A Unified Theory of Firm Selection and Growth*

Costas Arkolakis†

Yale University, NBER & CESifo

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Abstract

This paper studies the effects of marketing choice to firm growth. I assume that firm-level growth is the result of idiosyncratic productivity improvements while there is continuous arrival of new potential producers. A firm enters a market if it is profitable to incur the marginal cost to reach the first consumer and pays an increasing market penetration cost to reach additional consumers. The model is calibrated using data on the cross-section of firms and their sales across markets as well as the rate of incumbent firm-exit. The calibrated model quantitatively predicts firm exit, growth, and the resulting firm size distribution in the US manufacturing data. It also predicts a distribution of firm growth rates strikingly different from the one predicted by models postulating the independence of firm-size and growth, i.e. Gibrat's law.

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†Contact: Department of Economics, Yale University, 28 Hillhouse ave., New Haven, CT, 06511. Email: costas.arkolakis@yale.edu. web: http://www.econ.yale.edu/~ka265/research.htm

1 Introduction

In the last few decades, economists have performed a systematic investigation of the empirical patterns of turnover and growth of firms. A series of salient features of the data has been uncovered, indicating a robust cross-sectional distribution of firm sales as well as inverse relationships between the size of the firms and their exit and growth rates.[1] Gibrat's law –i.e. the independence of the expected growth rate of firms and their initial size– has been used as a key element of theories that attempt to explain such regularities.[2] While such theories are important in understanding the qualitative nature of firm-level facts, their inability to explain the fast growth of small firms and the stark implications of Gibrat's law resulted to their somewhat limited quantitative success.

This paper integrates a theory of marketing choice into a model of firm dynamics with Gibrat's law. It illustrates that the resulting model can quantitatively predict a number of key firm-level facts and can serve as a useful framework for applications of quantitative nature. In particular, I assume that firm-level growth is the result of idiosyncratic productivity improvements as in Luttmer (2007) and I model marketing costs using the formulation developed by Arkolakis (2010). If a firm is productive enough, it chooses to pay the entry cost to reach a positive fraction of consumers in a market. Additionally, firms have to pay a market penetration cost to reach more consumers in a market.

The proposed setup provides a generalization of previous theories of firm growth based on Gibrat's law. In particular, with constant marginal costs to reach additional consumers the model offers a dynamic extension of the Constant Elasticity of Substitution (CES) multi-market (international trade) setup of Melitz (2003)-Chaney (2008) with Gibrat's law embedded in firm behavior. However, with increasing market penetration costs the model generates a demand elasticity for the firms that declines with firm size and asymptotically tends to the CES demand elasticity.

Moreover, I show that with increasing market penetration costs the model addresses an important shortcoming of the Luttmer (2007) setup. In that model the small firms with negative growth rates are faced with an endogenous exit decision. In turn, endogenous exit implies that the variance of growth rates for small firms is lower, a fact that sharply contradicts empirical

[1] See Audretsch (1995), Sutton (1997), Caves (1998) and Axtell (2001) for a discussion of these findings.
[2] See for example Klette and Kortum (2004), Lentz and Mortensen (2008), and Luttmer (2007) among others.

observations. On the contrary, in the model with increasing market penetration costs the marketing choice for small firms is very volatile leading to an increase in the overall variance of the growth rates of these firms. In addition, because of the market penetration cost choice the model implies a strikingly different distribution of growth rates of small firms vis-a-vis a setup where Gibrat's law holds with some small firms growing at a very fast pace.

The paper also offers a new setup for the entry and exit of firms. Following Kortum (1997) and Eaton and Kortum (2001), I assume that the rate at which new ideas arrive is exogenously given. Each idea can be used by a monopolistically competitive firm to produce a differentiated good and (potentially) earn profits. These ideas become firms only if they are used in production while if not used they enter a "mothballing" state until the next production opportunity arises. This formulation makes the model more tractable than the standard sunk-entry cost setup and overcomes two of its important shortcomings: it implies that the average size of entrants and exitors is the same and that the hazard rates of entrants are high.[3]

To quantitatively assess the predictions of the model, I exploit the cross-sectional restrictions that the multi-market structure of the model imposes. Therefore, I use the same parameterization as in the related static calibration of Arkolakis (2010) since at each snapshot of time the dynamic model is identical to its static multi-market counterpart. In addition, the drift and the variance of the stochastic process governing firm growth have to be determined. These two parameters are calibrated to match two moments in the data: the elasticity of bilateral country trade with respect to trade costs and the survival rate of a cohort of firms from the US census. The elasticity of trade in the model is the shape parameter of the Pareto distribution, and in turn it endogenously arises in the model as a function of the drift and the variance of the stochastic process. This procedure allows me to calibrate the parameters of the model governing the growth of firms without using information on firm growth.

The calibrated model is used to explain the exit and growth of US manufacturing firms as reported by Dunne, Roberts, and Samuelson (1988) for a time span of 2 decades. I argue that assuming a sunk cost of entry implies a size of entrants larger than the size of exitors whereas the entry-exit process I propose closely matches the exit rates of both incumbent and new entrants. Additionally, the model with its benchmark parameterization can generate the small initial size

[3]For evidence on the average size of entrants and exitors and the hazard rates of US manufacturing firms see Dunne, Roberts, and Samuelson (1988). For exporters see Eaton, Eslava, Kugler, and Tybout (2008) and Ruhl and Willis (2008). The hazard rates that these authors report for entrants are high and decline upon entry.

and the growth rates of new entrants over time. The fit of the model is surprisingly good in light of the minimal information on firm exit and no information on firm growth that is used for the calibration.

I then evaluate the ability of the model to predict the size distribution of all US manufacturing firms as well as the size distribution of firms with different ages with data for the Portuguese economy. These distributions are also predicted by the model as a result of its consistent predictions for exit and growth of firms. This success reveals that the cross-sectional phenomena of firm sales are intimately linked to the dynamic ones requiring a "unified" theory to analyze them.

Having build and calibrated this simple theory of firm growth I now use it as a device to understand the subtle question of the relationship between firm size and growth. This question has preoccupied a large empirical literature that uses different measuring methodologies or econometric techniques to asses the relationship between firm size and growth. Nevertheless, it is of paramount theoretical nature, since the conclusion of the empirical analysis heavily depends on the assumptions for the true nature of the stochastic process and firm behavior. Using the predictions for the version of the theory that best matches the data I argue that the relationship of firm growth and size is inverse. In addition, I show that empirical tests that were used to falsify the inverse size growth relationship (such as measuring growth as suggested by Davis, Haltiwanger, and Schuh (1996) and Haltiwanger, Jarmin, and Schuh (2010)) can be misleading: they could record the growth rate of smaller firms as lower even in a model where small firms grow faster.

The above analysis follows a large tradition of models of firm dynamics with a continuum of heterogeneous firms. Such models are examined by Jovanovic (1982), Hopenhayn (1992), Klette and Kortum (2004), Luttmer (2007) among others.[4] In Jovanovic (1982), firm growth depends primarily on age rather than on size, which is a key difference from the framework of this paper.[5]

[4]Irarrazabal and Opromolla (2006) and Atkeson and Burstein (2010) develop two-country extensions of Luttmer (2007). Irarrazabal and Opromolla (2009) adapt a framework of entry and exit similar to the one in this paper. The authors retain the main assumptions of the fixed cost framework (without assuming sunk costs of exporting) and study the theoretically implied entry-exit patterns into individual destinations.

[5]However, in Jovanovic (1982) growth rates can increase or decrease with size (and age) depending on the shape of the cost function of the firms. Additionally, Rossi-Hansberg and Wright (2007) develop a model where plant growth and size are negatively related in different industries due to mean reversion in the accumulation of industry-specific human capital. This related work, while complementary to this paper is a considerable departure from the heterogeneous firm, analytically tractable framework that I consider.

In contrast to Jovanovic (1982), Hopenhayn (1992) and Luttmer (2007) but similarly to Klette and Kortum (2004) the model in this paper offers analytical relationships for the turnover and growth of firms.[6]

This paper is one of the very first to study the implication of marketing choice on firm growth. As argued in Arkolakis (2010) market penetration costs can capture other broader margins of firm expansion such as heterogeneous consumer tastes, multiple products etc. In related work Foster, Haltiwanger, and Syverson (2010) explore the implications of demand accumulation on firm growth. Another strand of the literature (see for example Cooley and Quadrini (2001) and Arellano, Bai, and Zhang (2008)) has looked at the impact of financial constraints on the firm-size growth relationship. Whereas I will argue that market penetration costs offer a set of strong and quantitatively consistent implications for firm growth it is important to note that these hold true for firm sales market-by-market whereas financial constraints explanations applies to the overall firm sales.

As in the celebrated work of Yule (1925) and Simon (1955), I use the two minimal sufficient conditions of random entry-exit and Gibrat's law to generate a cross-sectional distribution with Pareto right tails (see Reed (2001)). Random entry-exit is used in lieu of the assumption of a lower exit or reflective barrier and constitutes a major technical simplification compared to prior related work (see for example Luttmer (2007) and Luttmer (2008) or Gabaix (1999)).

The rest of the paper is organized as follows. Section 2 summarizes the quantitative evidence on firm selection, growth, and size distribution. In section 3, I develop the firm-dynamics framework and in section 4, I provide an analytical characterization of the theoretical predictions of the model. In sections 5 and 6, I calibrate the model and evaluate its predictions with the US census and Colombian export data. Section 7 discusses the implications of the theory for the growth-size debate and Section 8 concludes.

[6] Lentz and Mortensen (2008) and Bernard, Redding, and Schott (2009) develop models of firm dynamics extending the theories of Klette and Kortum (2004) and Hopenhayn (1992) respectively. In turn, their models borrow many of the qualitative features of these theories.

2 Quantitative Facts on Firm Selection, Growth and Size Distribution

This section summarizes the findings of a set of studies that present empirical regularities regarding the exit of firms, their growth, and their cross-sectional size distribution.

2.1 Firm Selection

In the rest of the analysis, the definition "incumbent cohort" includes all the firms that were in the market at a certain census year (normalized as year 0). The survivors of that cohort at year t are the firms from the cohort which also sell in the market at year t. "Entry cohorts" are the firms that enter the market between the current census and the previous one. Thus, by construction, incumbent cohorts include the surviving firms from all past entry cohorts as well as the firms of the current entry cohort. The quantitative facts on firm selection and firm growth, which are discussed below, are based on a set of facts collected by Dunne, Roberts, and Samuelson (1988) for US firms (henceforth DRS). All the data are based on means across manufacturing 4-digit SIC industries.

Figure 2 illustrates the fraction of exiting firms from incumbent and entry cohorts at subsequent censuses. The fraction of exitors is by construction 0% in year 0 of that cohort and increases as more firms of the cohort exit the market. Two facts clearly emerge. First, the exit rates in the census data are very large. After 15 years, only about a quarter of the incumbent cohort firms and around 12% of the entry cohort firms are still active. Second, the exit rates of entry cohorts are consistently higher than that of the incumbent cohort. Since the incumbent cohorts include firms from the current entry cohorts, exit rates of the entry cohorts account for a large part of the overall incumbent cohort exit.

2.2 Firm Growth

I now present evidence from DRS for the increase in the average size of incumbent and entry cohorts in order to illuminate the patterns of growth of incumbent versus new firms. In Figure 3, I plot the average sales of surviving firms from incumbent and entry cohorts. Notice that both Figures 2 and 3 suggest that firm behavior is roughly independent of the cohort choice.

The average size of incumbent cohort firms increases to around 3.2 times the size of all firms in the span of 15 years. Upon entry, the average size of entry cohorts firms is only about 1/3 of the average size of all firms. However, 15 years later the average size of the surviving entry cohort firms is around 30% larger than the size of all firms. Arguably, much of the growth of the average sales of firms, and especially firms in the entry cohort, is accounted by the fact that the exit rates are high. In relation to Figure 2 notice also that the higher exit rates of entrants are consistent with their smaller average size.

Whereas DRS report statistics aggregated by cohorts, a large literature has been devoted in understanding the relationship of firm growth and size using micro data. Research as early as Mansfield (1962) has demonstrated a robust inverse relationship between size and growth of surviving firms in the data. A similar inverse relationship between the variance of firm growth rates and the size of firms has been identified. Some evidence for publicly traded firms are presented by Amaral, Buldyrev, Havlin, Leschhorn, Mass, Salinger, Stanley, and Stanley (1997).[7]

The comprehensive econometric analysis by Evans (1987a) and Evans (1987b) shows that the negative growth-size relationship is robust to controlling for sample truncation caused by the exit of smaller firms. Nevertheless, the inverse size-growth relationship has been challenged by the work of Davis, Haltiwanger, and Schuh (1996) and Haltiwanger, Jarmin, and Schuh (2010). The authors point out the possibility that this finding in the data is generated by either mis-measurement of the initial size of the firm or reversion to the mean. Acknowledging these serious challenges in the measurement of the true relationship between size and growth, I will revisit this debate and use the predictions of the theory to establish an inverse size-growth relationship consistent with the above empirical findings.

2.3 Firm Size distribution

Since data for firm sales distribution are not provided by DRS, I appeal to different data sources. Detailed data on the size distribution of US manufacturing firms are reported by the Small Business Administration. These data report the number of firms at various size bins.

[7] Hart and Oulton (1996) point out that deviations from Gibrat's law appear to vanish when samples of large firms are considered. The large literature studying the empirical validity of Gibrat's law is reviewed by Sutton (1997) and Caves (1998). Evidence regarding the inverse relationship between firm variance of growth and size is summarized by Caves (1998) and Sutton (2002).

The sales distribution is plotted in Figures 4 and 5 where the sales of the firms (divided by mean sales) are in logarithms in both figures while the rank of the firm is not in logarithms in Figure 4. The first graph clearly illustrates that most of the firms in the data are very small and in particular the median firm appears to have less than 10% of the mean sales in the market. Figure 5 zooms in the top decile of the firms and indicates that the size distribution of the larger firms is approximately Pareto. By comparing these two figures one concludes that the distribution of sizes of *manufacturing* firms appears to exhibit large departures from the Pareto distribution, challenging the view postulated by Axtell (2001). These findings are very similar to the findings reported in Eaton, Kortum, and Kramarz (2010) for French manufacturing firms.

The decomposition of the firm size for firms of different age has been the topic of a recent contribution by Cabral and Mata (2003). The authors report non-parametric estimates for the employment size distribution of Portuguese firms proxying firm age with the largest worker tenure in the firm. Their estimates are illustrated in Figure 6 for a variety of age groups. They find that the sales of younger firms sales are concentrated in the lower ends of the size distribution (entrants are typically small) whereas the size distribution of the largest firms converges to a log-normal distribution.

The above facts constitute a collection of key quantitative findings for firm-dynamics. In the next section I lay out the elements of a simple model that can account for the firm-level facts summarized above.

3 The Model

The model introduces market penetration costs, as modeled in Arkolakis (2010), in a firm-dynamics setup similar to Luttmer (2007). I develop a multi-market version of this setup where the decisions of the firms are independent across markets. Thus, the predictions of the model hold for the sales of the firm in each market. Nevertheless, the multi-market structure of the model allows me to map its predictions to international trade models and use estimates from this literature for the calibration by considering distinct markets as actual countries.

3.1 Model Setup

Time is continuous and indexed by t. The importing market is denoted with an index j and the exporting market with i, where $i, j = 1, ..., N$. At each time t, market j is populated by a continuum of consumers of measure $L_{jt} = L_j e^{g_\eta t}$, where g_η is the growth rate of the population, $g_\eta \geq 0$. I assume that each good ω is produced by a single firm and each firm reaches consumers independently from other firms. Therefore, at a given point in time t, a consumer $l \in [0, L_{jt}]$ has access to a potentially different set of goods Ω_{jt}^l. Firms differ ex-ante only in their productivity, z, and their source market i. I consider a symmetric equilibrium where all firms of type z from market i choose to charge the same price in j, $p_{ijt}(z)$, and also reach consumers there with a certain probability, $n_{ijt}(z) \in [0, 1]$. The existence of a large number of firms implies that every consumer from j has access to the same distribution of prices for goods of different types. The existence of a large number of consumers in market j implies that the fraction of consumers reached by a firm of type z from i is $n_{ijt}(z)$ and their total measure is $n_{ijt}(z) L_{jt}$.

Each consumer from market j has preferences over a consumption stream $\{C_{jt}\}_{t \geq 0}$ of a composite good from which she derives utility according to

$$\left(E \int_0^{+\infty} \rho e^{-\rho t} C_{jt}^{\frac{\iota - 1}{\iota}} dt \right)^{\frac{\iota}{\iota - 1}} ,$$

where $\rho > 0$ is the discount rate and $\iota > 0$ is the intertemporal elasticity of substitution. The composite good is made from a continuum of differentiated commodities

$$C_{jt} = \left(\sum_{\upsilon=1}^{N} \int_0^{+\infty} c_{\upsilon jt}(z)^{(\sigma-1)/\sigma} \, dM_{\upsilon jt}(z) \right)^{\frac{\sigma}{\sigma-1}}$$

where $c_{\upsilon jt}(z)$ is the consumption of a good produced by a firm z in market υ and σ is the elasticity of substitution among different varieties of goods where $\sigma > 1$. $dM_{\upsilon jt}(z)$ is the density of goods of a given type z from market i that are actually sold to j. Since consumers from market j have access to the same distribution of prices, their level of consumption C_{jt} is the same.

Each household earns labor income w_{jt} from selling its unit labor endowment in the labor market and profits π_{jt} from the ownership of domestic firms. Thus, the demand for good z from

i by a consumer from market j is

$$c_{ijt}(z) = \frac{p_{ijt}(z)^{-\sigma}}{P_{jt}^{1-\sigma}} y_{jt} \ ,$$

where $y_{jt} = w_{jt} + \pi_{jt}$ and

$$P_{jt}^{1-\sigma} = \sum_{v=1}^{N} \int_{0}^{+\infty} p_{vjt}(z)^{1-\sigma} n_{vjt}(z) \, dM_{vjt}(z) \ . \tag{1}$$

Given the definition of the price index, P_{jt}, the budget constraint faced by each consumer is $C_{jt}P_{jt} = y_{jt}$. Thus, the total effective demand in market j for a firm of type z from i is

$$q_{ijt}(z) = n_{ijt}(z) L_{jt} \frac{p_{ijt}(z)^{-\sigma}}{P_{jt}^{1-\sigma}} y_{jt} \ . \tag{2}$$

3.2 Entry and Exit

An 'idea' is a way to produce a good ω. Each idea is exclusively owned and grants a monopoly over the related good. This exclusivity implies a monopolistic competition setup as in Dixit and Stiglitz (1977) and Melitz (2003). However, in my context ideas become firms only if they are used into production. If not, they enter a "mothballing" state until the next production opportunity arises. Once ideas are born, they can only die at an exogenous rate $\delta \geq 0$. In order to consider an economy that is consistent with balanced growth, I also assume that each market innovates at an exogenous rate $g_B \geq \delta$. This rate will be specified when I construct the balanced growth path and implies that the measure of existing ideas at each time t in i is $J_i e^{(g_B - \delta)t}$, where $J_i > 0$ is the initial measure of ideas in i.

New ideas arrive with an initial productivity, $\bar{z}_{it}$, where

$$\bar{z}_{it} = \bar{z}_i \exp(g_E t) \ ,$$

and $\bar{z}_i, g_E > 0$, and their productivity evolves over time as it will be specified in the next subsection. The parameter g_E is interpreted as the growth rate of the frontier of new ideas and all new ideas at time t enter with the same productivity. This specification incorporates a form of "creative destruction" since more recent ideas arrive with a higher productivity.[8] In fact, I show that, in the balanced growth path, there exists a lower productivity threshold of operation at each time t, z^*_{ijt}, and this threshold grows at a rate g_E. z^*_{ijt} is determined by the zero profit condition at each point in time: since there is no indivisible cost of production or entry, ideas with productivity higher than z^*_{ijt} are used into production and appear as firms in market j. If an idea is not productive enough to be profitable, it remains "mothballed" while waiting for the possibility to become profitable in the future (when its productivity surpasses z^*_{ijt} at a given time t).[9]

The assumption of this setup for entry and exit make this model substantially more tractable than that of Luttmer (2007) since there are no forward looking decisions for the firms. Nevertheless, the model retains the main desirable properties of the Luttmer setup as I illustrate in section 4.

3.3 Firms and Ideas

The productivity of an idea is the same in all markets and evolves, independently across ideas, according to

$$z_{t^b,a} = \bar{z}_i \exp\left(g_E t^b + g_I a + \sigma_z W_a\right) \ , \tag{3}$$

where $z_{t^b,a}$ is the labor productivity of the idea at age a that was born at time t^b. $W_a \sim \mathcal{N}(0,a)$ is a Brownian motion with independent increments and the parameter σ_z regulates the volatility of the growth of ideas. Note that the productivity of incumbent ideas is improving on average at a rate g_I. This evolution process for productivities is adapted by Luttmer (2007) and it implies that the expected growth of productivities is independent of firm size. Similar processes have been widely used to represent firm growth since Gibrat (1931). The Brownian motion assumption naturally emerges as the continuous time limit of a firm growth rate that is a

[8] Extending this simple case to one in which new entrants arrive with different productivities drawn from a non-atomic distribution is straightforward (see, for example, Reed (2002)). In particular, unless entrants are specified to be very large with a high probability the right tails of the distribution will be unaffected. In addition, the process of growth of ideas and firms is not affected by entry.

[9] In the one market model, allowing for free entry of ideas with a fixed amount of labor used for each new idea would imply a setup with identical predictions. The only difference would arise because profits would accrue to labor used for the setup cost as in Arkolakis, Demidova, Klenow, and Rodríguez-Clare (2008). However, such an extension would require further changes in the multi-market framework given that profits from entry arise from the operation into multiple markets.

discrete-random walk. Notably, the assumption of continuous time is not key for the results since what is important for predictions on firm growth is that the growth rates are normally distributed. It will simply be convenient in analytically characterizing the properties of the model.

I assume that products markets clear every period and firms produce using a constant returns to scale production function $q\left(z_{t^b,a}\right) = z_{t^b,a} l$, where l is the amount of labor used in production. Moreover, firms pay a market penetration cost which is a function of the number of consumers reached in a given market. I model these market penetration costs using the specification of Arkolakis (2010) derived from first principles as costs of marketing. I also assume these costs are incurred by the firms at each instant of time. The specification used for the costs of marketing in each market is employed to generalize the assumption used by previous models. For example, Melitz (2003), Luttmer (2007) assume that a per market fixed cost is required for the firm to operate at each point of time.[10]

The labor required for a firm to reach a fraction of consumers n in a market of population size L is

$$F\left(n, L\right) = \begin{cases} \frac{L^{\alpha}}{\psi} \frac{1-(1-n)^{-\beta+1}}{-\beta+1} & \text{for } \beta \in [0,1) \cup (1,+\infty) \\ -\frac{L^{\alpha}}{\psi} \log\left(1-n\right) & \text{for } \beta = 1 \end{cases} .$$

where $\alpha \in [0,1]$ and $\psi > 0$. If $\alpha < 1$, the market penetration costs to reach a certain number of consumers decrease with the population size of the market. The parameter β governs the convexity of the marketing cost function: higher β implies more convexity and steeper increases in the marginal cost to reach more consumers. For simplicity, I assume that labor from the destination market is hired for marketing purposes. This specification implies that the total market penetration cost paid by a firm from i that reaches n fraction of consumers in market j is $w_j F\left(n_{ij}, L_j\right)$.

In addition to the cost to reach consumers, the firm has to pay a variable trade cost modeled in the standard iceberg formulation. This iceberg cost implies that a firm operating in i and selling to market j must ship $\tau_{ij} > 1$ units in order for one unit of the good to arrive at the destination. For simplicity, I assume that $\tau_{ii} = 1$.

[10] A model that considers dynamics in marketing by examining state dependence of market penetration costs on previous marketing is left for future research. Drozd and Nosal (2008) and Gourio and Rudanko (2010) develop models where a representative firm's demand is modeled as marketing capital that accumulates over time. My modeling of marketing is static which implies a very tractable framework and allows me to study the effects of marketing on firm-level growth.

3.4 Firm Optimization

Given the constant returns to scale production technology and the separability of the marketing cost function across markets, the decision of a firm to sell to a given market is independent of the decision to sell to other markets. Total profits of a particular firm are the summation of the profits from exporting activities in all markets $j = 1, ..., N$ (or a subset thereof). Thus, at a given time t, the firm's problem is the same as in Arkolakis (2010), and firm z from i solves the following static maximization problem for each given market j:[11]

$$\begin{aligned}\pi_{ijt}(z) = & \max_{n_{ijt}, p_{ijt}} \left\{ n_{ijt} L_{jt} y_{jt} \frac{p_{ijt}^{1-\sigma}}{P_{jt}^{1-\sigma}} - n_{ijt} L_{jt} y_{jt} \frac{\tau_{ij} p_{ijt}^{-\sigma} w_{it}}{P_{jt}^{1-\sigma} z} - w_{jt} \frac{L_{jt}^{\alpha}}{\psi} \frac{1-[1-n_{ijt}]^{-\beta+1}}{-\beta+1} \right\} \\ & \text{s.t. } n_{ijt} \in [0,1] \ \ \forall t \ .\end{aligned}$$

For any β, the optimal decisions of the firm in the multi-market model are:

$$p_{ijt}(z) = \tilde{\sigma} \frac{\tau_{ij} w_{it}}{z} \tag{4}$$

where

$$\tilde{\sigma} = \frac{\sigma}{\sigma - 1} \ ,$$

and

$$n_{ijt}(z) = \max \left\{ 1 - \left(\frac{z_{ijt}^*}{z} \right)^{(\sigma-1)/\beta}, 0 \right\} \ . \tag{5}$$

z_{ijt}^* is defined by

$$z_{ijt}^* = \sup \{ z : \pi_{ijt}(z) = 0 \} \ , \tag{6}$$

and thus

$$z_{ijt}^* = \left[L_{jt}^{1-\alpha} y_{jt} w_{jt}^{-1} (\tilde{\sigma} \tau_{ij} w_{it})^{1-\sigma} \psi P_{jt}^{\sigma-1} / \sigma \right]^{-1/(\sigma-1)} \ . \tag{7}$$

Equation (7) reveals that apart from general equilibrium considerations, z_{ijt}^*, and thus the entry-exit decision of the firm, does not depend on the parameter β.

Substituting (4), (5) and (7) into the expression for sales per firm, (2), and multiplying it by the price, equation (4), the sales of firm z originating from market i in market j can be written

[11] Slighly abusing the notation, I denote the decision of the firm only as a function of its productivity z, supressing time of birth and age information. Given that the optimization decision is static, the current level of productivity is the only state variable. I keep the notation parsimonious throughout the text whenever possible.

as

$$r_{ijt}(z) \equiv p_{ijt}(z)\, q_{ijt}(z) = \begin{cases} L_{jt}^{\alpha} y_{jt} \frac{1}{\tilde{\psi}} \left[e^{\bar{c}_1 \ln\left(z/z_{ijt}^*\right)} - e^{\bar{c}_2 \ln\left(z/z_{ijt}^*\right)} \right] & \text{if } z \geq z_{ijt}^* \\ 0 & \text{otherwise} \end{cases}, \tag{8}$$

where

$$\bar{c}_1 = \sigma - 1,\ \bar{c}_2 = (\sigma - 1)\frac{(\beta - 1)}{\beta},\ \tilde{\psi} = \frac{\psi}{\sigma(1 - \bar{\pi})},$$

and $\bar{\pi} \equiv \pi_{it}/y_{it}$ is the fraction of profits out of total income. In the balanced growth path equilibrium, this fraction is constant and thus, I suppress its subscripts. Equation (8) reveals that for $\beta = 0$ all firms selling from i to j sell a minimum amount, $L_{jt}^{\alpha} y_{jt}/\tilde{\psi}$, while for $\beta > 0$ this amount is 0. Conditional on entry, more productive firms have higher sales as equation (8) indicates. These firms charge lower prices and thus sell more per consumer (i.e. at the intensive margin). In addition, if $\beta > 0$, they also reach more consumers (i.e. the extensive margin) as implied by equation (5). However, if $\beta = 0$, all entrants optimally choose $n_{ij} = 1$. Differences in β also reflect different growth patterns for firm sales as I will illustrate in section 4.

3.5 Balanced Growth Path Equilibrium

To solve for the cross-sectional distribution, I consider the stationary balanced growth path. I first define the productivity detrended by the rate of growth of the zero profit cutoff,

$$\begin{aligned} \phi_a &= \bar{z}_i \exp\left\{g_E t^b + g_I a + \sigma_z W_a\right\} / \exp\left\{g_E\left(t^b + a\right)\right\} \\ &= \bar{z}_i \exp\left\{(g_I - g_E)\, a + \sigma_z W_a\right\}\ . \end{aligned}$$

Given expression (8), the dynamic behavior of ϕ_a is sufficient to characterize the relative sales of an idea at each time a in the balanced growth path.

The logarithm of ϕ_a is a Brownian motion with a drift,

$$s_a = \ln \phi_a = \bar{s}_i + (g_I - g_E)\, a + \sigma_z W_a\ , \tag{9}$$

where $\bar{s}_i = \ln \bar{z}_i$. s_a will be used as a proxy for the productivity of an idea or the size of a firm after a years given that firms with larger s_a are (weakly) larger in sales, productivity and employment. The term $g_I - g_E$, i.e. the difference between the growth of incumbent ideas and the growth of the frontier of new ideas. Hereafter, I will denote this difference by μ. The probability density of the logarithm of productivities, $s_a = s$, for a given generation of ideas of age $a > 0$ from i is given by the normal density:[12]

$$f_i(s,a) = \frac{1}{\sigma_z\sqrt{a2\pi}} \exp\left\{ -\left(\frac{s - \bar{s}_i - \mu a}{\sigma_z \sqrt{a}} \right)^2 /2 \right\} . \tag{10}$$

This distribution will allow the model to match the findings of Cabral and Mata (2003) that the size distribution of old firms converges to a log-normal distribution.[13]

The distribution of sizes of one age cohort of ideas is not age-stationary. But considering the size distribution of all ideas, across different ages, delivers a stationary cross-sectional distribution of productivities of all ideas from i, $f_i(s)$. In a stationary equilibrium, with entry and exit of ideas, the dynamics of the probability density of each $s \neq \bar{s}_i$ and $\forall i$, are described by a Kolmogorov forward equation,[14]

$$-\mu f_i'(s) + \frac{1}{2}\sigma_z^2 f_i''(s) - g_B f_i(s) = 0 . \tag{11}$$

Intuitively, in a stationary steady state, the net changes at each point s of the distribution must equal the rate of reduction of the probability density at $s \in (-\infty, \bar{s}_i) \cup (\bar{s}_i, +\infty)$,

$$\underbrace{g_B}_{\substack{\text{rate of}\\ \text{reduction}}} = \delta + \underbrace{g_B - \delta}_{\substack{\text{entry rate}\\ \text{at point } \bar{s}_i}}.$$

[12] See for example Harrison (1985) p. 37. $f_i(s,a)$ can be derived as the solution of the differential equation $D_a f_i(s,a) = -\mu f_i'(s,a) + \frac{1}{2}\sigma_z^2 f_i''(s,a)$, with initial condition $f_i(s,a) = \Delta(s - \bar{s}_i)$, where $\Delta(.)$ is the Dirac delta function. Additionally, the realizations of the Brownian motion over different time periods, $s_{a_1}, s_{a_2}, ..., s_{a_n}$, follow a multivariate normal distribution with means $Es_a = s_0 + \mu a$ and covariances $Cov(s_a, s_{a'}) = \sigma_z^2[\min(a,a')]$. This feature can be used to implement further scrutiny on the model, or to pursue an alternative estimation of its parameters, by looking at the probability distribution of sales and entry exit decisions of individual firms overtime, for researchers that have access to this information.

[13] Amaral, Buldyrev, Havlin, Leschhorn, Mass, Salinger, Stanley, and Stanley (1997) study the distribution of sizes of US manufacturing firms in the compustat database and they also find evidence for lognormal distribution of sizes. Firms that typically end up appearing in the database are typically both older and larger since they are the firms that are publicly-traded.

[14] In an appendix available online, I provide a different proof by explicitly calculating $f(s) = \int_0^{+\infty} e^{-[g_B]a} f(s,a)\, da$. That proof, whereas it is more straightforward, provides less intuition on the exact forces that give rise to the cross sectional distribution of productivities across all ideas. Reed (2001) provides another proof using moment generating functions in which the intuition is also somewhat limited.

The net changes are due to the stochastic flows of productivities in and out of that point described by equation (9).

The density of productivities, $f_i(s)$, has to satisfy a set of conditions. The first requirement is that $-\infty$ is an absorbing barrier which implies the condition

$$\lim_{s \to -\infty} f_i(s) = 0 \; . \tag{12}$$

In addition $f_i(s)$, must be a probability density which implies that

$$f_i(s) \geq 0 \; , \;\; \forall s \in (-\infty, +\infty) \tag{13}$$

and

$$\int_{-\infty}^{\bar{s}_i} f_i(s)\, ds + \int_{\bar{s}_i}^{+\infty} f_i(s)\, ds = 1 \; . \tag{14}$$

Additionally, net inflows into the distribution must equal the net outflows:[15]

$$-\mu \left[f_i(\bar{s}_i-) - f_i(\bar{s}_i+) \right] + \frac{1}{2}\sigma_z^2 \left[f_i'(\bar{s}_i-) - f_i'(\bar{s}_i+) \right] = g_B \; . \tag{15}$$

The left-hand side is the net inflows into the distribution from point $\bar{s}_i$. The right-hand side is the outflows from the distribution due to new entry and random exit of ideas. By continuity, the first term in brackets is zero. However, entry of new ideas implies that the distribution is kinked at $\bar{s}_i$. Intuitively, the rate of change of the cdf changes direction at $\bar{s}_i$ because entry happens at that point. The solution of the above system is (see appendix A.2):

$$f_i(s) = \begin{cases} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2} e^{\theta_1 (s - \bar{s}_i)} & \text{if } s < \bar{s}_i \\ \frac{\theta_1 \theta_2}{\theta_1 + \theta_2} e^{-\theta_2 (s - \bar{s}_i)} & \text{if } s \geq \bar{s}_i \end{cases} \tag{16}$$

where

$$\theta_1 = \frac{\mu + \sqrt{\mu^2 + 2\sigma_z^2 g_B}}{\sigma_z^2} > 0 \; , \tag{17}$$

[15] This condition results by integrating (11) over all $s \in (-\infty, \bar{s}_i) \cup (\bar{s}_i, +\infty)$, i.e. considering the net inflows from point $\bar{s}_i$ to the rest of the distribution. Similar conditions are used in labor models to characterize the behavior of the distribution at a point of entry to or exit from a particular occupation (see for example Moscarini (2005) and Papageorgiou (2008)).

$$\theta_2 = -\frac{\mu - \sqrt{\mu^2 + 2\sigma_z^2 g_B}}{\sigma_z^2} > 0 \ . \tag{18}$$

The following assumption guarantees that a time-invariant distribution exists and an ever increasing fraction of ideas is not concentrated in either of the tails of the distribution:[16]

A 1 *: The rate of innovation is positive,* $g_B > 0$*.*

In particular, given that $g_B \geq \delta$, having $\delta = 0$ A1 implies that $g_B > \delta$. Using equation (18), A1 also implies that

$$\theta_2 \mu + (\theta_2)^2 \sigma_z^2 / 2 = g_B > 0 \ . \tag{19}$$

The resulting cross-sectional distribution of detrended productivities $\phi \in [0, +\infty)$ is the so-called double Pareto distribution (Reed (2001)) with probability density function:[17]

$$\hat{f}_i(\phi) = \begin{cases} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \frac{\phi^{\theta_1 - 1}}{\bar{z}_i^{\theta_1}} & \text{if } \phi < \bar{z}_i \\ \frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \frac{\phi^{-\theta_2 - 1}}{\bar{z}_i^{-\theta_2}} & \text{if } \phi \geq \bar{z}_i \end{cases} \tag{20}$$

The double Pareto distribution is illustrated in Figure 1. A closer look at the probability density of productivities (equation (20)) reveals that at each moment of time, a constant fraction of ideas $\theta_1 / (\theta_1 + \theta_2)$ is above the threshold $\bar{z}_i$. To keep all the expressions of the model as simple as possible, I assume for the rest of the paper that $1/\psi$ is sufficiently high so that $z_{ijt}^* > \bar{z}_{it}$, $\forall i$, t . Thus, the (detrended) cross-sectional distribution of operating ideas (i.e. firms) is Pareto at $[\bar{z}_i, +\infty)$ with shape parameter θ_2. Whereas A1 is necessary for a stationary distribution an additional assumption guarantees that the resulting distributions of firm productivities and sales have a finite mean:

A 2 *: Productivity and sales parameters satisfy*

$$g_B > \max\left\{\mu + \sigma_z^2/2 \ , \ (\sigma - 1)\mu + (\sigma - 1)^2 \sigma_z^2/2\right\} \ .$$

[16] Under the assumption $\mu > 0$, Pareto distribution emerges in the right-tail of the distribution for the limit case of $\sigma_z \to 0$. However, both $\mu < 0$ and $\sigma_z > 0$ will be essential features of the model in explaining the data as I illustrate in the calibration section.

[17] This distribution can also be thought of as a limit case of the distribution of firms derived by Luttmer (2007) when the exit cutoff goes to $-\infty$. However, in his case, this assumption would imply that firms never exit and that there is no selection in the model.

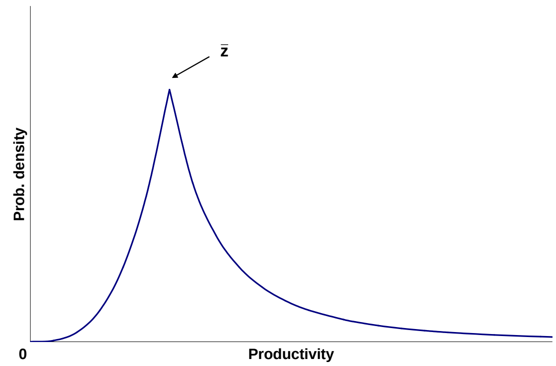

Figure 1: Double Pareto distribution

Assumption A2 implies that the entry rate of new ideas is larger than the growth rate of productivities and sales of the most productive incumbent firms. Notice that A2 and A1 (see equation (19)) also imply the common restriction that the Pareto shape coefficient, θ_2, is larger than both 1 and $\sigma - 1$. To summarize, A2 and A1 imply a set of restrictions –not necessarily independent– between μ, σ_z, σ and g_B.

I will now construct a balanced growth path equilibrium for this economy. To do so I assume that the entry rate of new ideas is

$$g_B = g_\eta (1 - \alpha) + \delta \ , \tag{21}$$

implying that the number of ideas above the entry point will be $\theta_1 / (\theta_1 + \theta_2) J_i e^{g_\eta (1-\alpha) t}$. Aggregate variables, w_{it}, C_{it} grow at a rate g_κ where

$$g_\kappa = g_E + g_\eta (1 - \alpha) / (\sigma - 1) \, . \tag{22}$$

The growth rate of the ideas and thus the varieties adds to the growth rate of the frontier of new productivities, g_E, with a rate that is larger when goods are less substitutable.[18]

Finally, notice that in the balanced growth path the cross-sectional distribution of firm sales and the bilateral trade shares, λ_{ij}, remain unchanged. This means that at each snapshot of

[18] The equilibrium also requires that the value of the aggregate endowment is finite. In order for this to happen the discount rate must exceed the rate of growth of the economy and thus preference and technology parameters must satisfy $\rho + \frac{1}{\iota} g_\kappa > g_\kappa + g_\eta$. This restriction and in particular the values of the parameters ρ and ι play no essential role in my analysis and will not be discussed altogether in what follows.

time, this model collapses to the endogenous cost model of Arkolakis (2010) when $\beta > 0$ and the fixed cost Chaney (2008) model when $\beta \to 0$.

Proposition 1 *Given A1-A2, and the values of g_κ, g_B given by the equations (22) and (21) respectively there exists a balanced growth path for the economy.*

Proof. *By assumption we have that $L_{it} = L_i e^{g_\eta t}$ and $J_{it} = J_i e^{g_\eta (1-\alpha) t}$, and $\bar{z}_{it} = \bar{z}_i \exp(g_E t)$. Define $z^*_{ijt} = z^*_{ij} e^{g_E t}$, such that $z^*_{ij} > \bar{z}_i$, $w_{it} = w_i e^{g_\kappa t}$, $C_{it} = C_i e^{g_\kappa t}$, $P_{it} = P_i$. Given these assumptions and definitions, the cross-sectional distribution of the productivities of operating firms is Pareto. For each cross section of the model, the share of profits in total income equals $\bar{\pi} = (\sigma - 1) / (\sigma \theta_2)$ (see Arkolakis (2010)) and the market share of i to j equals to*

$$\lambda_{ij} = (\tau_{ij})^{-\theta_2} w_i^{-\theta_2} J_i (\bar{z}_i)^{\theta_2} / \left[\sum_{\upsilon=1}^{N} (\tau_{\upsilon j})^{-\theta_2} J_\upsilon (\bar{z}_\upsilon)^{\theta_2} w_\upsilon^{-\theta_2} \right] . \tag{23}$$

In turn, the equilibrium variables w_{it}, P_{it}, z^*_{ijt} are characterized by the trade balance condition $w_i L_i = \sum_\upsilon \lambda_{i\upsilon} w_\upsilon L_\upsilon$, $\forall i$, the price index given by (1), $\forall i$, and the productivity cutoff condition given by (7) for $\forall i, j$. Simply substituting the guessed values of the variables into these equilibrium equations reveals that the guess is correct since the equations hold for $\forall t$. It also allows to solve for the values of z^*_{ij}, w_i, P_i using the same equations. Finally, C_i, can be solved using the budget constraint completing the construction of the balanced growth path. ■

Moreover, although it is not necessary for the existence of a balanced growth path, I will, in general, restrict the analysis to a parameterization that will allow me to match the facts on firm growth rates as a function of firm size. This parameterization will imply that the productivity growth of firms is not too negative, so that there is positive growth, on average, in the extensive margin of consumers for the smaller firms.

R 1 *: Productivity and sales parameters satisfy $\mu(\sigma - 1) + (\sigma - 1)^2 \sigma_z^2 > 0$.*

This restriction will hold true in the calibration.

4 Theoretical Predictions of the Model

I will now proceed to describe the theoretical properties of the model and sketch the connection to the empirical findings presented in section 2. The proofs of the main results and claims are relegated to the theoretical appendix.

To facilitate exposition I will define some additional notation. Aside from the fact that there is exogenous death of ideas, the productivity of an idea can be considered at a given time $\tilde{t}$ as a new process starting from current productivity $z_{\tilde{t}}$. For convenience, I define a proxy of the relative "size" of an idea from a given origin i to a given destination j when a years have elapsed from some reference time $\tilde{t}$ as,

$$s_{ija} \equiv \ln \frac{z_{\tilde{t}+a}}{z^*_{ij\tilde{t}+a}} \text{ , } a \geq 0. \tag{24}$$

s_{ija} follows a Brownian motion with initial condition s_{ij0}, drift μ, and standard deviation σ_z. Notice that given the expression for sales, equation (8), the variable s_{ij0} and the aggregate variables summarize current firm behavior in market j. In particular, if $s_{ij0} < 0$ the firm does not currently sell in market j.

4.1 Entry and Exit of Firms

I first derive analytical relationships for firm and cohort survival rates that provide a more intuitive interpretation of the workings of firm-selection in the model. In particular, the survival function for a firm of initial size $s_{ij0} = s_0 > 0$, $S_{ij}\left(a|s_{ij0} = s_0\right)$, is defined as the probability of selling in market j after a years conditional on initial size in the market, and is given by

$$S_{ij}\left(a|s_{ij0} = s_0\right) = e^{-\delta a} \Phi \left(\frac{s_0 + \mu a}{\sigma_z \sqrt{a}} \right) \text{ .} \tag{25}$$

where $\Phi\left(\cdot\right)$ denotes the cdf of the normal distribution. This expression implies that firms with larger initial size in a market, $s_{ij0} = s_0 > 0$, have higher probability of selling in this market next period. Notice that the survival function only depends on the size of the firm in the market, s_0, implying that the probability of exit of a firm in a market depends on its relative size there.

Integrating the firm survival rates across different initial sizes the model delivers an analytical characterization of the survival rates of a given cohort of firms from i that sell to j,

$$S_{ij}\left(a\right) = e^{-\delta a} \left[\Phi \left(\frac{\mu}{\sigma_z} \sqrt{a} \right) + e^{a\left(\frac{\theta_2^2 \sigma_z^2}{2} + \theta_2 \mu\right)} \Phi \left(-\frac{\theta_2 \sigma_z^2 + \mu}{\sigma_z} \sqrt{a} \right) \right] \text{ .} \tag{26}$$

This expression is strictly decreasing in the cohort age a if $\mu < 0$.[19] The expression depends on the dispersion of the productivity distribution governed by θ_2. This parameter determines the number of firms whose productivity is close to the threshold of exit, z^*_{ijt}, at each time and thus the number of firms that are likely to exit in the future.

Overall, the model is qualitatively consistent with the evidence on firm exit illustrated in section 2. In particular, consistent with Figure 2 the model generates high attrition for new entrants since these firms typically enter with small size, in line with the empirical evidence summarized by Caves (1998). In addition the model is consistent with the decreasing survival rates of a given cohort of firms illustrated in Figure 2. The key driving force of these results is the selection forces adapted through the use of stochastic process as in Luttmer (2007) but by dispensing of the assumption of a sunk cost of entry. Notice that given the process for individual productivities, equation (24), and the resulting size distribution of ideas (20), the parameters β and σ do not play any role in entry and exit but will play a critical role for understanding firm growth and firm size distribution. I now turn to analyze a critical departure of the model from previous literature generated by the key assumption of market penetration costs: the substantially different growth patterns for small versus large firms.

4.2 Firm Growth

Given the process that embeds Gibrat's law in productivity growth and the CES demand specification, two distinct forces act so that Gibrat's law does not hold for all firms in the model: the selection effects and the market penetration technology. I turn to analyze each force separately.

4.2.1 Firm Selection and Firm Growth

I first examine the changes in the mean and the variance of the natural logarithm of sales for the case of $\beta \to 0$.[20] The moments of the logarithm of sales function can be obtained using the moment generating function, derived in appendix A.3.4. In this case, I define the growth over

[19] Since the empirically relevant case will turn out to be $\mu < 0$ I will mainly discuss the prediction of the model under this restriction in the main text.

[20] The use of these statistics by applied economists is quite common in the literature. For example, Hall (1987) and Evans (1987b) empirically study the effects of selection on expected growth and variance of (primarily) employment as well as sales of firms.

the period of a years as $\hat{G}_{ija} = \log r_{ij\tilde{t}+a}(s_{ija}) - \log r_{ij\tilde{t}}(s_{ij0})$. The expected firm growth given initial size is[21]

$$E\left(\hat{G}_{ija}|s_{ija} > 0, s_{ij0} = s_0\right) = (\alpha g_\eta + g_\kappa)\, a + (\sigma - 1)\,\mu a + (\sigma - 1)\,\sigma_z \sqrt{a} m\left(-\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right). \quad (27)$$

where $m(x) = \varphi(x)/\Phi(-x)$ is the inverse Mills ratio, with $\varphi(x)$ the pdf of the standard normal distribution. The third term of this expression appears because of selection and is decreasing in size, s_0, and converging to 0 for large s_0 (see appendix A.1, property F4). Thus, the force of selection by itself implies that growth rates are declining in initial size but since larger firms are unaffected by this force Gibrat's law approximately holds for the largest firms.

The variance of firm growth given initial size is

$$\mathcal{V}\left(\hat{G}_{ija}|s_{ija} > 0, s_{ij0} = s_0\right) = (\sigma - 1)^2\,\sigma_z^2 a\left\{1 - m\left(-\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right)\left[m\left(-\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right) + \frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right]\right\}.$$

The term in the brackets incorporates the effects of selection and can be shown that it is increasing in its argument, $\frac{s_0+\mu a}{\sigma_z\sqrt{a}}$.[22] In turn, $\mathcal{V}$ is increasing in s_0 and in fact for large s_0 it converges to $(\sigma - 1)^2\,\sigma_z^2 a$. Straightforward intuition implies that given that the normal distribution of growth rates is unimodal, censoring of the negative growth rates will reduce the variance of firm growth rates.

A number of instructive conclusions can be derived from the above derivations. First, the selection mechanism alone implies that surviving small firms grow faster than larger firms. Second, the same mechanism also implies that the variance of firm growth would decrease with firm size, an implication in sharp contrast to the robust inverse relationship discussed in section 2. Thus, the model with the selection mechanism alone is unable to generate the salient features

[21] The correction for the selection bias is different from the specification of Heckman (1979) in that entry and sales decision are perfectly correlated in my case (both driven by productivity shocks). Partial correlation can be generated, for example, if there exists randomness in a term that would influence entry but is not perfectly correlated to sales. The obvious candidate term in this model is the parameter $1/\psi$ in the costs of entry. In such an event the second term of equation (27) would depend on the covariance of the uncertainty of the shocks to entry and sales. The econometric techiques developed to adjust for selection bias by Heckman (1979) could be appropriate for this case. Such an approach has been used by Evans (1987b).

[22] The proof can be found in Sampford (1953). More generally, the result that the left truncated variance is decreasing in the truncation point (and thus is increasing in the size of the firm) holds for all distributions with logconcave pdf (see An (1998)). This set of distributions includes the normal. Logconcavity implies unimodality of the distribution but not the other way. The sales of the firm r_{ijt} are lognormally distributed but the log normal distribution is neither logconcave nor logconvex in its entire domain. Thus, an analytical relationship for the variance of growth G_t can be obtained its variance is not in general monotonic in size.

on firm growth.[23]

4.2.2 Market Penetration Technology and Firm Growth

I now turn to study the effects of different specifications in the market penetration technology on the growth of firms, independently from the selection effect. To do so, I characterize the instantaneous growth rate of the firm, which is not affected by entry and exit. This analysis can be performed by applying Ito's lemma to expression (8) for firms with initial size $s_{ij0} = s_0 > 0$,[24]

$$\frac{dr_{ij\tilde{t}}(s_0)}{r_{ij\tilde{t}}(s_0)} = \left[\alpha g_\eta + g_\kappa + \mu \frac{h'(s_0)}{h(s_0)} + \frac{1}{2}\sigma_z^2 \frac{h''(s_0)}{h(s_0)}\right] da + \left[\sigma_z \frac{h'(s_0)}{h(s_0)}\right] dW \tag{28}$$

where

$$h(s_0) = e^{\bar{c}_1 s_0} - e^{\bar{c}_2 s_0}$$

In equation (28) the first and second parenthetical terms represent the (instantaneous) growth, $E(dr/r)$, and the standard deviation of growth of a firm of size s_0 respectively.[25] Proposition 2 characterizes the relationship between the instantaneous growth rates of firms of size $s_{ij0} = s_0$ in a given destination for different values of β:

Proposition 2 *Given A1-A2 and R1,*

a) If $\beta \to 0$ the growth rate of all firms is the same.

b) There exist a $\beta' \in (0, +\infty)$, such that $\forall\ \beta > \beta'$, it is $\partial(E(dr/r))/\partial s_0 < 0$, and $\forall \beta < \beta'$, $\partial(E(dr/r))/\partial s_0 > 0$ for all firms with $s_0 > 0$.

Proof. *The proof is somewhat instructive for the effects of market penetration technology on growth. To prove part (a) of proposition 2 I use De l' Hospital rule to compute the terms in expression (28) for $s_{ij0} = s_0$,*

$$\frac{h'(s_0)}{h(s_0)} \xrightarrow{\beta \to 0} (\sigma - 1)\ , \tag{29}$$

[23] In the Klette and Kortum (2004) model, the variance unconditional on survival is inversely proportional to firm size. The decrease in the variance with firm size happens since the sales of the firm are proportional to the number of goods that the firm has. Since each good has the same variance, the total variance of firm sales is inversely proportional to firm size in that model.

[24] Since the Brownian motion paths are not differentiable with probability 1 and exhibit infinite variation for any given time interval standard calculus does not apply. The application of Ito's Lemma requires the sales function to have a continuous second derivative (see for example Oksendal (2003), ch. 4). The function $h(s)$ is continuously differentiable at any order for $s > 0$ but it does does not attain continuous derivatives at $s = 0$.

[25] See Dixit and Pindyck (1994) chapter 3 for the details of Ito's lemma and related derivations.

$$\frac{h''(s_0)}{h(s_0)} \overset{\beta \to 0}{\longrightarrow} (\sigma - 1)^2 . \tag{30}$$

To prove part (b) I look at the derivative of the first parenthetical term in expression (28) with respect to s_0. In appendix A.4, I show that the sign of this derivative is negative if and only if

$$\beta \geq \frac{(\sigma - 1)^2 \sigma_z^2}{2\left[\mu(\sigma - 1) + (\sigma - 1)^2 \sigma_z^2\right]} > 0 . \tag{31}$$

Thus, if R1 is not satisfied there exists no value of β for which the growth rates are decreasing in size. Notice that the growth rate for very large firms, $s_0 \to \infty$, is the same as the growth rate of all firms for $\beta \to 0$. ■

For $\beta > 0$, the model with endogenous market penetration costs also predicts an inverse relationship between the sales of firms in a market and the instantaneous variance of their growth rates for that market as illustrated in the next proposition:[26]

Proposition 3 *Given A1-A2*

a) If $\beta \to 0$, the instantaneous variance of the growth rate of sales of firms in a destination is independent of their initial size.

b) If $\beta > 0$, the instantaneous variance of the growth rate of sales of firms in a destination is higher the smaller their initial size.

Proof. *See appendix A.5.* ■

Overall, the endogenous cost model, with a high enough β, predicts an inverse size-growth relationship and also has the potential to overcome the shortcoming of the fixed cost model by correctly predicting the size-variance of growth relationship. As we will see in the calibrated examples the distribution of growth rates that the models with different β predict are strikingly different.

[26] Recent explanations that can generate this behavior include the availability of different technologies for small firms, and financial restrictions, suggested by Luttmer (2008) and Cooley and Quadrini (2001) respectively. The novelty of my approach is to suggest a demand based explanation so that its predictions apply to each market. Still, the tractability and generality of predictions of this demand-based model go beyond previous models in various dimensions that I explore.

5 Calibration

The goal of this section is to determine the parameters of the model without using information on the growth of firm sales. This calibration seems to be appropriate since the main purpose of the exercise is to evaluate the ability of the model to *predict* the relationship between firm size and firm growth whereas micro-data on individual firm-growth rates are limited.[27]

For the calibration I will consider the distinct markets as different countries in order to use estimates from the international trade literature. As a rule, I choose parameters that affect the cross-section of country trade flows and firm sales using the results of the estimation of Eaton and Kortum (2002) and information from the French exporting dataset of Eaton, Kortum, and Kramarz (2010) respectively as exploited for the static version of this model in Arkolakis (2010). The parameters that determine the balanced growth path are calibrated by looking at information from the US manufacturing census and US macroeconomic aggregates. US data are easily accessible for these statistics. To calibrate the stochastic process of firm productivities, additional information on firm exit rates is also taken from the US census. Finally, when I parametrize different versions of the model I only change the parameters β, $\sigma - 1$. These parameters affect the predictions of the model for firm growth and firm-size distribution but do not impact the prediction of the model for firm selection (since they do not affect the evolution and steady state distribution of firm *productivities*). Table 1 provides a summary of the model parameterization that I discuss in detail below and the sources used.

5.1 Parameters from the static model

For the calibration of the parameters that determine the cross-section of sales I follow Arkolakis (2010) since each cross-section of the dynamic model is identical to that setup. This paragraph briefly describes his procedure. The parameter α governs firm entry as a function of the population of the market. Is set to $\alpha = .44$ to match the entry of French exporting firms into markets with different population size. The parameter β and the ratio $\tilde{\theta} = \theta_2 / (\sigma - 1)$, jointly determine the cross-sectional sales heterogeneity. The choice of $\beta = .915$ and $\tilde{\theta} = 1.645$ implies that the model matches the size advantage in the domestic market (France) of prolific exporters

[27] More details about the data and discussion on the parameterization are provided in appendix B and in an online appendix.

Benchmark Param.	Value	Source
Cross Section		**Cross-sectional exporting data**
α	0.44	Arkolakis (2010)
θ_2	8.28	Eaton & Kortum (2002)
σ	6.02	Sales advantage of prolific exporters in France:
β	.915	Arkolakis (2010), Eaton, Kortum & Kramarz (forth)
Balanced Growth		**US macroeconomic aggregates**
g_η	0.0122	US population growth
g_κ	0.02	US GDP growth
g_E	0.0187	US GDP growth
Idiosyncratic Product.		**US manufacturing Census data**
δ	0.0089	Death rate of firms with 500+ employees
g_I	0.0024	Exit rates of 1963 cohort from Dunne, Roberts
σ_z	0.0664	& Samuelson (1988) (& the value of θ_2)

Table 1: Model Parameterization

compared to firms that export little or not at all.

Moreover, I exploit the mapping to the static model in order to calibrate the values of θ_2 and σ, given the value of $\tilde{\theta}$. The parameter θ_2 is key in determining the aggregate elasticity of bilateral trade flows and the welfare properties of a wide class of multi-market models as argued by Arkolakis, Costinot, and Rodríguez-Clare (2010). Thus, I use the estimate for the trade elasticity of Eaton and Kortum (2002), $\theta_2 = 8.28$, and I retain this parameter fixed when I look at the predictions of the fixed cost model. Given $\theta_2 = 8.28$ the value of σ that is consistent with $\tilde{\theta} = 1.645$ is $\sigma = 6.02$. This value of σ is in the ballpark of the estimates of Broda and Weinstein (2006) and also implies a markup of around 20% which is consistent with values reported in the literature (see Martins, Scarpetta, and Pilat (1996)).[28] For the fixed cost model, $\beta = 0$, I set $\tilde{\theta} = 1.49$, which is the benchmark calibration of Arkolakis (2010) for that model, which given $\theta_2 = 8.28$ requires $\sigma = 6.55$ for that model. The effect of changing $\tilde{\theta}$ in the fixed cost model is discussed in the specific exercises.

[28] Whereas I use the mapping of my model to the Eaton and Kortum (2002) setup to use their estimate of 8.28 Luttmer picks the value of the coefficient to be 9.56 to match the estimate of the upper tail of the size distribution by Axtell (2001), $\tilde{\theta} = 1.06$, with a choice of $\sigma = 10$. This difference makes for most of the (small) differences in the calibrated parameters, g_I, g_E and σ_z that I obtain below versus the ones used by Luttmer (2007).

5.2 Parameters governing dynamics

To determine the value of the parameters that govern the aggregate dynamics of the model, I use macroeconomic data for the US economy and US manufacturing census data from DRS. The parameters g_η, g_κ, g_E govern primarily aggregate dynamics. g_κ determines the growth rate of aggregate variables (e.g. aggregate income and consumption), g_η the growth rate of firms and their average sales over time and g_E the growth of the frontier of ideas. The growth of the population from 1960 onwards in the US is around 1.22% and the growth rate of real GDP per capita is around 2%. Thus, I set $g_\eta = .0122$ and $g_\kappa = .02$. Given the definition of g_κ, the growth of the technological frontier of new ideas is

$$g_E = g_\kappa - g_\eta \frac{(1-\alpha)}{(\sigma-1)} = 0.0187 \ .$$

The parameters δ, g_I and σ_z, which govern firm dynamics, must also be specified. In the model, δ regulates the exogenous death rate. Given that the probability of endogenous exit for firms with large size is (practically) 0, I calibrate δ by looking at the death rate of these firms. This information is obtained by the US Manufacturing Census during the period 1996-2004, where the tabulation of the largest manufacturing firms is the one for 500 or more employees. The data indicate an average exit rate of 0.89% per year for these firms, in turn, δ is set at 0.0089.

The parameters g_I and σ_z govern productivity (and thus firm-) dynamics. Note that, θ_2, which is an explicit function of these two parameters (equation (18)), was calibrated to the value of 8.28. Thus, to jointly calibrate g_I and σ_z, I need one moment from the data which is a function of these two parameters in the model. I get this information from the data by looking at the cohort exit rates of US manufacturing firms as reported by DRS. I use the exit rate of 42% in the first 4 years for the first cohort analyzed by DRS. Using equations (26) and (18) as well as the empirical values for the elasticity of trade and the cohort exit rates a simple method of moments implies the values $g_I = 0.24\%$ and $\sigma_z = 6.64\%$. This parameterization implies that $\mu = g_I - g_E = -1.63\%$ for the incumbent firms.

Notice that μ, σ_z, and δ are present in equations (26) and (18) while $\sigma - 1$ and β do not affect these relationships. Additionally, for the *firm* statistics generated by the model the difference

$\mu = g_I - g_E$ is important and not the value of the parameters separately.[29] Thus, while the fixed cost model entails a different σ (and β) parameterization, all the rest of the model parameters are kept constant across different comparison exercises.

6 Quantitative Results

In the previous section, the model was calibrated so that it is consistent with a balanced growth of the economy, statistics on French exporters size-advantage, and the exit rates of one US-cohort. I now turn to look at the predictions of the model for firm exit, growth and the firm-size distribution. Effectively, all of the predictions of the model below are out of sample predictions.

6.1 Firm Exit

In the next two subsections, I study the implications of the model for the patterns of exit and sales growth for incumbent and entry cohorts. To compare the outcomes of the model with the DRS statistics I look at the predictions of the model every five years, effectively conducting a five-year census in the model generated data. To measure entrants and exitors I resemble as closely as possible the DRS methodology and classify a firm that exits in a census-year and re-enters in a later census as an exitor when it leaves and as an entrant when it re-enters.[30]

The predictions of the calibrated model for the cumulative exit rates of incumbent and new cohorts are illustrated in Figure 7. The model closely matches the exit rate of incumbent cohorts for a period of two decades as well as the entry cohort exit rates for a period of 15 years.The high exit rates for the first cohort years is the implication of the high concentration of firms close to the cutoff productivity, given that both entry and exit happens there. The fact that the fraction of firms concentrated close to this cutoff is naturally higher for entry cohorts is the feature of the model that accounts for the differences among entry and incumbent cohorts.

The success of the model in generating the exit patterns in the US census data challenges the view that sunk costs are *necessary* to explain the entry and exit behavior of firms. Whereas

[29] Thus, there is an additional degree of freedom in the calibration. I use this extra degree and chose g_E so that g_κ is the same in all model specifications even if σ's might differ (equation 22).

[30] See footnote 13 in DRS.

the existence of a sunk cost would not refrain the model from generating the exit rates observed in the US data this existence would nevertheless imply that the average size of entrants is larger than the average size of exitors. Different calibrations of the Luttmer (2007) model (analyzed in Appendix A.6) suggest that the model requires the average size of entrants to be about 15%-25% larger than the average size of exitors in order to match the exit rates in the data. In contrast, in the DRS data, the average reported size of entrants and exitors is about the same –the average difference reported across four censuses in DRS is less than 3%–.[31] For these reasons, as well as to keep the model more tractable, the entry-exit process that I introduce appears to be a valuable tool for the analysis of the entry and exit patterns of firms.

The sole driving force in explaining the entry-exit patterns of firms is the dynamic nature of firm productivity. The parameters β and σ do not affect the predictions regarding entry and exit but will play a crucial role for the determination of firm growth and the size distribution.

6.2 Firm Growth

I now illustrate the predictions of the model for the average size of incumbent and new cohorts in Figure 8. The benchmark model can match the small contribution of the entering cohort but eventually overpredicts the size of the survivors in this entry cohort (and underpredicts the average size of firms in the incumbent cohort). Thus, the model probably implies more growth for the new entrants/small firms than what is seen in the DRS data.

The main tension between the fixed cost model and the data is that the model overpredicts the size of the entrants. Decreasing $\tilde{\theta}$ (increasing σ) increases the dispersion of sizes between small and large firms for both incumbent and entry cohorts. As illustrated in the online appendix, for $\tilde{\theta} = 1.25$ the fixed model achieves satisfactory predictions in this dimension but for higher and lower $\tilde{\theta}$ the model substantially deviates from the data. As it will be obvious from the discussion later on, different values of $\tilde{\theta}$ does not improve other key shortcomings of the fixed cost model.

Particularly illuminating for the prediction of the two models is a graph of the ratio of final

[31] The data studied by Eaton, Eslava, Kugler, and Tybout (2008) verify the findings of DRS for exporters into individual size. For the trade data they show that the average size of entrants and exitors into individual markets is almost the same and typically very small. A model with large sunk costs generates hazard rates for entry cohorts that are initally increasing, as argued by Ruhl and Willis (2008). This implication is counterfactual both for the DRS data and the trade data.

to initial sizes as a function of firm initial sizes (as percentiles), effectively a scatterplot of firm growth and initial size. Such a graph shows how likely is for a firm to grow at a certain rate as a function of its initial size. In the left panel of Figure 9, I plot such a scatterplot for the endogenous cost model and in the right panel for the fixed cost model.

Starting from the simpler case of the model with $\beta = 0$, firms with very negative productivity growth select out of the market, the more so the lower their initial size. Whereas this selection mechanism implies that the expected growth rates is inversely related to size (expression (27)) in the absence of the marketing margin it implies that the variance of the distribution of firm growth rates increases with firm size, as discussed in subsection 4.2.1.

Introducing the marketing choice of firms implies that the distribution of growth rates is fundamentally different, with the case of $\beta = .915$ featuring some firms with phenomenally high growth rates. Despite the fact that negative productivity growth implies that firms are more likely to exit, the more so the smaller they are, the marketing choice may still lead firms to reach very small fraction of consumers (so that their growth ratio could be arbitrarily low). In turn, this large volatility of the marketing margin implies that the variance of firms declines with size as discussed in proposition 3.

In Figure 10 I illustrate the predictions of the model regarding the variance of growth rates using data for publicly traded US manufacturing firms analyzed by Amaral, Buldyrev, Havlin, Leschhorn, Mass, Salinger, Stanley, and Stanley (1997) tabulated at employment bins (1-10 employees, 10-100 etc). For the largest firms, the standard deviation implied by the model converges to $(\sigma - 1)\,\sigma_z = 33\%$, which is around 2 to 3 times what is reported in the data (see also Davis, Haltiwanger, Jarmin, and Miranda (2007) and Comin and Mulani (2007)).[32] In general, the endogenous cost model overpredicts the variance of growth rates observed in the data (except for the size bin of 10-100 employees), but captures the inverse relationship of variance of firm growth and firm size. The fixed cost model predicts a (slightly) increasing relationship.

[32] This shortcoming of the model is the topic of Luttmer (2008).

6.3 Firm Size Distribution

As argued above, the predictions of the model for the overall firm size distribution are the ones analyzed by Arkolakis (2010). For completeness, I report the predictions of both models in Figure 11. Both models can match the Pareto right tail of the distribution but only the model with $\beta > 0$ can explain the existence of many small firms in the data (deviations from Pareto). In the model with $\beta > 0$ small firms endogenously reach very few consumers leading to a failure of Gibrat's law and also to a size distribution more curved than the Pareto for that model.

Figures 12 and 13 illustrate the predictions of the model with $\beta > 0$ and $\beta = 0$ for the size distribution of firms with different ages.[33] Due to selection and the stochastic evolution of productivities, the size distribution shifts to the right with age. The endogenous cost model *quantitatively* generates the main features in the Brazilian data of Cabral and Mata (2003): it implies enough dispersion for the distribution of young firms whereas the distribution of older firms eventually approximates a log-normal distribution. Two differences arise with the fixed cost model, which falls short of predicting the employment size distribution observed in the Portuguese data. First, given that entrants are concentrated around the entry point, the absence of the marketing decision implies a small dispersion in the size distribution, and thus almost all young firms are concentrated around the minimum employment size. Second, because of the fact that selection forces act only on the left tail of the distribution, absent of the marketing choice the shape of the distribution of old firms is not symmetric. This choice primarily affects the left tail of the distribution and thus it remedies the shortcomings of the fixed cost model by generating a roughly symmetric distribution for the older firms. Having discussed in detail the predictions of the model, I now explore the lessons that arise from the model for the relationship between firm size and growth.

7 The Size-Growth Debate

Do small firms grow faster? This question has been at the center of an academic and policy debate over the past three decades. Evidence for violations of the independence of firm growth

[33] Cabral and Mata (2003) use the largest tenure of the firm to approximate firm age. To generate statistics from the model as comparable as possible to the one of the authors I consider a firm as an exitor the first year it has zero sales. Thus, a 15 year old firm is one that operates continuously for 15 years. I also calibrate ψ to generate the average size of Portuguese firms during the time period.

rate and size (i.e. violations of Gibrat's law) have been reviewed in section 2.2 and are cited as justification for differential treatment for small business (see for example Birch (1981), Birch (2010) and the discussion in Neumark, Wall, and Zhang (2011)). Methodologically, Davis, Haltiwanger, and Schuh (1996) and Haltiwanger, Jarmin, and Schuh (2010) challenge the inverse size-growth relationship in the basis of the interaction of size classification and possible regression to the mean –the tendency of firms that experience a growth shock in one period to experience an opposite shock in the next one–.

Two key premises in my analysis imply a theoretical setup immune to that critique. First, the productivity growth in the model is a random walk not a mean reverting process. Second, in the theoretical model there is no miss-classification bias in the sense that small firms can temporarily be located in a different size class either because of regression to the mean or because of some measurement error.

In the previous section, I have argued that a model where Gibrat's law for firm sizes fails can explain a number of features of the US census data, better than a model where Gibrat's law holds. In summary, given the fact that the endogenous cost model is successful in quantitatively predicting a number of "out of sample" moments, and the model implies higher growth rates for small firms than the model with Gibrat's law the findings of this paper strongly support the view that small firms grow faster.

Nevertheless, in Figure 9 I have used the ratio of final to initial size as a measure of firm growth. Davis, Haltiwanger, and Schuh (1996) and Haltiwanger, Jarmin, and Schuh (2010) criticize this metric for firm growth on the basis that it will identify temporarily small firms (either because of misclassification or from a temporary shock to their long run size) as growing. They instead propose measuring firm initial size as its mean size in the two periods and computing the growth of the firm based on that. Using this metric they find that the size growth relationship disappears in many of their specifications.

To illustrate the potential fallacies that might arise from their empirical methodology I apply their metric in the model generated data. In Figure 14 I compute the ratio of growth for firms of different deciles with growth and initial size of the firm computed with the base year metric or with the average size year metric of as a function Davis, Haltiwanger, and Schuh (1996). Whereas I have argued that small firms grow faster, many of them at very high growth rates, when I use their metric of growth the inverse size growth relationship erroneously disappears (in fact, measured growth rates across categories are very slightly increasing). The reason is that small firms that grow fast (and many times really fast) are being misclassified in high initial sizes, a weakness of their metric also pointed out by Neumark, Wall, and Zhang (2011).[34] Thus, the alternative metric for firm growth misclassifies small firms *exactly* when small firms grow faster.

It is important to notice that for the endogenous cost model firms located at the first 2 deciles are firms with very low sales (or correspondingly employment size less than 1). Naturally, the sales of these firms exhibit really high growth rates, potentially much more than what is observed in the data for employment firms. Thus, the predictions of the model for the growth of firms in these two deciles might be overestimating what is reported in the data for US employment firms etc. However, such phenomenal growth rates are reported for the trade data by Eaton, Eslava, Kugler, and Tybout (2008) and Eaton, Eslava, Krizan, Kugler, and Tybout (2008).

Whereas the prediction for an inverse size-growth relationship is similar to the findings of previous empirical work, the approach of using an "overidentified" calibrated model in generating the answer is unique. Interestingly, the model based on a random walk for firm productivity is consistent with the low persistence of firm sales, a key feature observed in the data. For example, running an Arellano and Bond (1991) regression on the model generated data of firm-sales implies a autoregression coefficient much less than one (around .7) for the endogenous cost model (this coefficient is around .8 for the fixed cost model). The modeling elements beyond the random walk productivity and in particular the negative productivity drift, firm selection and the market penetration cost imply that the model generated sales resemble closely the sales of firms that are observed in the data.

8 Conclusion

This paper develops a simple unified framework as a device to analyze firm selection and growth. The framework is based on idiosyncratic firm productivity shocks and the modeling of marketing choice at the firm-level. The fit of the model to a number of firm statistics in the data is

[34] Neumark, Wall, and Zhang (2011) find an inverse relationship between growth rates and firm size using data for the entire U.S. private sector from 1992 to 2004. In their careful analysis they avoid various misleading interpretations of the data pointed out by Davis, Haltiwanger, and Schuh (1996) and Haltiwanger, Jarmin, and Schuh (2010).

surprisingly good given the parsimonious calibration that was suggested. This success implies that modeling marketing costs could be a promising avenue for understanding firm dynamics.

A key modeling simplification that I used is that firms incur the marketing cost to reach consumers each period, without being able to build continuing customer relationships. This simplification made the analysis highly tractable by keeping firm efficiency as the only current decision state of the firm. Other literature, such as the customer capital models of Drozd and Nosal (2008) and Gourio and Rudanko (2010), or the buyer-seller matching model of Eaton, Eslava, Krizan, Kugler, and Tybout (2008) are developing the first steps in modeling this accumulation process. Whereas clearly more work should be done in this direction modeling continuing customer relationship at the firm level may lead to better models that account for differences in the short run versus long run behavior of the firms and the importance of age in their decisions.

A Model Appendix

A.1 Preliminary definitions and facts

In the various proofs and derivations of this appendix I am going to use the following definitions and well known facts for the Normal distribution quoted as **properties F**.

F 1 *The simple normal distribution with mean 0 and variance 1 is given by* $\varphi(x) = \frac{e^{-x^2}/2}{\sqrt{2\pi}}$.

F 2 *The cdf of the normal is given by* $\Phi\left(\frac{x-\mu}{\sigma_z}\right) = \frac{1}{\sigma_z\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left\{-\frac{(\tilde{x}-\mu)^2}{2\sigma_z^2}\right\}d\tilde{x}$. *Using change of variables* $\upsilon = (\tilde{x}-\mu)/\sigma_z$ *which implies* $d\upsilon = d\tilde{x}/\sigma_z$ *it is also true that*

$$\Phi\left(\frac{x-\mu}{\sigma_z}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x-\mu}{\sigma_z}} e^{-\frac{(\tilde{x})^2}{2}}d\tilde{x}$$

F 3 *Because of the symmetry of the normal distribution,* $\varphi(x) = \varphi(-x)$ *and* $\Phi(x) = 1-\Phi(-x)$.

F 4 *The inverse mill's ratio of the Normal,* $\varphi(x)/\Phi(-x)$, *is increasing in* x, $\forall x \in (-\infty, +\infty)$.

F 5 $\Phi(x+\tilde{c})/\Phi(x)$, *with* $\tilde{c} > 0$, *is decreasing in* x, $\forall x \in (-\infty, +\infty)$.

F 6 *The error function is defined by:* $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-(\tilde{x})^2}d\tilde{x}$.

F 7 $\Phi(x) = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$, *where* $\Phi(x)$ *is the cdf of the standard normal cdf*

F 8 *The error function is odd:* $\operatorname{erf}(-x) = -\operatorname{erf}(x)$. *Also* $\lim_{x\to+\infty}\operatorname{erf}(x) = 1$.

F 9 $\int e^{-\tilde{c}_1x^2+\tilde{c}_2x}dx = e^{(\tilde{c}_2)^2/4(\tilde{c}_1)}\sqrt{\pi}\operatorname{erf}\left(\frac{2\tilde{c}_1x-\tilde{c}_2}{2\sqrt{\tilde{c}_1}}\right)/\left(2\sqrt{\tilde{c}_1}\right)$, *for some constants* $\tilde{c}_1$, $\tilde{c}_2 > 0$

A.2 Deriving the Stationary Distribution of Productivities

A simple guess for the solution of the Kolmogorov equation (11) is $f(s) = A_1e^{\theta_1 s} + A_2e^{-\theta_2 s}$ where θ_1 and $-\theta_2$ are given by the two solutions of the quadratic equation $\frac{1}{2}\sigma_z^2\theta_i^2 - (g_I - g_E)\theta_i - g_\eta(1-\alpha) = 0$, where $i = 1, 2$. Using condition (12) set $A_2 = 0$ for $s < \bar{s}_i$ and using the requirement that $f(s)$ is a probability density set $A_1 = 0$ for $s \geq \bar{s}_i$.

Finally, from the characterization of the flows at the entry point (15), I pick A_1, A_2 such that

$$\frac{1}{2}\sigma_z^2\left(A_1\theta_1e^{\theta_1\bar{s}_i} + A_2\theta_2e^{-\theta_2\bar{s}_i}\right) = g_\eta(1-\alpha) \ ,$$

which in combination with (14) that gives

$$\int_{-\infty}^{\bar{z}_i'} A_1 e^{\theta_1 s} ds + \int_{\bar{z}_i'}^{+\infty} A_2 e^{-\theta_2 s} ds = 1 \ ,$$

imply that

$$A_1 = \frac{\theta_1\theta_2}{\theta_1+\theta_2} e^{-\theta_1 \bar{s}_i} \ , \ A_2 = \frac{\theta_1\theta_2}{\theta_1+\theta_2} e^{\theta_2 \bar{s}_i} \ .$$

Notice, that the solutions also satisfy the first term in the LHS of (15) since the above solutions imply that $f(\bar{s}_i-) = f(\bar{s}_i+)$. In other words the distribution is continuous, but the derivative has a kink at $\bar{s}_i$.

A.3 Survival and Hazard Rates

A.3.1 Firm Survival in a market

The objective is to compute the probability that a firm will be selling in a market after a years (so that $s_{ija} \geq 0$), conditional on the initial productivity of the firm today, $s_{ij0} = s_0$. I denote this probability by $S(a|s_0)$ and thus, using expression (10),

$$S(a|s_0) = e^{-\delta a} \int_0^{+\infty} \frac{e^{-\frac{\left(\frac{s_a - s_0 - \mu a}{\sigma_z \sqrt{a}}\right)^2}{2}}}{\sigma_z \sqrt{a 2\pi}} ds_a$$

which using change of variables implies:

$$S(a|s_0) = e^{-\delta a} \Phi\left(\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right) \ . \tag{32}$$

A.3.2 Cohort Survival Rates

The expression to be derived is expression (26): the probability that an incumbent firm which is currently operating, $s_{ij0} \geq 0$, also operates after time a has elapsed, $s_{ija} \geq 0$. If I denote this probability as $\Pr(s_{ija} \geq 0 | s_{ij0} \geq 0)$, then taking in account random death the cohort survival rate is $S_{ij}(a) = e^{-\delta a} \Pr(s_{ija} \geq 0 | s_{ij0} \geq 0)$. I first derive the probability,

$$\Pr(s_{ija} \geq 0 | s_{ij0} \geq 0) = \int_0^{+\infty} \int_0^{+\infty} \Pr(s_{ija} = s_a | s_{ij0} = s_0) \frac{\Pr(s_{ij0} = s_0)}{\Pr(s_{ij0} \geq 0)} ds_a ds_0$$

$$= \int_0^{+\infty} \frac{\Pr(s_{ij0} = s_0)}{\Pr(s_{ij0} \geq 0)} \int_0^{+\infty} \Pr(s_{ija} = s_a | s_{ij0} = s_0)\, ds_a ds_0 \tag{33}$$

Using equation (16) the conditional density of productivities is given by

$$\frac{\Pr(s_{ij0} = s_0)}{\Pr(s_{ij0} \geq 0)} = \theta_2 e^{-\theta_2(s_0 - 0)} \ . \tag{34}$$

The inner integral of expression (33) is given by equation (32). Thus, by replacing expressions (32), (34) in (33) and using integration by parts,

$$\Pr(s_{ija} \geq 0 | s_{ij0} \geq 0) = \Phi\left(\frac{\mu\sqrt{a}}{\sigma_z}\right) + \int_0^{+\infty} e^{-\theta_2 s_0} \frac{1}{\sigma_z\sqrt{a}} \varphi\left(\frac{s_0 + \mu a}{\sigma_z\sqrt{a}}\right) ds_0 \ .$$

Using the definition of the error function F6 and property F9 the integral of the last expression equals to

$$\frac{e^{-\frac{1}{2}\frac{\mu^2}{\sigma_z^2}a}}{\sigma_z\sqrt{a2\pi}} \left| e^{\frac{\left(\frac{\mu}{\sigma_z^2}+\theta_2\right)^2}{4\frac{1}{2\sigma_z^2 a}}} \frac{\sqrt{\pi}}{\sqrt{2\frac{1}{2\sigma_z^2 a}}} \operatorname{erf}\left(\frac{2\frac{1}{2\sigma_z^2 a}x + \frac{\mu}{\sigma_z^2} + \theta_2}{2\sqrt{\frac{1}{2\sigma_z^2 a}}}\right) \right|_{x=0}^{x=+\infty}$$

$$= e^{\frac{\sigma_z^2 a}{2}(\theta_2)^2 + \theta_2 \mu a} \Phi\left(-\left(\frac{\mu}{\sigma_z} + \theta_2\sigma_z\right)\sqrt{a}\right) \ .$$

where I used property F7 for the last equality. Combining the expressions with the random death term gives the survival function, $S_{ij}(a)$, expression (26).

In the online appendix I show that $S_{ij}(a)$ is increasing in μ, and if $\mu < 0$, $S_{ij}(a)$ is decreasing in a, $DS_{ij}(a) < 0$. The results are applications of the properties of the normal distribution.

A.3.3 Expected Sales

The purpose is to compute the expected value of expression (8) where I will use the definition of $s_{ija} = \ln z_{\tilde{t}+a}/z^*_{ij\tilde{t}+a}$ (for simplicity denoted by s_a). I focus on deriving the expected value of the two terms inside the brackets since the terms outside the brackets are deterministic. Of course since the term s_{ija} follows a simple Brownian motion with a drift μ and a volatility σ_z I can consider each term separately and calculate $E(e^{\bar{c}_i s_{ija}} | s_{ija} \geq 0, s_{ij0} \geq 0)$ where $\bar{c}_1 = \sigma - 1$, $\bar{c}_2 = (\sigma - 1)/\tilde{\beta}$ with $\tilde{\beta} = \beta/(\beta - 1)$. Then the terms can be combined and multiplied by the values of the deterministic parameters. Regarding the expected values for $\bar{c}_i$, $i = 1, 2$,

$$E\left(e^{\bar{c}_i s_{ija}} | s_{ija} \geq 0, s_{ij0} = s_0\right) = \int_0^{+\infty} \frac{e^{\bar{c}_i s_a}}{\sigma_z \sqrt{a 2\pi}} \frac{e^{-\frac{\left(\frac{s_a - s_0 - \mu a}{\sigma_z \sqrt{a}}\right)^2}{2}}}{\Phi\left(\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right)} ds_a$$

$$= \frac{e^{\frac{-(s_0)^2 - 2\mu a s_0 - \mu^2 a^2}{\sigma_z^2 a 2}}}{\Phi\left(\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right) \sigma_z \sqrt{a 2\pi}} \int_0^{+\infty} \exp\left\{-\frac{1}{\sigma_z^2 a 2}(s_a)^2 + \frac{2s_0 + 2\mu a + \bar{c}_i \sigma_z^2 a 2}{\sigma_z^2 a 2} s_a\right\} ds_a$$

using property F8 and F9 the above expression equals to:

$$\frac{e^{\frac{-(s_0)^2 - 2\mu a s_0 - \mu^2 a^2}{\sigma_z^2 a 2}}}{\Phi\left(\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right) \sigma_z \sqrt{a 2\pi}} e^{\frac{\left(\frac{2s_0 + 2\mu a + \bar{c}_i \sigma_z^2 a 2}{\sigma_z^2 a 2}\right)^2 \sigma_z^2 a 2}{4}} \frac{\sqrt{\pi}}{2\sqrt{1/\left(\sigma_z^2 a 2\right)}} \left[1 + \operatorname{erf}\left(\frac{s_0 + \mu a + \bar{c}_i \sigma_z^2 a}{\sqrt{\sigma_z^2 a 2}}\right)\right]$$

Using F7 this last expression gives

$$E\left(e^{\bar{c}_i s_{ija}} | s_{ija} \geq 0, s_{ij0} = s_0\right) = \exp\left\{\frac{(\bar{c}_i)^2 \sigma_z^2}{2} a + \bar{c}_i \mu a + \bar{c}_i s_0\right\} \frac{\Phi\left(\frac{s_0 + \mu a + \bar{c}_i \sigma_z^2 a}{\sigma_z \sqrt{a}}\right)}{\Phi\left(\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right)} \tag{35}$$

A.3.4 Expected Log Sales

Here I sketch the derivations for the expected growth and variance of growth of the log sales of firms described in section 4.2.1. To derive that I have to derive moments of the natural logarithm of sales of the firm after time a has elapsed for $\beta \to 0$,

$$\ln L_{j\tilde{t}+a}^{\alpha} y_{j\tilde{t}+a} \frac{1}{\tilde{\psi}} + (\sigma - 1) s_{ij\tilde{t}+a} \; . \tag{36}$$

The first term is deterministic so that derivations are easy. To compute the moments of the second term I can compute the moment generating function (MGF) of this term. I start by computing the moment generating function of some variable $\tilde{s}_a$ that is normally distributed as $(\sigma - 1) s_{ij\tilde{t}+a}$ but with different parameters. Let the mean be $\tilde{\mu}$, the variance $\tilde{\sigma}^2$ and the lower threshold $\tilde{x}$ the values of which I will specify below. The MGF is (for some $\tilde{c} \in R$)

$$E\left(e^{\tilde{c} s_a} | s_0, s_a \geq 0\right) = \frac{1}{\tilde{\sigma}\sqrt{2\pi}} \frac{\int_{\tilde{x}}^{\infty} e^{\tilde{c} x} e^{-\frac{1}{2}\left(\frac{x - \tilde{\mu}}{\tilde{\sigma}}\right)^2} dx}{1 - \Phi\left(\frac{\tilde{x} - \tilde{\mu}}{\tilde{\sigma}}\right)}$$

$$= e^{-\frac{(\tilde{\mu})^2 - \left[(\tilde{\sigma})^2 \tilde{c} + \tilde{\mu}\right]^2}{2(\tilde{\sigma})^2}} \frac{\int_{\tilde{x}}^{\infty} \frac{1}{\tilde{\sigma}\sqrt{2\pi}} e^{-\frac{\left[x - (\tilde{\sigma})^2 \tilde{c} - \tilde{\mu}\right]^2}{2(\tilde{\sigma})^2}} dx}{1 - \Phi\left(\frac{\tilde{x} - \tilde{\mu}}{\tilde{\sigma}}\right)} = e^{\tilde{\mu}\tilde{c} + \frac{\tilde{\sigma}^2 \tilde{c}^2}{2}} \frac{1 - \Phi\left(\frac{\tilde{x} - \tilde{\mu} - \tilde{\sigma}^2 \tilde{c}}{\tilde{\sigma}}\right)}{1 - \Phi\left(\frac{\tilde{x} - \tilde{\mu}}{\tilde{\sigma}}\right)}$$

where in the last equality I used the definition of the normal distribution F2. I can now adjust the parameters of the distribution so that they correspond to the current firm sales size and the underlying stochastic process: $\tilde{\mu} = (\sigma - 1) s_0 + (\sigma - 1) \mu a$, $\tilde{\sigma} = (\sigma - 1) \sigma_z \sqrt{a}$, $\tilde{x} = 0$. Finally, I can compute the moments of the second term of equation (36) by computing the successive derivatives of the MGF wrt to $\tilde{c}$.

A.4 Proof of Proposition 2

To prove the proposition I need to show that $\partial \left((g_I - g_E) \frac{h'(s)}{h(s)} + \frac{\sigma_z^2}{2} \frac{h''(s)}{h(s)} \right) \Big/ \partial s \leq 0$. Extended derivations for this proposition given in an online appendix imply that it is equivalent to show that

$$(g_I - g_E) (\sigma - 1) \left(\frac{1 - \tilde{\beta}}{\tilde{\beta}} \right) e^{-s\frac{(\sigma-1)}{\beta}} + \frac{\sigma_z^2}{2} (\sigma - 1)^2 \frac{1 - \tilde{\beta}^2}{\tilde{\beta}^2} e^{-s\frac{(\sigma-1)}{\beta}} \leq 0$$

so that for $\tilde{\beta} = \beta / (\beta - 1)$ I need to show (notice that $e^{s\frac{(\sigma-1)}{\beta}} \geq 1$, for $s \geq 0$)

$$-\left(\tilde{\beta}\right)^2 \left[(g_I - g_E) (\sigma - 1) + (\sigma - 1)^2 \frac{\sigma_z^2}{2} \right] + (g_I - g_E) (\sigma - 1) \tilde{\beta} + (\sigma - 1)^2 \frac{\sigma_z^2}{2} \leq 0 \ .$$

This expression after some manipulations gives the condition in equation (31). Notice that if $(g_I - g_E) (\sigma - 1) + (\sigma - 1)^2 \sigma_z^2 < 0$ there does not exist a $\beta \in [0, +\infty)$ that satisfies the inequality.

A.5 Proof of Proposition 3

The proof of the proposition uses Ito's Lemma. In particular, the variance of the instantaneous growth rate of firms is given by the square of the second bracketed term in expression (28). Given (29) this term equals to $\sigma_z^2 (\sigma - 1)^2$ for $\beta \to 0$. For the second part of the proposition, given $\beta > 0$, the derivative of the term is always negative. Thus, the instantaneous variance of growth rates of firms selling to a destination is inversely related to their size there. In the limit for $s_{ij0} \to +\infty$ the term tends to $\sigma_z^2 (\sigma - 1)^2$ completing the proof of the proposition.

A.6 Sunk Costs

I consider different calibration of the sunk cost model and the implied difference for the entrants and exitors that they imply. The key equation in the model of Luttmer (2007) is equation (19). The exercise that I perform is to consider the average difference of exitors to entrants that will imply a 60% 5-year exit rate for the entry cohort. For the calibrated parameters that I consider the model requires an average size of entrants to exitors that is around 22% higher whereas with the calibration of Luttmer this number is around 25%. If given the calibration of the rest of the parameters in Luttmer I choose a $\sigma = 6.5$ instead of a $\sigma = 10$ that he considers the average size difference drops to 15%.

A.7 Age and Size in the US Data

In addition to the results on the size distribution of firms of different ages the model implies that firms reach a large size at a reasonable age compared to what is observed in the US data. Considering firms of age 100 years or less, the benchmark model predicts that around 1.2 in 1,000 firms are of the size of 2,500 employees or more. The corresponding number for the population of the US manufacturing firms is around 3.8 in every 1,000. The data are from the Small Business Administration, sba.gov, for 2005. The model was calibrated to the mean employment size of US manufacturing firms in 2005 (47.3 employees) to generate this number by appropriately choosing the value of the parameter ψ.

B Data Appendix

B.1 Census Data

The statistics for the data on US manufacturing firms are constructed using data provided by the census and the paper of Dunne, Roberts, and Samuelson (1988). See also online excel file for details.

B.2 Exit and Entry in the Data and the Model

In the data, firms that stop to operate oftentimes seize to exist. Also the reverse is true: firms that may be recorded as selling in consecutive censuses years do not necessarily sell in all the years between the censuses, which is closer to the interpretation of exit given in this paper. To avoid potential bias imposed by the way I model entry-exit I generate statistics in the simulated data by following the procedure of Dunne, Roberts, and Samuelson (1988) (see for example Dunne, Roberts, and Samuelson (1988) footnote 13). Thus, if a firm stop selling in a census and returns in the next one is considered as a new entrant (and included in the entry cohort).

Notice that reentry is also frequent in the trade data. Using some estimates kindly provided by Maurice Kugler for the Colombian exporting data around 15% of the times that a firm exits a given destination it sells back in the destination in some or all the three consecutive years after the year of exit. This pattern of re-entry is consistent with the model.

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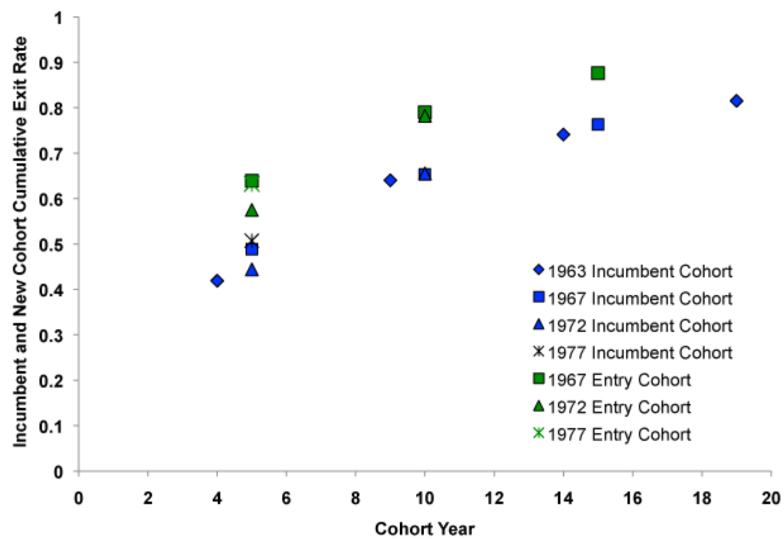

Figure 2: Incumbent and Entry Cohorts Exit Rate in the US Manufacturing Census

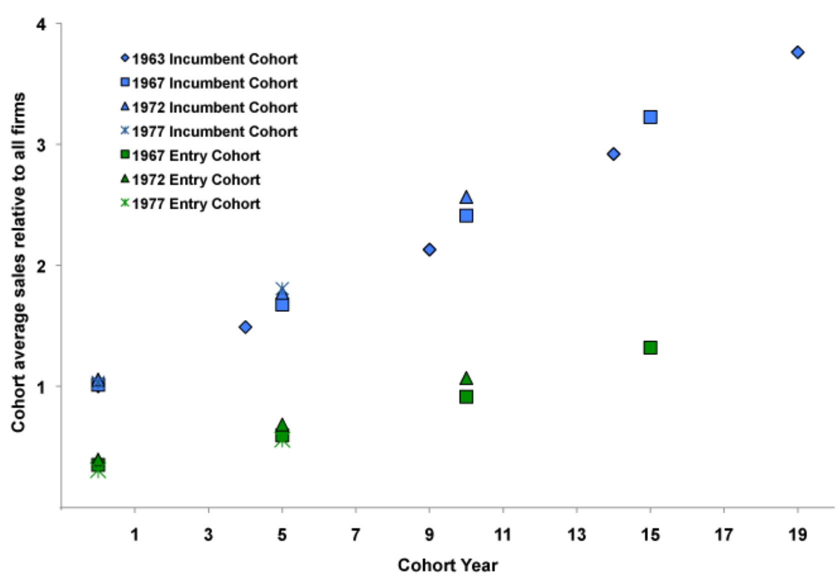

Figure 3: Incumbent and Entry Cohort Average Sales (of surviving firms) in the US Manufacturing Data

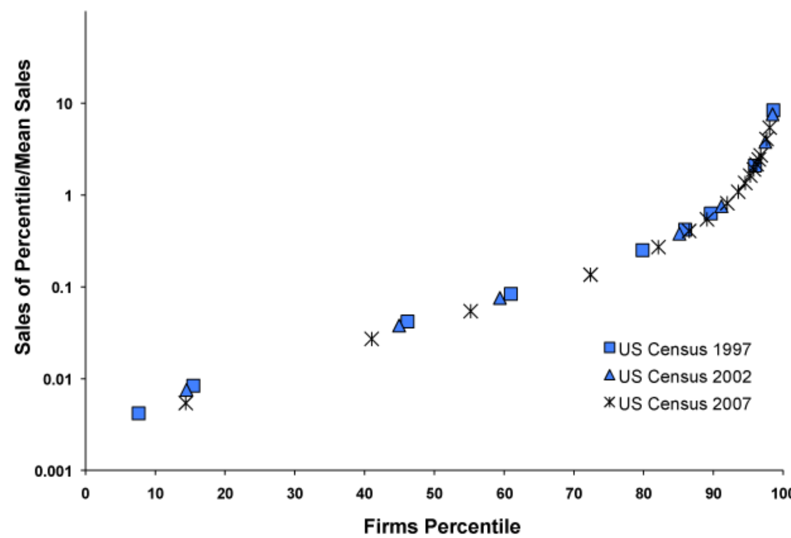

Figure 4: Distribution of Sales in the US manufacturing census.
Source: Small Business Administration. Note: I use the maximum point of each bin and the number of firms included in the bin to plot the sales size of the firms and the corresponding percentiles.

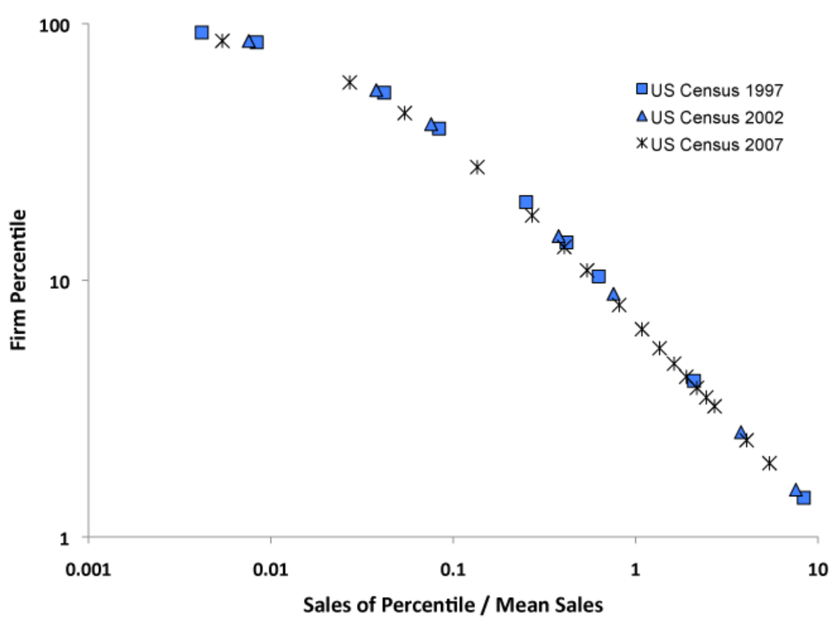

Figure 5: Distribution of Sales in the US manufacturing census.
Source: Small Business Administration. Note: I use the maximum point of each bin and the number of firms included in the bin to plot the sales size of the firms and the corresponding percentiles.

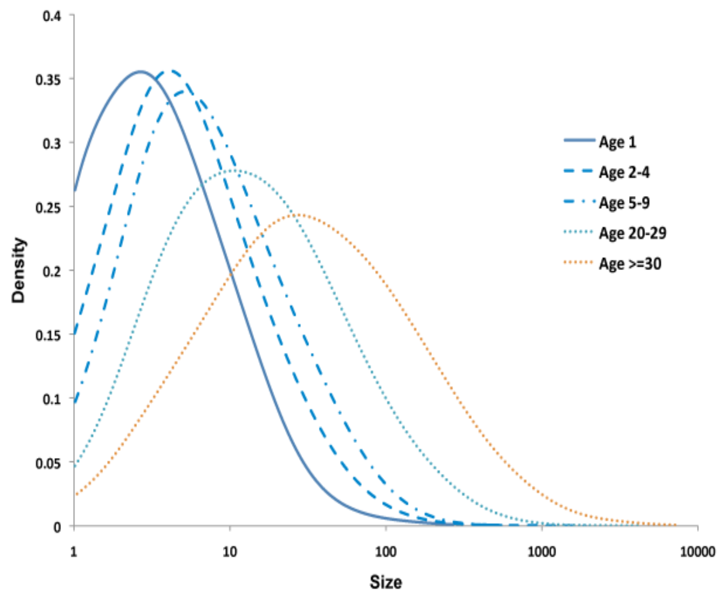

Figure 6: Distribution of Sales by Age Cohort. Source Cabral and Mata (2003).

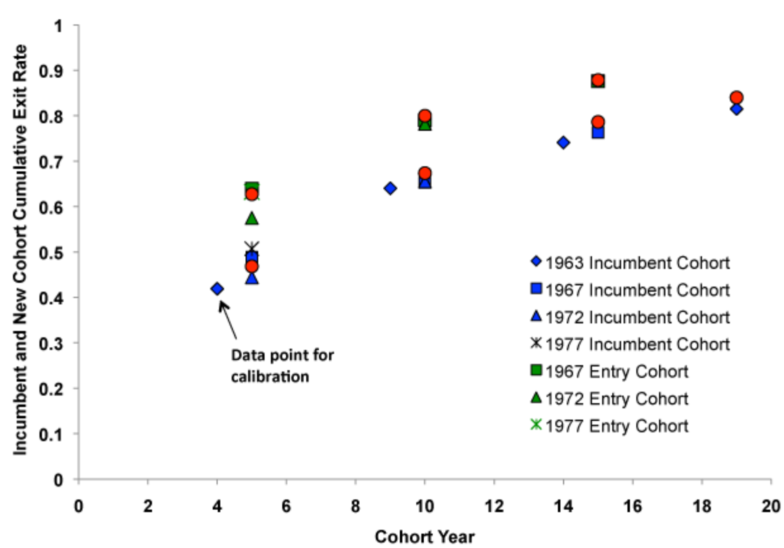

Figure 7: Incumbent and Entry Cohort Exit in the Data and the Model.

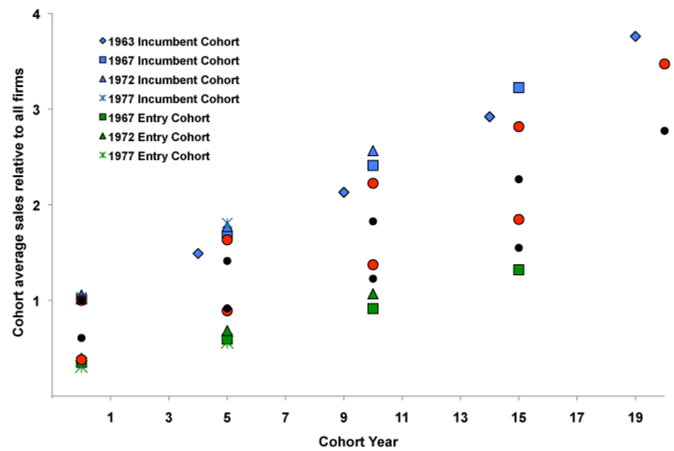

Figure 8: Average Size of Incumbent and Entry Cohorts in the US census data and the model

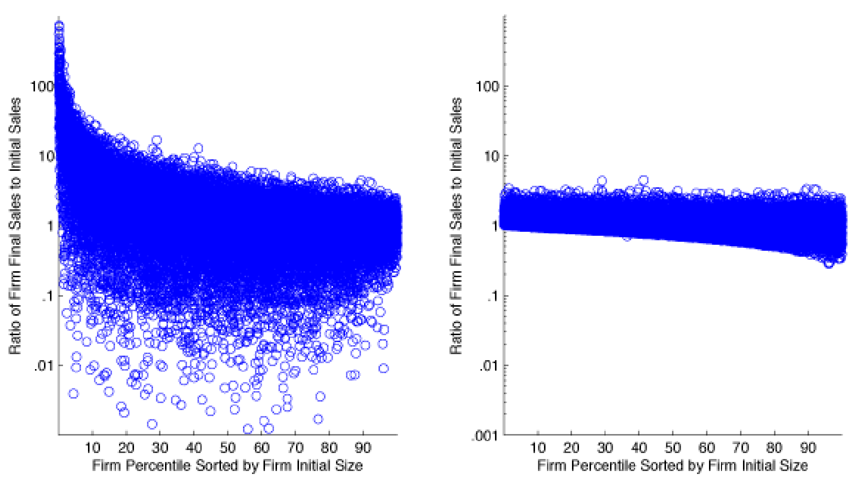

Figure 9: Firm Percentiles and Firm Growth Rates in the Model

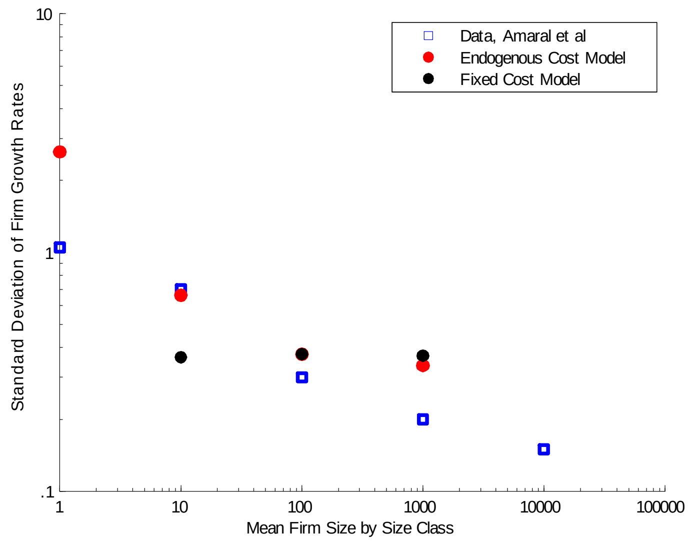

Figure 10: Standard Deviation of Firm Growth Rates and Firm Initial Employment size. Source: Data for US Manufacturing firms in compustat tabulated by Amaral, Buldyrev, Havlin, Leschhorn, Mass, Salinger, Stanley, and Stanley (1997).

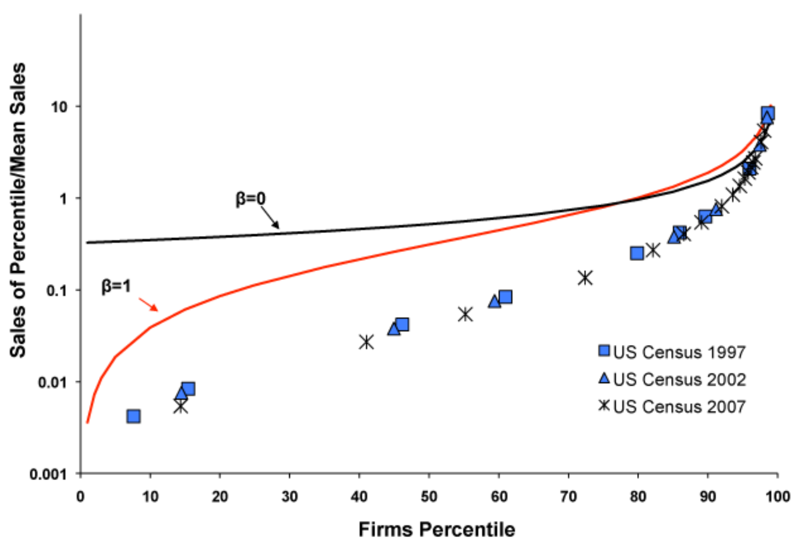

Figure 11: Size Distribution of US Manufacturing FIrms and the Model. Source: Small Business Administration, www.sba.gov

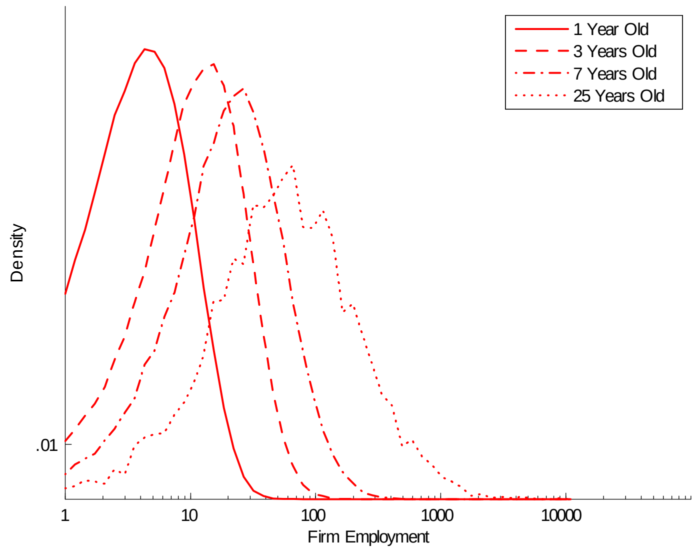

Figure 12: Employment Distribution for Firm with Different Ages in the Endogenous Cost Model.

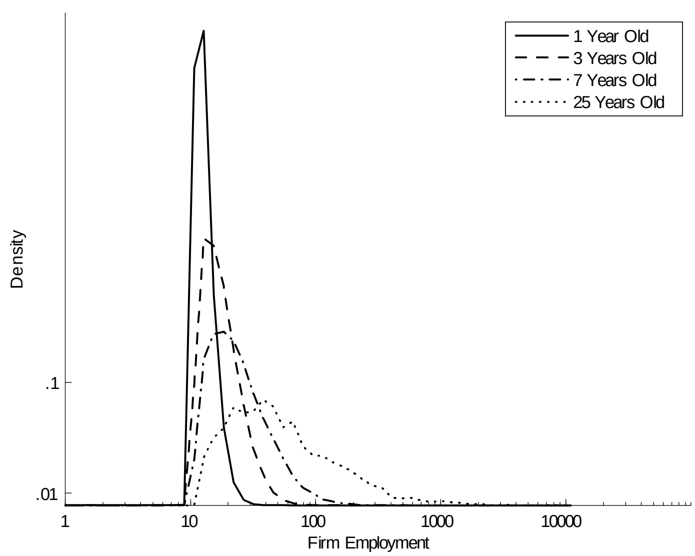

Figure 13: Employment Distribution for Firm with Different Ages in the Fixed Cost Model.

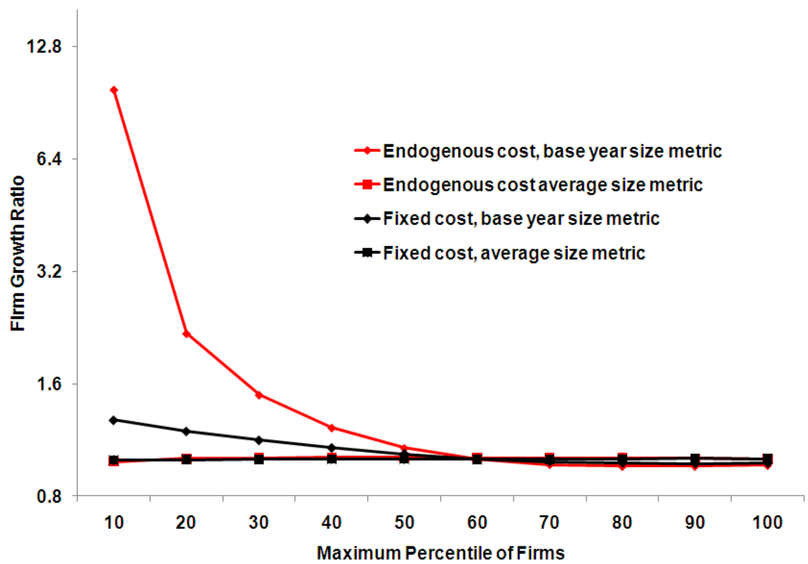

Figure 14: Firm Growth Rates with Firm Size as Base Year Size and Mean Base and Final Year Size