective results are given in the Appendix. Figure 2 summarizes the results in two diagrams showing the evolution of optimal prices as the stock of each variety changes. Part (a) depicts the optimal price path of variety 1, while part (b), the optimal path of variety 2. We observe that, for each variety, decreasing the stock level of this variety (and keeping constant the level of the other variety) increases the optimal price of this variety. Moreover, holding the stock level of a given variety constant and decreasing that of the other variety implies a decrease in the price of the first variety, if the market is covered, while this price remains constant when there is a positive probability of no sale. Further, we examine the effect of a decrease of the discount factor, \(\delta\), on the optimal prices. The results, for \(c = 6\), \(s = 8\) and \(\delta = 0.8\), are presented in the Appendix. As expected, when the firm becomes more impatient (\(\delta\) decreases), it sets prices that ensure a sale with probability one for lower stock levels. In other words, for any stock level of variety \(i\), the market is covered for lower stock levels of variety \(j\) \((i, j = 1, 2)\). For instance, when
δ = 0.9 and k_i = 1, optimal prices are such that there is a positive probability of no sale for every k_j = 1,...,10. When the discount factor decreases to δ = 0.8, optimal prices are such that a sale with probability one is ensured for every k_j ≥ 6. In particular, we find that, in the cases where the probability of no sale remains positive after δ has dropped to 0.8, this probability decreases. It also decreases for these stock levels, for which the market is not covered when δ = 0.9, but becomes covered when δ = 0.8. In all the remaining cases, the market is covered both for δ = 0.9 and for δ = 0.8, therefore, the probability of no sale remains equal to zero when the discount factor decreases. It follows that the probability of no sale does not increase when the discount decreases from 0.9 to 0.8. This is because this probability depends positively on the sum of the two prices, which either decreases or remains unchanged as δ decreases. However, regarding the effect on each one of the prices, we find that, in contrast to the one-variety case where the optimal prices cannot increase as δ decreases, there is no such a monotonic price response in the two-variety case. We have formed the price difference between p_i(k_1, k_2), for δ = 0.9, and p_i(k_1, k_2), for δ = 0.8, in order to examine the price changes, i = 1, 2. We denote this difference by ∆p_i. We present the results for k_1, k_2 = 1, 2,...,10 in Table 2, where a positive entry indicates that a price decreases when the discount factor decreases from 0.9 to 0.8 and a negative entry indicates that a price increases. It is clear from Table 2 that there are three possibilities regarding the response of prices to a decrease of the discount factor: i) prices remain unchanged, ii) both prices decrease and iii) the price of the variety with higher stock increases while the price of the variety with lower stock decreases. Which of these will be observed, depends jointly on the absolute stock levels and the asymmetry between them.\(^{27}\) However, note that, for the stock pairs for which the market remains not covered after the decrease of δ, both prices decrease.

5 Conclusion

In this paper we have examined the dynamic pricing of a monopolist who sells a finite stock of two distinct varieties of some product over an infinite selling horizon. The seller’s optimal pricing decisions are derived based on a stochastic dynamic programming formulation. In each period, the seller faces unit demand from a customer with random preference for one variety or the other. The objective of the firm is to maximize the present value of future expected profits and in doing so it has to take into account the trade-off between increasing the probability of a sale and, thus, obtaining the stream of profits earlier in time and increasing the revenue from selling a given unit. The existence of two varieties significantly enriches the problem since in setting the optimal price for one variety the firm should consider the available stock of both varieties. Hence, in contrast to the single-product setting, the question that the seller faces is

\(^{27}\)Detailed discussion of how prices change depending on the capacity levels is presented in the Appendix.
not only whether it is more profitable to sell a unit today or not, but in addition which variety he would rather sell. Consequently, optimal prices determine, in addition to the instantaneous revenue and the overall probability of a sale, the probability of a purchase of a given variety. The evolution of observed prices in the market will depend jointly on the optimal pricing rule and also on the (random) sequence of consumers preferences.

For the case where there is positive stock of only one variety left, we extend the results of Das Varma and Vettas (2001) by establishing that the optimal price path is nondecreasing. Specifically, for certain parameter values (e.g. relatively low transportation cost or large stock), the seller may optimally wish to ensure a purchase even from a buyer with a strong preference for the other variety. Then, the seller charges a price that leads to a sale with probability one and, as long as it is optimal to cover the whole market, the price path is constant. This path becomes increasing in the number of units that have been sold, when the probability of a sale is lower than one.

With a positive stock of both varieties, the market may also be either covered or not. We find that, all else kept equal, the market may be covered (zero probability of no sale) for high stock levels and become not covered (positive probability of no sale) as the available stock decreases.
Once the probability of no sale becomes positive, it remains positive and nondecreasing for all remaining periods. When the seller optimally lets some buyers leave empty-handed, the pricing of one variety is independent from the pricing decision regarding the other variety. Consequently, the optimal price of a certain variety, for a given own-stock level, equals the optimal price for the same stock level when only this variety is available. When the market is covered, the intertemporal change in one of the prices is accompanied by an equal in magnitude but opposite in sign change of the other price. This is because the monopolist internalizes the effect of the one variety price on the other and never leaves the indifferent buyer with positive utility. We characterize the optimal price paths and establish their properties parametrically, for small and symmetric initial stock levels, and numerically, for a larger range of stock pairs. We establish that the price of a product tends to decrease in its own capacity level and to increase in the other product’s capacity. Finally, numerical calculations suggest that the probability of a sale either increases or remains equal to one, when the discount factor decreases. Importantly and in contrast to the one-variety case where optimal prices decrease as the discount factor decreases, the response of the optimal prices to a decrease of the discount factor in the two-variety case is not monotonic. Depending on the absolute level of available stock and on the degree of asymmetry between the two stock levels, both prices may remain unchanged or decrease, or the price of the low-stock variety may decrease while that of the high-stock variety increases.

In this paper we have proposed a dynamic model of pricing differentiated products. The intertemporal link is driven by the stock (inventories) of each variety that the seller has. While problems that fall into this general discussion have remained largely unexplored in the literature, they appear both interesting from a theoretical viewpoint and quite important for business pricing decisions. The general theme here is that the inventories or capacity constraints for all products may affect dynamically all prices. Thus, there appears to be a significant room for further research in this area, which will shed light on various important aspects of the strategies of multi-product firms.
Appendix

Proof of Proposition 3. We want to show that \( V(k_1, k_2) = V(k_1) + V(k_2) \), given that when the market is not covered for certain stock levels \((k_1, k_2)\), it will not be covered for any lower levels of stock (i.e. in all subsequent periods).

Let \( k_1 = m \) and \( k_2 = n \). Since \( V(m, n) = V(n, m) \), we will present the analysis for \( V(m, n) \), where \( m = 1, 2, \ldots, n \) and \( n = 1, 2, \ldots, k \).

1. Let \( m = 1 \). We can show inductively that \( V(1, n) = V(1) + V(n) \).

   i) For \( n = 1 \), \( V(1, 1) = 2V(1) \).
   Writing (20) for \( V(1, 1)\) and after further manipulation we obtain
   \[
   V(1, 1) = \frac{\sqrt{c(1-\delta) + 2\delta s + 2\delta^2 V(1) - \sqrt{c(1-\delta)}}}{2\delta^2}
   \]

   We use (12) to substitute for \( V(1)\) in \( \sqrt{c(1-\delta) + 2\delta s + 2\delta^2 V(1)}\), which after some simplifications becomes
   \[
   \sqrt{c(1-\delta) + 2\delta s + 2\delta^2 V(1)} = 2\sqrt{c(1-\delta) + \delta s - \sqrt{c(1-\delta)}}
   \]

   Therefore we obtain
   \[
   V(1, 1) = \frac{(2 \sqrt{c(1-\delta) + \delta s - 2\sqrt{c(1-\delta)}})^2}{2\delta^2} = \frac{2(\sqrt{c(1-\delta) + \delta s - \sqrt{c(1-\delta)}})^2}{2\delta^2} = 2V(1).
   \]

ii) Assume that \( V(1, n-1) = V(1) + V(n-1) \).

iii) We can show that \( V(1, n) = V(1) + V(n) \).

From (20), we have that

\[
V(1, n) = \frac{2c(1-\delta) + 2\delta s + \delta^2 V(n) + V(1, n-1) - Z}{2\delta^2}, \quad \text{(A1)}
\]

where

\[
Z = \sqrt{4c(1-\delta) \left[ c(1-\delta) + 2\delta s + \delta^2 (V(n) + V(1, n-1)) \right] - \delta^4 (V(n) - V(1, n-1))^2}. \quad \text{(A2)}
\]

Take, first, the last term under the root, \( \delta^4 (V(n) - V(1, n-1))^2 \). Since \( V(1, n-1) = V(1) + V(n-1) \), this term equals \( \delta^4 (V(n-1)+V(1)-V(n))^2 \). After substituting for the value functions, from (12), and simplifying, we obtain

\[
\delta^4 (V(n) - V(1, n-1))^2 = 4c(1-\delta) \left( \sqrt{c(1-\delta) + \delta s + \delta^2 V(n-1) - \sqrt{c(1-\delta) + \delta s}} \right)^2.
\]

Substituting back into (A2) we have
for $V$ and the market is covered in all periods, we have that $m = 2$. Let

iii) We can show that

Substituting back into (A1) and simplifying we obtain that $V(1, n) = V(1) + V(n)$. 

2. Let $m = 2$. Following the same steps as before we can show that $V(2, n) = V(2) + V(n)$.

i) From the previous analysis we know that $V(2, 1) = V(2) + V(1)$.

ii) Assume that $V(2, n - 1) = V(2) + V(n - 1)$.

From (20), $V(2, n)$ equals

$$V(2, n) = \frac{2c(1 - \delta) + 2\delta s + \delta^2 V(1, n) + V(2, n - 1)}{2\delta^2} - \frac{Z}{2\delta^2}, \quad (A3)$$

where $Z = \sqrt{4c(1 - \delta) \left[ c(1 - \delta) + 2\delta s + \delta^2 (V(1, n) + V(2, n - 1)) \right] - \delta^4 (V(1, n) - V(2, n - 1))^2}$. \quad (A4)

From the analysis of $V(1, n)$, we know that $V(1, n) = V(1) + V(n)$. Hence, after substituting for $V(1, n)$ and $V(2, n - 1)$ and simplifying as before, we obtain

$$Z = \sqrt{4c(1 - \delta) \left( \sqrt{c(1 - \delta) + \delta s + \delta^2 V(n)} + \sqrt{c(1 - \delta) + \delta s + \delta^2 V(n - 1) - \sqrt{c(1 - \delta)^2}} \right).$$

Substituting $Z$ back to (A3) and simplifying accordingly, we obtain $V(2, n) = V(2) + V(n)$.

3. In the same inductive way one can show that $V(m, n) = V(m) + V(n)$, for $m = 3, \ldots, n$. For each $m$, $V(m, n)$ is a function of $V(m - 1, n)$ and $V(m, n - 1)$. Then one uses the result of the previous step, that $V(m - 1, n) = V(m - 1) + V(n)$, to substitute for $V(m - 1, n)$ and proceed as we did in steps 1 and 2 above.

**Proof of Remark 1.** The case of stock $(2, 2)$ is presented in the text. If the stock is $(3, 3)$ and the market is covered in all periods, we have that

$$p_1(3, 3) = p_2(3, 3) = \frac{(2s - c)}{2}$$

$$p_1(2, 3) = \frac{(2s - c)}{2} - \frac{\delta (V(1, 3) - V(2, 2))}{4}$$

$$p_1(1, 3) = \frac{(2s - c)}{2} - \frac{\delta (V(3) - V(1, 2))}{4}.$$
We want to show that \( p_1(1,3) \geq p_1(2,3) \geq p_1(3,3) \). By Lemma 2, \( V(1,2) \geq V(3) \) and so \( p_1(1,3) \geq p_1(3,3) \). First, we will show that \( p_1(1,3) \geq p_1(2,3) \). Substituting for the optimal prices from (24) we obtain \( V(2,2) - V(1,3) \leq V(1,2) - V(3) \). Substituting for the value functions on the LHS (by 24), we obtain

\[
\frac{\delta(V(1,2) - V(3))}{2} - \frac{\delta^2(V(1,2) - V(3))^2}{8c} \leq V(1,2) - V(3).
\]

Let \( A \equiv V(1,2) - V(3) \) so that we have \( A \left( \frac{\delta}{2} - \frac{\delta^2}{8c} - 1 \right) \leq 0 \). This is always true since \( A \geq 0 \) and \( \left( \frac{\delta}{2} - \frac{\delta^2}{8c} - 1 \right) < 0 \).

Furthermore, we show that \( p_1(2,3) \geq p_1(3,3) \), which is equivalent to showing that \( V(2,2) - V(3) \geq 0 \). By Lemma 2, \( V(2,2) \geq V(4) \) and, since \( V(4) \geq V(3) \), we obtain \( V(2,2) - V(3) \geq 0 \). Hence, \( p_1(2,3) \geq p_1(3,3) \) and it follows that \( p_1(1,3) \geq p_1(2,3) \geq p_1(3,3) \).

**Numerical Examples**

1. Numerical example for the following parameter values: \( c = 6, s = 8, \delta = 0.9 \).

   It follows that the market will not be be covered as long as:

   i) \( p > s - c = 2 \), in the one-variety case, and

   ii) \( p_1(k_1, k_2) + p_2(k_1, k_2) > 2s - c = 10 \), when there is positive stock of both varieties.

   In the case where there is positive stock of only one variety, we denote the remaining stock by \( k = 1, 2, ...; 20 \) and compute recursively the optimal prices, using equation (9), and the value function, using (12), for every \( k \). Starting from \( k = 1 \) and for each capacity level, we first compute the optimal price, \( p(k) \), which depends only on the value function when capacity is one unit less. Then we use this price to compute the value function for the same level of stock, \( V(k) \), which is subsequently used as an input for computing the optimal price for available stock \( k + 1 \).

   In computing \( p(1) \), we use the fact that \( V(0) = 0 \). If for some \( k \), the optimal price falls below \( s - c = 2 \), we set \( p(k) = p(k + 1) = ... = p(20) = 2 \), as for these stock levels the market will be covered and there will always be a sale with probability one.

   In the case where there is positive stock of both varieties we let \( k_1 = 1, 2, ...; 20 \) and \( k_2 = 1, 2, ..., 10 \). Within each table we keep \( k_2 \) constant and compute the optimal prices and the value function for each capacity pair as \( k_1 \) increases. We allow \( k_2 \) to increase across tables. Having computed the optimal prices and the value function when there is only one variety left, we first examine the two-variety case, for \( k_2 = 1 \). The optimal prices \( p_1(k_1, 1) \) and \( p_2(k_1, 1) \) depend on \( V(1) \) and \( V(n - 1) \) and starting from \( k_1 = 1 \), we are able to compute recursively \( p_1(k_1, 1) \), \( p_2(k_1, 1) \) and \( V(k_1, 1) \) for every \( k_1 = 1, 2, ..., 20 \). In the same way we use \( V(k_1, 1) \) for computing \( p_1(k_1, 2) \), \( p_2(k_1, 2) \) and \( V(k_1, 2) \) for every \( k_1 = 1, 2, ..., 20 \). For every capacity pair we use (17) and (18) to find \( p_1(k_1, k_2) \) and \( p_2(k_1, k_2) \), respectively, and present them in the second and the third
column of the respective table. But these are the relevant expressions only if the probability of no sale is positive. Therefore, we check whether \( p_1(k_1, k_2) + p_2(k_1, k_2) > 2s - c = 10 \), which ensures that prices are high enough not to cover the market. If this condition is satisfied, we use (20) to compute the value function for the specific level of capacities (last column in each table). If the condition is not satisfied, we compute again the optimal prices using (22) and (23) and denote them by \( p_1^c(k_1, k_2) \) and \( p_2^c(k_1, k_2) \), where the superscript \( c \) refers to the market being ‘covered’ (columns five and six). Then we compute the value function, for the cases where the probability of no sale is zero, using (24).

<table>
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<th>Optimal prices and Value Function for ( k=1,2,...,20 )</th>
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2. Numerical example for the following parameter values: $c = 6$, $s = 8$, $\delta = 0.8$. We follow the same procedure as before and obtain the following results:

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The one variety case for $k = 1, 2, ..., 20$

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Optimal price and value functions for $k = 1, 2, ..., 20$ and $\delta = 0.8$

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Optimal price and value functions for $k = 1, 2, ..., 20$ and $\delta = 0.8$

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The impact, on prices, of a decrease of $\delta$, from 0.9 to 0.8.

We observe, from Table 2 in the main text that, when there is only one unit left of some variety, independently of the stock of the other variety, both prices decrease after a decrease in the discount factor, while for symmetric stock levels from $(5,5)$ to $(10,10)$, prices remain unchanged. Further, the price of the variety from which there is smaller stock decreases and the price of the higher-stock variety increases, depending on the degree of asymmetry between stock levels and on their absolute values. For instance the above holds for $m = 8, 9, 10$ and every $1 < n \leq 10, n \neq m$. As $m$ decreases from 7 to 2, the above does not hold for some values of $n$ and the number of cases where it is not true increases as $m$ decreases. Specifically, the price of the low-stock variety decreases, while the price of the high-stock variety increases:

i) for $m = 6, 7$ and $2 < n \leq 10, n \neq m$. When $n \leq 2$, both prices decrease.

ii) for $m = 5$ and $3 < n \leq 10, n \neq m$. When $n \leq 3$, both prices decrease.

iii) for $m = 4$ and $4 < n \leq 10$ . When $n \leq 4$, both prices decrease.

iv) for $m = 3$ and $5 < n \leq 10$. When $n \leq 5$, both prices decrease.

v) for $m = 2$ and $7 < n \leq 10$. When $n \leq 7$, both prices decrease.

References


