Sincere and sophisticated players in the envy-free allocation problem

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Abstract

We consider the manipulation of envy-free solutions for the allocation of indivisible goods and money when some agents are sincere, i.e., unconditionally report their true preferences, and the other are strategic. We show that strategic agents non-cooperatively coordinate on the envy-free allocations that are not Pareto dominated for them by any other envy-free allocation. Independently of the envy-free solution that is operated: (i) a "pessimistic" agent, i.e., an agent who expects his worst-case scenario equilibrium payoff, has no welfare loss if she commits to be sincere, and (ii) an "optimistic" agent, i.e., an agent who expects his best-case equilibrium payoff, generically has an incentive to be strategic. This suggests that in our environment dominance of truthful revelation is generically unrelated with an agent's incentive to be sincere.

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Keywords: manipulation of envy-free solutions; sincere and strategic agents; indivisible goods; mechanism design; no-envy.

1 Introduction

We consider the problem of fairly allocating a set of indivisible goods (objects) and an amount of money among some agents who collectively own, or are re-

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sponsible for, these resources. We assume that each agent receives exactly one object and consumptions of money add up to an amount we refer to as the budget. Examples are the allocation of tasks and salary among workers (Crawford and Knoer, 1981), the allocation of rooms and contributions to rent among roommates (Abdulkadiroğlu et al., 2004), and the dissolution of a partnership. Our central notion of fairness is no-envy (Foley, 1967; Varian, 1974), i.e., the requirement that no agent prefer the consumption of any other agent to her own. We are interested in "envy-free solutions," i.e., functions that associate with each problem an envy-free allocation for it. We study the manipulation, in a complete information setting, of envy-free solutions when some agents are sincere, i.e., unconditionally report their true preferences, and the other are strategic. Our main result, Theorem 1, characterizes the limit Nash equilibrium outcomes of the direct revelation game for the strategic agents associated with each envy-free solution. This set is welfare equivalent to the set of envy-free allocations for the true preferences that are not Pareto dominated for the strategic agents by any other envy-free allocation. Envy-free allocations are Pareto efficient in our environment (Svensson, 1983). Thus, manipulation of an envy-free solution when some agents are sincere induces no efficiency loss.

The first purpose of our study is to evaluate the performance of envy-free solutions in the presence of a behavioral type of interest, i.e., sincere agents. In the related problem of school choice, where parents report preferences on public schools, Pathak and Sönmez (2008) argue that parents' sophistication is not homogeneous. Some parents may participate in extensive discussion of the best strategies given the mechanism adopted by a school district. Some other may report their true preferences without further thought. One can envision a similar situation in our environment.[1] Moreover, one can interpret sincerity in our model as an extreme form of risk-aversion: for each agent, her true preference relation is a Maximin strategy in the manipulation game associated with any envy-free solution (Propositions 1 and 5).

We identify two interesting features of the equilibrium outcomes from the manipulation of each envy-free solution. First, each of these solutions provides a safety net for sincere agents. They guarantee these agents, at least, the minimum welfare among all envy-free allocations for the true preferences. It is well known that at each envy-free allocation, the welfare of a truthful agent is

[1] In experimental settings, recent studies have documented the propensity of some subjects to provide truthful reports in strategic communication games (Cai and Wang, 2006; Sanchez-Pages and Vorsatz, 2009)

bounded below by that she would obtain should all the agents share her preferences (Moulin, 1990; Beviá, 2010). This lower bound generically does not bind among the envy-free allocations for the true preferences.[2] Thus, the surprising part of our result is that generically, strategic agents cannot force a truthful agent to approximate her "all-profile minimum welfare," and can only force her to receive her true-profile minimum.[3] Such a happy conclusion can be related to the manipulation of one-side-optimal "stable" solutions in marriage markets. There, the manipulation of the solution that selects the best stable matching for one side of the market, say men, leads to stable allocations whenever men are truthful (Roth, 1984).

Second, strategic agents take advantage, to some extent, of sincere agents. Indeed, they non-cooperatively coordinate to extract, from sincere agents, the maximum possible "surplus" among envy-free allocations that they could unanimously agree on (Theorem 1). This can be related again to the manipulation of one-side-optimal stable solutions in marriage markets.[4] There, women can achieve her best stable allocation in a strong equilibrium when the men-optimal stable solution is operated and men truthfully report their preferences (Gale and Sotomayor, 1985).[5] Moreover, this is the unique outcome under dominance solvability (Alcalde, 1996). A striking difference with our results, is that we characterize Nash equilibrium behavior without any other refinement. Recall that in matching markets, when preferences are strict, the women optimal stable matching is preferred by all women to any other stable matching. Nevertheless, each stable matching is an equilibrium of the direct revelation game associated with the men-optimal stable solution when men truthfully report their preferences (Gale and Sotomayor, 1985). Similarly, the outcomes from the manipulation of the "Boston mechanism" in school choice problems when some parents are sincere and the other strategic, contains outcomes that are Pareto dominated for the strategic parents by another equilibrium outcome (Pathak and Sönmez, 2008).

[2]For each agent there is a unique list of bundles, one for each object, among which the agent is indifferent and whose associated consumptions of money add up to the budget; the agent gets her Maximin payoff in a certain profile only if there is an allocation of these bundles that is envy-free with respect to true preferences.

[3]Here "all-profile minimum welfare" refers to the minimum welfare the agent can get in an envy-free allocation among all profiles in which her preferences are equal to her true preferences.

[4]Thanks to Lars Ehlers for bringing my attention to the connection between the reversal property in matching markets and the current paper.

[5]The symmetric statement holds for the women-optimal stable solution.

The second purpose of our study is to investigate an agent's incentive to be sincere. Unconditionally reporting her true preferences has important consequences for an agent. It implies that the agent gives up her strategic advantage and this is common knowledge. Thus, a complete evaluation of the agent's incentives when a solution is operated calls for the determination of the manipulation outcomes that would ensue if the agent is sincere. Only then can one judge the incentives for unconditional truthful revelation of preferences by comparing these outcomes with the outcomes from the manipulation of the solution when the agent is strategic.

Our results reveal that independently of the envy-free solution that is operated, an agent's decision to be sincere depends on how pessimistic or optimistic the agent is. In order to formalize this intuition we introduce utility representation in our otherwise ordinal model. We define the "price of sincerity for a pessimistic agent" as the ratio of the lowest equilibrium payoff when the agent is strategic, to the lowest equilibrium payoff when the agent is sincere. Surprisingly, for each envy-free solution, as long as another agent is strategic, the price of sincerity for a pessimistic agent is one (Proposition 2). That is, if the agent expects to coordinate in her worst equilibrium outcome, she may prefer to be sincere, commit to it by allowing for the verification of her report, and then avoid the strategic effort of selecting her best response to the other players' actions.

Likewise, we define the "price of sincerity for an optimistic agent" as the ratio of the highest equilibrium payoff when the agent is strategic, to the highest equilibrium payoff when the agent is sincere. The result here is that for each envy-free solution, generically, this ratio is greater than one (Propositions 3 and 6). That is, for almost all preference profiles, an optimistic agent would never have the incentive to commit to report her true preferences. The striking part of this result is that it holds independently of the envy-free solution that is operated. It is well known that no envy-free solution in our environment is "strategy-proof" (Tadenuma and Thomson, 1995a).[6] Nevertheless, there are envy-free solutions that reach some level of incentive compatibility. Let i be an agent. There is an envy-free solution for which it is a dominant strategy for agent i to report her true preference relation (Andersson et al., 2011). Since the solution is envy-free, our result implies that agent i's price of sincerity if she is optimistic is generically greater than one! Thus, even though truth telling is

[6] A solution is strategy-proof if for each preference profile it is a dominant strategy to report her true preferences for each agent.

a dominant strategy for agent i, this agent would generically benefit by coordinating in an equilibrium outcome that cannot be sustained with a truthful report (Section 5). Indeed, one cannot expect that agent i will be truthful when this solution is operated and the agent is optimistic, contradicting the common belief that dominance of truthful revelation for an agent induces her sincerity.[7]

The rest of the paper is organized as follows. Section 2 illustrates our results with two examples. Section 3 presents the model. Section 4 presents our results. Section 5 discusses our results. Proofs are collected in the Appendix.

2 Examples

2.1 Dissolving a partnership

Consider two agents, say Ann and Bob, who collectively own a company and decide to dissolve their partnership. They have to decide who gets the company and how much he or she pays for it. We normalize Ann and Bob's valuation of staying out of the company to zero. Their valuations for keeping the company are shown in Table 1.

	Ann	Bob
value	\$1.2 million	\$1 million

Table 1: Valuations

We assume that Ann and Bob's preferences are quasi-linear, that is, they are represented by the aggregate value of their allotment, i.e., getting or not the company, plus/minus their transfer/payment.

There is a continuum of envy-free allocations in this problem. At each of these allocations Ann receives the company and pays Bob an amount between \$0.5 and \$0.6 million. If Ann pays Bob less than \$0.5 million, Bob would prefer to get Ann's allotment. If she pays more than \$0.6 million, she would prefer Bob's allotment. At no envy-free allocation Bob receives the company (in our model no-envy implies Pareto efficiency).

[7]The terminology of pessimistic and optimistic price of sincerity is inspired on the "Price of Anarchy (PoA)" and the "Price of Stability (PoS)" (Schulz and Moses, 2003; Anshelevich et al., 2004; Koutsoupias and Papadimitriou, 2009). The difference of the price of sincerity and these ratios is that we bound individual gains and losses and the PoA and PoS bound social losses.

Suppose that an arbitrator is asked to dissolve the partnership. Consider first the case in which the arbitrator knows the agents' valuations. Call the maximum of these valuations $v_{\max}$ and the minimum $v_{\min}$. In general, as described above, if $v_{\max} > v_{\min}$, there is a continuum of envy-free allocations. At each of these allocations the agent with maximal valuation receives the company and pays the other agent an amount between $\frac{1}{2}v_{\min}$ and $\frac{1}{2}v_{\max}$. If $v_{\max} = v_{\min}$, then there is essentially one envy-free allocation, at which any agent may receive the company and pays $\frac{1}{2}v_{\max} = \frac{1}{2}v_{\max}$ to the other agent. The arbitrator may consider the cental point of the envy-free set a salient fair allocation: the maximum-value agent receives the company and pays $\frac{1}{4}(v_{\min} + v_{\max})$ to the other agent (Tadenuma and Thomson (1995b) propose this selection and extend it to the n-agent and n-object case).

Now, with no knowledge of Ann and Bob's valuations, the arbitrator, with the same objective as above, may operate the following mechanism: (i) ask both agents for their valuations; (ii) assign the company to the maximum-value agent and ask him or her to pay $\frac{1}{4}(v_{\min} + v_{\max})$ to the other agent.

One can easily see that if both agents are strategic, each envy-free allocation for true preferences is a "limit Nash equilibrium" outcome of the arbitrators' mechanism.[8] Moreover, each limit Nash equilibrium is envy-free (Corollary 1; see also Tadenuma and Thomson (1995a)).

Let us now consider the case in which one agent, say Ann, is sincere. Then, there is a unique limit Nash equilibrium outcome: Ann receives the company and pays \$0.6 million to Bob. This allocation is envy-free! Thus, Ann's welfare is not worse than the minimum welfare she would obtain if she were strategic. However, Bob takes advantage of Ann's sincerity. If Ann were strategic, she would be better off if they were to coordinate in a better equilibrium for her. We show in our main results that independently of the number of agents in the problem, these are two common features of all envy-free solutions.[9]

Now, imagine that the arbitrator operates the alternative mechanism that

[8] Existence of pure strategy Nash equilibria in this game may be compromised by the way the arbitrator breaks ties for equal reports. As in a standard Bertrand game, a sensible solution to this problem is to consider limit Nash equilibria (see Section 3 for details). For instance, here for each $v \in [\$0.5 \text{ million}, \$0.6 \text{ million}]$, the reports $v_{\text{Ann}} = v + \varepsilon$ and $v_{\text{Bob}} = v$ are an ε-equilibrium. A limit Nash equilibrium outcome is the limit as $\varepsilon \to 0$ of a sequence of ε-equilibrium outcomes.

[9] In the two-agent case if one agent is sincere and the other strategic, then the sincere agent gets in equilibrium her all-profile minimum welfare among envy-free allocations. This is not necessarily the case for more than two agents.

for each report selects Ann's preferred envy-free allocation. More precisely, let v_A and v_B be Ann and Bob's reports, respectively. If $v_A \geq v_B$, the mechanism recommends that Ann receive the company and pay $\frac{1}{2}v_B$ to Bob. If $v_B > v_A$, the mechanism recommends that Bob receive the company and pay $\frac{1}{2}v_B$ to Ann. One can easily verify that reporting her true valuation is a dominant strategy for Ann. However, if Ann unconditionally reports her true valuation and Bob is strategic, there is a unique limit Nash equilibrium outcome. This is exactly the outcome from any envy-free mechanism: Ann will receive the company and pay \$0.6 million to Bob. One can expect that if Ann is optimistic, she will misrepresent her preferences and try to coordinate in a better equilibrium for her. Again, our results reveal that this is a common feature of all envy-free solutions independently of the number of agents who are involved in the problem (Propositions 3 and 6).

2.2 Allocating rooms and contributions to rent among three roommates

Consider a set of roommates $N \equiv \{1,2,3\}$ who collectively lease a house with rooms $A \equiv \{1,2,3\}$. The rent for the house is \$1200. Each roommate is to receive exactly one room and pay an amount of money for it. Individual payments of rent should add up to \$1200, so there is no surplus or deficit. The roommates' valuations for the rooms are as follows:

	Room 1	Room 2	Room 3
roommate 1	**\$500**	\$400	\$400
roommate 2	\$400	**\$600**	\$400
roommate 3	\$400	\$400	**\$700**

Table 2: Valuations

We assume that preferences are quasi-linear. That is, they are represented by the aggregate value of the agent's consumption. For example, if roommate 1 receives room 1 and pays \$300 for it, her aggregate value is 200.

There is a continuum of envy-free allocations for this problem. Let $i \in N$. Recall that envy-free allocations are Pareto efficient in this environment (Svensson, 1983). Thus, agent i receives room i at each envy-free allocation. Let $(r_i)_{i \in N}$ be the individual payments of rent at an envy-free allocation. Since at each of these allocations no agent prefers the consumption of any other agent, these payments are those satisfying for each $i \neq j$ the linear inequality constraint $v_i^i - r_i \geq v_j^i - r_j$ where v_j^i is agent i's value of room j. Since $r_1+r_2+r_3 = 1200$, we can solve this inequality system for r_1 and r_2 (Figure 1 (a)). The set of pairs (r_1, r_2) associated with envy-free allocations is a polygonal with non-empty interior in $\mathbb{R}_+^2$ (Figure 1 (a)).

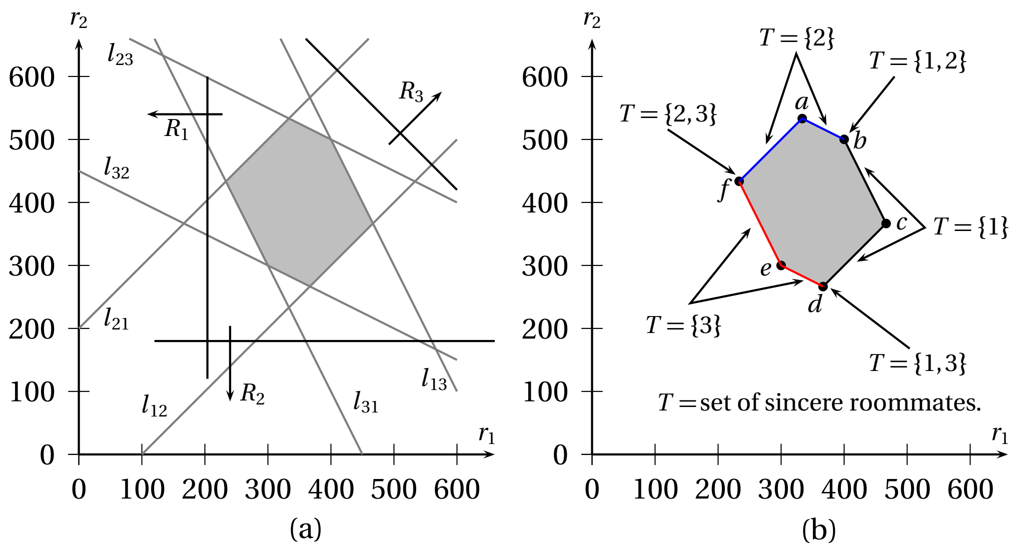

Figure 1: (a) Envy-free allocation of rooms $A \equiv \{1,2,3\}$ and contributions to rent among three roommates $N \equiv \{1,2,3\}$ when $r_1+r_2+r_3 = 1200$ and valuations are given in Table **??**. Let l_{ij} be the linear inequality constraint $v_i^i - r_i \geq v_j^i - r_j$ where v_j^i is agent i's value of room j. The figure displays for each pair $(i, j) \in N \times A$, the constraint l_{ij} with its label inside the half space satisfying the inequality constraint. The shaded polygonal area contains all the combinations of rent payments for agents 1 and 2, (r_1, r_2), such that the allocation where each agent i receives room i and rent payments are $(r_1, r_2, 1200 - r_1 - r_2)$ is envy-free. Roommate 1's indifference curves are vertical lines (roommate 1 prefers smaller payments r_1); roommate 2's indifference curves are horizontal lines (roommate 2 prefers smaller payments r_2); roommate 3's indifference curves are lines with constant $r_1 + r_2$ (roommate 3 prefers higher aggregate payments by agents 1 and 2). (b) equilibrium outcomes from the manipulation of any envy-free solution when the set of agents T is sincere.

An envy-free solution associates with each valuation vector $(v_j^i)_{i\in N, j\in A}$ an envy-free allocation for it. As in Section 2.1, one can think of a solution as the judgement of an arbitrator who endorses the envy-free principle and considers all configurations of values. It is known that the set of outcomes from the direct revelation game induced by an envy-free solution at some valuation vector is exactly the set of envy-free allocations for the true valuations (Beviá, 2010; Velez, 2011; Fujinaka and Wakayama, 2012). Our main result, Theorem 1, implies that if some agents are sincere, the set of manipulation outcomes is sharply reduced to a subset of the "faces" of the envy-free set (Figure 1 (b)). More precisely, if some agents, say $T \subsetneq N$ are sincere, the set of limit equilibrium outcomes from the manipulation of any envy-free solution are the Pareto dominant allocations for $N \setminus T$ inside the envy-free set for true the preferences. For instance, if agent 3 is sincere, these limit equilibrium outcomes are those in the segments connecting f and e and e and d in Figure 1 (b). If both agents 1 and 3 are sincere, then the unique limit equilibrium outcome is the preferred allocation for agent 2 in the envy-free set, i.e., d.

This example also illustrates our results concerning incentives for the unconditional truthful revelation of preferences. First, each agent's worst-case limit equilibrium payoff from the manipulation of an envy-free solution are the same when the agent is sincere and there is at least another strategic agent and when, ceteris paribus, the agent is strategic (Proposition 2). Second, this example is generical. For almost all preference profiles, each agent prefers the best limit equilibrium outcome from the manipulation of an envy-free allocation when the agent is strategic to the best such an equilibrium when, ceteris paribus, the agent is sincere (Propositions 3 and 6).

3 The Model

3.1 Environment, solutions, and properties of solutions

We consider the problem of allocating a finite set of objects $\boldsymbol{A}$ and an amount $\boldsymbol{M} \in \mathbb{R}$ of an infinitely divisible good, which we refer to as "money," among a group of agents $\boldsymbol{N}$. We assume that the number of agents and objects are equal, i.e., $n \equiv |N| = |A|$. Generic objects are denoted α and β. Agents consume bundles in $\mathbb{R} \times A$. The generic consumption bundle is (x_α, α). The domain of preferences on $\mathbb{R} \times A$ is $\mathscr{R}$. Agent i's generic preference is R_i and the generic preference profile is $R \equiv (R_i)_{i \in N}$. As usual, I_i and P_i are the symmetric and asymmetric parts of R_i, respectively. We assume that preferences satisfy two properties: (i) **money-monotonicity**, i.e., for each $\alpha \in A$ and each $\{x_\alpha, x'_\alpha\} \subseteq \mathbb{R}$ such that $x'_\alpha > x_\alpha$, $(x'_\alpha, \alpha) P_i (x_\alpha, \alpha)$, and (ii) **no object is infinitely better than any other**, i.e., for each $\{\alpha, \beta\} \subseteq A$ and each $x_\beta \in \mathbb{R}$, there is $x_\alpha \in \mathbb{R}$ such that

$(x_\beta, \beta)\, I_i\, (x_\alpha, \alpha)$.[10] Our results also hold when preferences are restricted to be **quasi-linear**, i.e., preferences $R_i \in \mathscr{R}$ such that for each two bundles (x_α, α) and (x_β, β) such that $(x_\alpha, \alpha)\, I_i\, (x_\beta, \beta)$, and each $\delta \in \mathbb{R}$, $(x_\alpha + \delta, \alpha)\, I_i\, (x_\beta + \delta, \beta)$. We denote the domain of quasi-linear preferences by $\boldsymbol{\mathscr{Q}}$.

We introduce a normalized representation of preferences in order to evaluate an agent's incentive to be sincere by means of a comparison of equilibrium payoffs when the agent is sincere and when the agent is strategic. This representation is also useful in evaluating genericity statements about the agents' behavior in terms of a familiar topology in a functional space. Let $\alpha^* \in A$ be a reference object and $\varphi : \mathbb{R} \to \mathbb{R}_{++}$ be a strictly increasing, continuous, and bounded function. For each $R_i \in \mathscr{R}$, let $u[R_i] : \mathbb{R} \times A \to \mathbb{R}_{++}$ be the continuous representation of R_i such that for each $x_{\alpha^*} \in \mathbb{R}$, $u[R_i](x_{\alpha^*}, \alpha^*) = \varphi(x_{\alpha^*})$. Whenever possible, we write u_i for $u[R_i]$. Let $\boldsymbol{\mathscr{U}} \equiv \{u_i[R_i] : R_i \in \mathscr{R}\}$ be the space of normalized utility representations for $\mathscr{R}$ endowed with the supremmum metric $|\cdot|_\infty$.

Quasi-linear preferences can be seen as members of a much simpler space. Indeed, there is a one-to-one relation between this domain and $\mathbb{R}^{n-1}$: for each $R_i \in \mathscr{Q}$, there is a unique representation $(x_i, \alpha) \in \mathbb{R} \times A \mapsto x_i + v^i_\alpha$ where $v^i \equiv (v^i_\alpha)_{\alpha \in A} \in \mathbb{R}^A$ is such that $\sum_{\alpha \in A} v^i_\alpha = 0$. Thus, one can evaluate the genericity of a property in the quasi-linear domain in terms of the Euclidean topology and the Lebesgue measure on $\mathbb{R}^{n-1}$.

We assume that each agent receives one object and some amount of money. An **allocation** is a pair $z \equiv (x, \mu) \in \mathbb{R}^A \times A^N$ such that $\sum_{\alpha \in A} x_\alpha = M$ and $\mu : N \to B$ is a bijection. The consumption of money associated with object α at z is x_α. Agent i's allotment at z is $z_i \equiv (x_{\mu(i)}, \mu(i))$. Let $\boldsymbol{Z}$ be the set of all allocations.

Let $R \in \mathscr{R}^N$. For each $i \in N$ and each $R'_i \in \mathscr{R}$, the profile (R_{-i}, R'_i) is obtained from R by replacing R_i by R'_i. For each $K \subseteq N$, R_K is the subprofile $(R_i)_{i \in K}$. Analogously, for each $R'_K \in \mathscr{R}^K$, (R_{-K}, R_K) is obtained from R by replacing R_K by R'_K. We denote $K + i$ the set $K \cup \{i\}$.

We are interested in systematic ways of selecting allocations for each possible configuration of preferences. A **solution**, generically denoted by f, associates with each $R \in \mathscr{R}^N$ a feasible allocation $f(R) \in Z$.

Let $R \in \mathscr{R}^N$. An allocation $z \in Z$ is **envy-free for $\boldsymbol{R}$** if no agent prefers some other agent's consumption at z to her own (Foley, 1967; Varian, 1974). Let $\boldsymbol{F(R)}$ be the set of *envy-free* allocations for R; it is well known that under our assump-

[10] Money-monotonicity implies continuity, i.e., weak upper and lower contour sets are closed in the product topology on $\mathbb{R} \times A$ induced by the Euclidean and discrete topologies.

tions, this set is non-empty (Alkan et al., 1991; Velez, 2012a); moreover, since there are as many agents as objects, *envy-free* allocations are Pareto efficient (Svensson, 1983).[11] A solution f is *envy-free* if it selects an *envy-free* allocation for each preference profile.

3.2 Manipulation of a solution

Our main interest is the manipulation of solutions when some agents may be sincere. We consider the direct revelation game associated with a solution f when a group of agents $T \subseteq N$ is sincere, true preferences are $R \in \mathscr{R}^N$, and the set of admissible preferences is $\mathscr{D} \subseteq \mathscr{R}$. Formally, this is a game for the set of players $N \setminus T$ in which each player's strategy space is $\mathscr{D}$ and the outcome function is $f(R_T, \cdot)$.

One can easily see that the set of pure strategy Nash equilibria of the game above may be empty.[12] The issue here is that each solution acts as a tie-breaker in the profiles for which multiple allocations are welfare equivalent to the selected allocation. Similarly to a Bertrand competition game, a solution may induce discontinuities in payoffs, conducing to the non-existence of pure strategy Nash equilibria.

There are several approaches to defining a sensible prediction for these manipulation games. The most convenient for our purpose is the notion of limit Nash equilibrium.[13] An **ε-equilibrium** is a strategy profile $R^{\varepsilon}_{N\setminus T}$ from which no strategic agent can change her report and obtain an allocation that is preferred to the bundle obtained by adding ε of money to her consumption at $R^{\varepsilon}_{N\setminus T}$. An allocation is a **limit Nash equilibrium outcome** if it is the limit as $\varepsilon \to 0$ of a sequence of ε-equilibrium outcomes.[14] We denote the set of limit Nash equi-

[11] An allocation $z \in Z$ is Pareto efficient $R \in \mathscr{R}^N$ if there is no other allocation $z' \in Z$ such that for each $i \in N$, $z'_i \, R_i \, z_i$ and for at least one $j \in N$, $z'_j \, R_j \, z_j$.

[12] See Footnote 8 for an example.

[13] Alternative approaches are the extension of the equilibrium concept to include the tie breaking rule, or sharing rule (Simon and Zame, 1990), or the discretization of consumptions of money. One can also consider multivalued rules and define an extension of Nash equilibrium to games with multivalued outcome functions (Tadenuma and Thomson, 1995a; Beviá, 2010). Limit equilibria captures the intuitive idea that the non-existence of equilibria is only caused by the tie-breaking role played by solutions, while applying to all single-valued solutions.

[14] Since preferences are money monotone, and thus continuous, our notion of limit equilibria is equivalent to the standard notion with respect to any continuous utility representations of preferences. That is, given a profile of utilities $u \equiv (u_i)$ representing R, an ε-equilibrium with respect to u is a profile of strategies $R^{\varepsilon}_{N\setminus T}$ such that for each $i \in N \setminus T$ and each $R'_i \in \mathscr{R}$,

librium outcomes from the manipulation of a solution f at $R \in \mathscr{R}^N$ when the domain of admissible preferences is $\mathscr{D}$ and the set of sincere agents is T by $\boldsymbol{\mathcal{O}(N \setminus T, \mathscr{D}^{N\setminus T}, f(R_T,\cdot), R_{N\setminus T})}$.

3.3 Incentives for unconditional truthful revelation of preferences

Our model allows us to evaluate an agent's incentive to be sincere, i.e., unconditionally report her true preferences, when a solution is operated. In order to do so, we need to compare the possible payoffs for the agent if she decides to be sincere with the payoffs when she is strategic. From the multiple approaches to compare payoff sets, we concentrate on two benchmark measures that can be associated to pessimistic and optimistic views of the agent.

Consider first an agent who expects to coordinate on his worst possible payoff equilibrium if she decides to be strategic. In order to measure the incentive to be sincere for such an agent, we define the **price of sincerity for a pessimistic agent (PSP)** for agent i at f, $T \subseteq N \setminus \{i\}$, and R, when the domain of admissible preferences is $\mathscr{D} \subseteq \mathscr{R}$ as the ratio of the agent's worst-case normalized equilibrium payoff when she is strategic to that when she is sincere.[15]

$$\boldsymbol{PSP_i[f,\mathscr{D},T,R]} \equiv \frac{\min\{u_i(z_i) : z \in \mathcal{O}(N \setminus T, \mathscr{D}^{N\setminus T}, f(R_T,\cdot), R_{N\setminus T})\}}{\min\{u_i(z_i) : z \in \mathcal{O}(N \setminus (T+i), \mathscr{D}^{N\setminus(T+i)}, f(R_{T+i},\cdot), R_{N\setminus(T+i)})\}}.$$

Consider now an agent who expects to coordinate on his best possible payoff equilibrium if she decides to be strategic. In order to measure the incentive to be sincere for such an agent, we define the **price of sincerity for an optimistic agent (PSO)** for agent i at f, $T \subseteq N \setminus \{i\}$, and R, when the domain of admissible preferences is $\mathscr{D} \subseteq \mathscr{R}$ as the ratio of the agent's best-case normalized equilibrium payoff when she is strategic to that when she is sincere.

$$\boldsymbol{PSO_i[f,\mathscr{D},T,R]} \equiv \frac{\max\{u_i(z_i) : z \in \mathcal{O}(N \setminus T, \mathscr{D}^{N\setminus T}, f(R_T,\cdot), R_{N\setminus T})\}}{\max\{u_i(z_i) : z \in \mathcal{O}(N \setminus (T+i), \mathscr{D}^{N\setminus(T+i)}, f(R_{T+i},\cdot), R_{N\setminus(T+i)})\}}.$$

$u(f(R_T, R^{\varepsilon}_{N\setminus T})) \geq u(f(R_T, R^{\varepsilon}_{N\setminus(T\cup\{i\})}, R'_i)+\varepsilon$. A limit equilibrium with respect to u is the limit as $\varepsilon \to 0$ of a sequence of ε-equilibrium outcomes with respect to u.

[15] We use min instead of inf in the definition of *PSP* and *PSO*, because our results show that the minimum in the respective sets always exist.

4 Results

A Maximin strategy for an agent in a game, is a strategy that gives the agent her best worst-case scenario outcome among all strategies. Our first result states that an agent's true preference relation is one of her Maximin strategies in the revelation game associated with any envy-free solution at any preference profile. We collect all proofs in the Appendix.

Proposition 1 (Truth is Maximin strategy)**.** *Let f be an envy-free solution, $i \in N$, and $R_i^0 \in \mathscr{R}$. Then,*

$$R_i^0 \in \underset{R_i \in \mathscr{R}}{\arg\max} \left\{ \inf\{u[R_i^0](z_i) : z \in f(R_i, R_{-i}), R_{-i} \in \mathscr{R}^{N \setminus i}\} \right\}.$$

Proposition 1 allows us to interpret sincere agents in our model as subjects who exhibit an extreme form of risk aversion.

The following theorem, our main result, characterizes the limit Nash equilibrium outcomes of the game associated with an envy-free solution at a profile R when a group of agents is sincere. First, the set is non-empty. Second, the set is welfare equivalent to the set of envy-free allocations for the true preferences that are not Pareto dominated by any other envy-free allocation for the strategic agents.

Theorem 1. *Let f be an envy-free solution, $R \in \mathscr{R}^N$, and $T \subsetneq N$. Then,*

(i) $\mathscr{O}(N \setminus T, \mathscr{R}^{N\setminus T}, f(R_T, \cdot), R_{N\setminus T}) \neq \emptyset$

(ii) If $z \in \mathscr{O}(N \setminus T, \mathscr{R}^{N\setminus T}, f(R_T, \cdot), R_{N\setminus T})$, then $z \in F(R)$ and there is no $z' \in F(R)$ such that for each $i \in N \setminus T$, $z'_i \, R_i \, z_i$ and for some $j \in N \setminus T$, $z'_j \, P_j \, z_j$.

(iii) If $\hat{z} \in F(R)$ and there is no $z' \in F(R)$ such that for each $i \in N \setminus T$, $z'_i \, R_i \, \hat{z}_i$ and for some $j \in N \setminus T$, $z'_j \, P_j \, \hat{z}_j$, then there is $z \in \mathscr{O}(N \setminus T, \mathscr{R}^{N\setminus T}, f(R_T, \cdot), R_{N\setminus T})$ such that for each $i \in T$, $\hat{z}_i \, I_i \, z_i$ and for each $j \in N \setminus T$, $\hat{z}_j = z_j$.

Our proof of Theorem 1 follows from three lemmas of independent interest. These results provide an expedite test to determine whether an allocation is a limit Nash equilibrium from the manipulation of an envy-free solution when some agents are sincere. They state necessary and sufficient conditions in terms of a binary relation on the set of agents, which is associated with each allocation and the true preferences. Let z be an allocation and R a preference profile. Intuitively, agent i dominates agent j in terms of the binary relation associated with R and z when agent i's consumption at z can be connected to that of agent j through a chain of indifferences among the agents and their consumptions at z.

Formally, the binary relation $\trianglerighteq (R,z)$ is defined as follows: for each pair $\{i,j\} \subseteq N$, $i \trianglerighteq(R,z)\, j$ if there is a list of agents $i_1, i_2, \ldots, i_k$ such that $z_{i_1}\, I_{i_1}\, z_{i_2}, \ldots, z_{i_{k-1}}\, I_{i_{k-1}}\, z_{i_k}, z_{i_k}\, I_{i_k}\, z_{i_1}$, $i_1 = i$, and $i_k = j$.[16]

The use of this binary relation for the study of the manipulation of envy-free solutions in $\mathscr{R}$ was pioneered by Velez (2011) for the money-Rawlsian envy-free solutions. These solutions select for each preference profile, the envy-free allocations that maximize the minimum individual consumption of money recalibrated by a family of increasing functions. Following this approach, Fujinaka and Wakayama (2012) study the manipulation of the solution that achieves the maximum welfare for a given agent.[17] The study of this solution proves to be of great theoretical interest. It delivers a characterization of the situations in which an agent is able to manipulate any envy-free solution (Fujinaka and Wakayama, 2012, Maximal Manipulation Theorem (MMT)).

Our first two lemmas are applications of the MMT. They state necessary conditions for an allocation to be a limit Nash equilibrium from the manipulation of an envy-free solution when some agents are sincere. First, each limit Nash equilibrium outcome is envy-free for true preferences.

Lemma 1. *Let f be an envy-free solution, $R \in \mathscr{R}^N$, and $T \subsetneq N$. Then,*

$$\mathscr{O}(N \setminus T, \mathscr{R}^{N\setminus T}, f(R_T, \cdot), R_{N\setminus T}) \subseteq F(R).$$

Our second lemma states a necessary condition for an allocation z to be a limit Nash equilibrium outcome from the manipulation of an envy-free solution at R in terms of $\trianglerighteq(R,z)$: the consumption of each sincere agent can be linked with the consumption of a strategic agent through a chain of indifferences, with respect to true preferences, among the agents and their consumptions at z.

Lemma 2. *Let f be an envy-free solution, $R \in \mathscr{R}^N$, and $T \subsetneq N$. If $z \in \mathscr{O}(N \setminus T, \mathscr{R}^{N\setminus T}, f(R_T, \cdot), R_{N\setminus T})$, then for each $i \in T$ there is $j \in N \setminus T$ such that $i \trianglerighteq(R,z)\, j$.*

[16] The binary relation $\trianglerighteq (R,z)$ was introduced, in its equivalent form for the set of objects, by Alkan (1994). It has been at the center of the analysis of incentives in the manipulation of envy-free solutions (Velez, 2011; Fujinaka and Wakayama, 2012; Andersson et al., 2011).

[17] Andersson et al. (2011) independently study this solution in the quasi-linear domain of preferences.

Our third lemma is essentially a converse to Lemmas 1 and 2. It states that, essentially, each envy-free allocation for preferences R, z, at which the consumption of each sincere agent can be linked with a strategic agent through a chain of indifferences, with respect to true preferences, among the agents and their consumptions at z, is the outcome from the manipulation of any envy-free solution at R. Here, "essentially" means that if z is not such an equilibrium outcome, then there is an equilibrium outcome that is welfare equivalent to z.

Lemma 3. *Let f be an envy-free solution, $R \in \mathscr{R}^N$, and $T \subsetneq N$. Let $\hat{z} \in F(R)$ be such that for each $i \in T$ there is $j \in N \setminus T$ such that $i \trianglerighteq (R, \hat{z}) j$. Then, there is $z \in \mathscr{O}(N \setminus T, \mathscr{R}^{N \setminus T}, f(R_T, \cdot), R_{N \setminus T})$ such that for each $i \in T$, $\hat{z}_i \, I_i \, z_i$ and for each $j \in N \setminus T$, $z_j = \hat{z}_j$.*

There has been a wide range of approaches to study the manipulation of envy-free solutions. Under the diverse predictions for these manipulation games considered in the literature, the results have been uniform: the set of outcomes from the manipulation of each envy-free solution when all agents are strategic is the set of envy-free allocations for true preferences (Tadenuma and Thomson, 1995a; Āzacis, 2008; Beviá, 2010; Velez, 2011; Fujinaka and Wakayama, 2012). The following corollary of Theorem 1 states that the aforementioned result also holds when one considers limit Nash equilibria as the prediction in these manipulation games. Our study is the first to consider this equilibrium concept in this environment. We omit the straightforward proof.

Corollary 1. *Let f be an envy-free solution and $R \in \mathscr{R}^N$. Then,*

$$\mathscr{O}(N, \mathscr{R}^N, f, R) = F(R).$$

We now consider a pessimistic agent's incentives for truthful revelation of preferences. Our main result here is that if there is another strategic agent, a pessimistic agent would not be hurt by committing to unconditionally report her true preferences.

Proposition 2. *Let f be an envy-free solution, $R \in \mathscr{R}^N$, $i \in N$, and $T \subsetneq N \setminus \{i\}$. Then,*

$$PSP_i[f, \mathscr{R}, T, R] = 1.$$

Proposition 2 allows us to conclude that a pessimistic agent does not expect to benefit from strategizing in the manipulation of an envy-free solution. It is worth noting that this is different from the agent exhibiting an extreme form of risk aversion. Maximin strategies bound the worst-case payoff for an agent for all possible reports of the other agents. By contrast, a pessimistic agent is a welfare maximizing agent who, aware of the equilibrium play, has a pessimistic expectation about what the outcomes from manipulation will be.

Proposition 2 follows from two results of independent interest that we state next. Our first result, a corollary of Lemma 3, states that any achievable consumption for a sincere agent is achievable if, ceteris paribus, the agent becomes strategic. This implies that the worst-case scenario equilibrium for a sincere agent is no worse than the the worst-case scenario equilibrium if the agent is strategic.

Corollary 2. *Let f be an envy-free solution, $R \in \mathscr{R}^N$, and $T \subsetneq N$. Let $i \in T$ and $z \in \mathscr{O}(N \setminus T, \mathscr{R}^{N\setminus T}, f(R_T, \cdot), R_{N\setminus T})$. Then, there is*

$$z' \in \mathscr{O}(N \setminus (T \setminus \{i\}), \mathscr{R}^{N\setminus(T\setminus\{i\})}, f(R_{T\setminus\{i\}}, \cdot), R_{N\setminus(T\setminus\{i\})}),$$

such that $z'_i = z_i$.

One may think that Corollary 2 is trivial, since a strategic agent has always her true preferences as a possible strategy. However, true preferences may not be a best response, for a strategic agent, to the strategies of the other players that sustain an allocation as an equilibrium outcome when the agent is sincere. Here is were Lemma 3 comes into play and allows us to identify a limit Nash equilibrium with the desired properties.

The following lemma is a converse to our worst-case interpretation of Corollary 2. It states that as long as there is another strategic agent, the worst-case scenario equilibrium for a strategic agent is no worse than the worst-case scenario equilibrium if the agent is sincere.

Lemma 4. *Let f be an envy-free solution, $R \in \mathscr{R}^N$, $i \in N$, and $T \subsetneq N \setminus \{i\}$. Let $z \in \mathscr{O}(N \setminus T, \mathscr{R}^{N\setminus T}, f(R_T, \cdot), R_{N\setminus T})$. Then, then there is*

$$\hat{z} \in \mathscr{O}(N \setminus (T \cup \{i\}), \mathscr{R}^{N\setminus(T\cup\{i\})}, f(R_{T\cup\{i\}}, \cdot), R_{N\setminus(T\cup\{i\})}),$$

such that $z_i \, R_i \, \hat{z}_i$.

It is straightforward to see that Proposition 2 follows from Corollary 2 and Lemma 4.

We now study the incentives for unconditional truthful revelation for an optimistic agent. The following corollary to Lemma 3 states that an agent's preferred envy-free allocation for the true preferences is always among the outcomes from the manipulation of any envy-free solution when the agent is strategic.

Corollary 3. *Let f be an envy-free solution, $R \in \mathscr{R}^N$, $i \in N$, and $T \subseteq N \setminus \{i\}$. Then,* $\arg\max\{u[R_i](z_i) : z \in F(R)\} \cap \mathscr{O}(N \setminus T, \mathscr{R}^{N\setminus T}, f(R_T,\cdot), R_{N\setminus T}) \neq \emptyset$.

The following proposition states that generically the price of sincerity for an optimistic agent is greater than one.

Proposition 3. *Let f be an envy-free solution, $R \in \mathscr{R}^N$, $i \in N$, and $T \subseteq N \setminus \{i\}$.*

1. *For each $R \in \mathscr{R}^N$, $PSO_i[f, \mathscr{R}, T, R] \geq 1$.*
2. *The set of utility profiles $u[R]$ with $R \in \mathscr{R}^N$ for which*

$$PSO_i[f, \mathscr{R}, T, R] > 1,$$

is an open dense set in the product topology of $\mathscr{U}^N$.

Corollary 3 and Proposition 3 allow us to conclude that a sincere agent generically gives up the opportunity to obtain her preferred envy-free allocation for the true preferences. Thus, for almost all preference profiles, independently of the envy-free solution that is operated, an optimistic agent would never give up her strategic advantage.

We finally investigate to what extent strategic agents can take advantage of sincere agents. Let $R \in \mathscr{R}^N$ and $z \in F(R)$. Suppose that z is not a limit equilibrium of the game associated with an envy-free solution f at R. From Theorem 1 we know that z has to be Pareto dominated for the strategic agents by another envy-free allocation. The following proposition strengthens this result. It states that there is a limit equilibrium of the game associated with f at R, $\hat{z}$, that all strategic agents prefer to z. Moreover, there is at least a sincere agent who prefers z to $\hat{z}$.

Proposition 4. *Let f be an envy-free solution, $R \in \mathscr{R}^N$, and $T \subsetneq N$. If $z \notin \mathscr{O}(N \setminus T, \mathscr{R}^{N\setminus T}, f(R_T,\cdot), R_{N\setminus T})$, then there is $\hat{z} \in \mathscr{O}(N \setminus T, \mathscr{R}^{N\setminus T}, f(R_T,\cdot), R_{N\setminus T})$ and $i \in T$ such that $z_i\, P_i\, \hat{z}_i$ and for each agent $j \in N \setminus T$, $\hat{z}_j\, P_j\, z_j$.*

A corollary of Proposition 4 is that if agent i is the only sincere agent and z is not a limit equilibrium of the game associated with an envy-free solution f at R, then there is an equilibrium $\hat{z}$ such that agent i prefers z to $\hat{z}$ (all strategic agents prefer $\hat{z}$ to z). We omit the straightforward proof.

Corollary 4. *Let f be an envy-free solution, $R \in \mathscr{R}^N$, and $i \in N$. If $z \notin \mathscr{O}(N \setminus \{i\}, \mathscr{R}^{N\setminus\{i\}}, f(R_i,\cdot), R_{N\setminus\{i\}})$, then there is $\hat{z} \in \mathscr{O}(N \setminus \{i\}, \mathscr{R}^{N\setminus\{i\}}, f(R_i,\cdot), R_{N\setminus\{i\}})$ such that $z_i\, P_i\, \hat{z}_i$ and for each agent $j \in N \setminus \{i\}$, $\hat{z}_j\, P_j\, z_j$.*

5 Discussion

5.1 Quasilinear preferences

All of our results in Section 4 hold if one assumes that both true preferences and admissible preferences are quasi-linear. If one interprets our model as an arbitration problem, this would correspond to the case in which the arbitrator knows that preferences are quasi-linear, but has no additional information.

Proposition 1 holds in a stronger form when both true preferences and admissible preferences are quasi-linear. The following proposition states that, in this case, not only truth is a Maximin strategy in the game associated with an envy-free allocation at some profile, but also it is the unique Maximin strategy.

Proposition 5 (Truth is unique Maximin strategy in quasi-linear domain)**.** *Let f be an envy-free solution, $i \in N$, and $R_i^0 \in \mathscr{Q}$. Then,*

$$\left\{R_i^0\right\} = \underset{R_i \in \mathscr{Q}}{\arg\max} \left\{\inf\{u[R_i^0](z_i) : z \in f(R_i, R_{-i}), R_{-i} \in \mathscr{Q}^{N\setminus i}\}\right\}.$$

One can easily see that our proof of Lemmas 1 and 2 go through with no modification when true preferences and admissible preferences are quasi-linear. Our proof of Lemma 3 requires to be adjusted, however. There, we construct a sequence of ε-equilibria, R^ε, whose outcomes converge, as $\varepsilon \to 0$, to an allocation z satisfying the properties stated in the lemma. Each of these R^ε is defined by means of two equations which imply that generically $R^\varepsilon \in (\mathscr{R} \setminus \mathscr{Q})^{N\setminus T}$. One can prove that the sequence of quasi-linear preferences satisfying only the first of these conditions is a sequence of ε-equilibria when the domain of admissible preferences is $\mathscr{Q}$. Moreover, the outcomes of this sequence also converge to z as $\varepsilon \to 0$. Thus, Lemma 3 also holds in the quasi-linear domain. Consequently, Theorem 1, Propositions 2 and 4, and Corollaries 1-4 hold whenever true preferences and admissible preferences are in $\mathscr{Q}$.

When preferences are quasi-linear we can evaluate genericity statements not only in a topological sense, but also with respect to the Lebesgue measure. The following proposition is the counterpart of Proposition 3 in $\mathscr{Q}$. It states that the price of sincerity for an optimistic agent in $\mathscr{Q}$ is generically greater than one. That is, for almost all preference profiles, a sincere agent gives up the possibility to obtain her best allocation among all envy-free allocations for the true preferences. It is worth noting that genericity here holds with respect to the Euclidean topology and the Lebesgue measure on $\mathscr{Q}^N$.

Proposition 6. *Let f be an envy-free solution, $i \in N$, and $T \subseteq N \setminus \{i\}$.*

1. *For each $R \in \mathcal{Q}^N$, $PSO_i[f, \mathcal{Q}, T, R] \geq 1$.*

2. *The set of preference profiles $R \in \mathcal{Q}^N$ for which*

$$PSO_i[f, \mathcal{Q}, T, R] > 1,$$

   is an open dense set of $\mathbb{R}^{(n-1)\times n}$.

3. *The set of preference profiles $R \in \mathcal{Q}^N$ for which*

$$PSO_i[f, \mathcal{Q}, T, R] = 1,$$

   has Lebesgue measure zero in $\mathbb{R}^{(n-1)\times n}$.

5.2 Dominant strategies and truthful revelation

Let f be a solution. Assume that the domain of admissible preferences is $\mathcal{R}$. Let $i \in N$. Agent i's true preference relation is a dominant strategy in the manipulation game of f if for each $R \in \mathcal{D}^N$ and each $R'_i \in \mathcal{D}$, $f_i(R)\, R_i\, f(R_{-i}, R'_i)$. It is well known that no envy-free solution is such that for each agent her true preference relation is a dominant strategy (Tadenuma and Thomson, 1995a). There are envy-free solutions that achieve some partial form of incentive compatibility, however. Define a solution, which we refer to as the i-optimal envy-free solution, as follows: from the set of all envy-free allocations for a profile, select one of agent i's preferred allocations (these allocations are welfare equivalent for all agents; Alkan et al., 1991). It turns out that true preferences are a dominant strategy for agent i when this solution is operated (Andersson et al., 2011).[18] Notice here the similarity with marriage markets, where the men and women optimal stable solutions make it a dominant strategy for men and women to report their true preferences, respectively (Dubins and Freedman, 1981; Roth, 1982).

In order to completely understand agent i's incentives to truthfully reveal her preferences when the i-optimal envy-free solution is operated, one has to

[18] The agent-optimal envy-free solutions are dominant ("least manipulable") in the family of envy-free solutions with respect to the following comparative notion of incentive compatibility: solution 1 is more manipulable than solution 2 if whenever an agent can manipulate solution 2, the agent can manipulate solution 1 as well (Andersson et al., 2011).

determine the consequences of her unconditional truthful behavior. Our results reveal that, if it is common knowledge that agent i is sincere, then the outcomes from the manipulation of the i-optimal envy-free solution are the same as those from the manipulation of any envy-free solution (Theorem 1). Independently of the solution that is operated, agent i's decision to unconditionally report her true preferences depends on how optimistic or pessimistic the agent is. If the agent is pessimistic, one may expect she would be sincere (Proposition 2). If the agent is optimistic, one may expect she will be strategic (Propositions 3 and 6). This suggest that there is no connection between dominance of truthful revelation and the decision to be sincere for an agent in our environment.

Appendix

Proof of Proposition 1. Let $\{R_i^0, R_i\} \subseteq \mathscr{R}$ and $R_{-i} \in \mathscr{R}^{N\setminus\{i\}}$. We prove that there is $R'_{-i} \in \mathscr{Q}^{N\setminus\{i\}}$ such that $f_i(R_{-i}, R_i^0)\, R_i^0\, f_i(R'_{-i}, R_i)$. Since $R_i^0 \in \mathscr{R}$, then there is $x \equiv (x_\alpha)_{\alpha\in A} \in \mathbb{R}^A$ such that $\sum_{\alpha\in A} x_\alpha = M$ and for each pair $\{\alpha,\beta\} \subseteq A$, $(x_\alpha,\alpha)\, I_i^0\, (x_\beta,\beta)$. Since envy-free allocations satisfy the Identical Preferences Lower Bound (IPLB), then for each $\alpha \in A$, $f_i(R_{-i}, R_i^0)\, R_i^0\, (x_\alpha,\alpha)$ (Moulin, 1990; Beviá, 2010). For each $j \in N\setminus\{i\}$, let $R'_j \in \mathscr{Q}$ be such that for each pair $\{\alpha,\beta\} \subseteq A$, $(x_\alpha,\alpha)\, I'_j\, (x_\beta,\beta)$. Let $z \equiv (y,\mu) \equiv f(R'_{-i}, R_i)$. Recall that $z \in F(R'_{-i}, R_i)$. Since envy-free allocations satisfy the IPLB, then for each $j \in N\setminus\{i\}$, $y_{\mu(j)} \geq x_{\mu(j)}$. Thus, $y_{\mu(i)} \leq x_{\mu(i)}$ and $(x_{\mu(i)}, \mu(i))\, R_i^0\, (y_{\mu(i)}, \mu(i))$. Thus, $f_i(R_{-i}, R_i^0)\, R_i^0\, f_i(R'_{-i}, R_i)$.

Then $\inf\{u[R_i^0](f(R_{-i}, R_i^0)) : R_{-i} \in \mathscr{R}^{N\setminus\{i\}}\} \geq u[R_i^0](f_i(R'_{-i}, R_i^0))$ and $u[R_i^0](f_i(R'_{-i}, R_i^0)) \geq \inf\{u[R_i^0](f(R_{-i}, R'_i)) : R_{-i} \in \mathscr{R}^{N\setminus\{i\}}\}$. Thus,

$$R_i^0 \in \underset{R_i\in\mathscr{R}}{\arg\max}\left\{\inf\{u[R_i^0](z_i) : z \in f(R_i, R_{-i}), R_{-i} \in \mathscr{R}^{N\setminus i}\}\right\}.$$

$\square$

The following results play an important role in our proofs. We state them for completeness. See the respective papers for the proof.

Lemma 5 (Decomposition Lemma; Alkan et al., 1991)**.** *Let $R \in \mathscr{R}^N$, $z \equiv (x,\mu) \in F(R)$, and $\hat{z} \equiv (\hat{x},\hat{\mu}) \in F(R)$. Then, both μ and $\hat{\mu}$ are bijections between:*

(i) $\{i \in N : z_i\, P_i\, \hat{z}_i\}$ and $\{\alpha \in A : x_\alpha > \hat{x}_\alpha\}$.

(ii) $\{i \in N : z_i\, I_i\, \hat{z}_i\}$ and $\{\alpha \in A : x_\alpha = \hat{x}_\alpha\}$.

(iii) $\{i \in N : \hat{z}_i\, P_i\, z_i\}$ and $\{\alpha \in A : \hat{x}_\alpha > x_\alpha\}$.

Theorem 2 (Maximal Manipulation Theorem; Fujinaka and Wakayama, 2012)**.** *Let* $\mathscr{D} \in \{\mathscr{R}, \mathscr{Q}\}$*,* f *an envy-free solution,* $R \in \mathscr{D}^N$*,* $i \in N$*, and* $z \equiv (x, \mu)$ *such that* $z \in \arg\max\{u[R_i](z_i) : z \in F(R)\}$*. Then, for each* $\varepsilon > 0$ *there is* $R_i^\varepsilon \in \mathscr{D}$ *such that* $f(R_{-i}, R_i^\varepsilon)\, R_i\, (x_{\mu(i)} - \varepsilon, \mu(i))$*.*

Proof of Lemma 1. Let $z \in \mathscr{O}(N \setminus T, \mathscr{R}^N \setminus T, f(R_T, \cdot), R_{N\setminus T})$ and $R^*_{N\setminus T}$ an action profile that sustains z as an equilibrium outcome. We prove that $z \in F(R)$. Suppose by contradiction that there is $\{i, j\} \subseteq N$ such that $z_j\, P_i\, z_j$. Suppose without loss of generality that z_j is agent i's preferred bundle in $(z_k)_{k\in N}$. Since z is *envy-free* for the reported preferences, then $i \in N \setminus T$. Let $R' = (R_T, R_i, R^*_{N\setminus(T\cup\{i\})})$. We claim that there is $z' \in F(R')$ such that $z'_i\, P_i\, z_i$. Suppose first that $j \trianglerighteq (R', z)\, i$. Then, there is a way to reshuffle objects at z so that agent i receives $\mu(j)$ and the resulting allocation is envy free for R' (just reshuffle along the chain that defines $j \trianglerighteq (R', z)\, k$). Let $N' \equiv \{k : k \trianglerighteq (R', z)\, i\}$. Suppose now that $N \setminus N' \neq \emptyset$. A perturbation argument easily shows that one can find z' by extracting some money from $N \setminus N'$, and distributing it among N' so all agents in N' benefit. Since $i \in N'$, then $z'_i\, P_i\, z_i$. By the Maximal Manipulation Theorem (Fujinaka and Wakayama, 2012), for small ε, no allocation close to z is an ε*-equilibrium*. This is a contradiction.[19] □

Proof of Lemma 2. Let $z \in \mathscr{O}(N \setminus T, \mathscr{R}^N \setminus T, f(R_T, \cdot), R_{N\setminus T})$ and $R^*_{N\setminus T}$ an action profile that sustains z as an equilibrium outcome. Let $i \in T$. We prove that there is $j \in N \setminus T$ such that $i \trianglerighteq (R, z)\, j$. Suppose by means of contradiction that $N' \equiv \{k : i \trianglerighteq (R', z)\, k\}$ is such that $N' \cap N \setminus T = \emptyset$. Let $j \in N \setminus T$ and $R' = (R_T, R_j, R^*_{N\setminus(T\cup\{i\})})$. Since f is *envy-free* and by Lemma 1, $z \in F(R)$, then $z \in F(R')$. Then, one can construct $z' \in F(R')$ such that $z'_i\, P_i\, z_i$ by extracting an amount of money from the agents in N' and distributing it among $N \setminus N'$ so all agents in $N \setminus N'$ benefit. By the Maximal Manipulation Theorem (Fujinaka and Wakayama, 2012), for small ε, no allocation close to z is an ε*-equilibrium*. This is a contradiction. □

Proof of Lemma 3. Let $\hat{z} \equiv (\hat{x}, \hat{\mu}) \in F(R)$ be such that for each $i \in T$ there is $j \in N \setminus T$ such that $i \trianglerighteq (R, \hat{z})\, j$.

Let $\varepsilon > 0$. We construct an ε-equilibrium of the game associated with f at R, $z \in Z$, such that for each $j \in N \setminus T$, $z_j \approx \hat{z}_j$. Let $\delta > 0$ and $m \in \mathbb{R}$ be such that $m > n^2\delta$ and for each $\alpha \in A$ and each $i \in T$, $(\alpha, \hat{x}_\alpha + m)\, P_i\, \hat{z}_i$. Let $i \in N \setminus T$ and $\alpha \equiv \hat{\mu}(i)$. Let R_i^δ be such that for each $\beta \in A \setminus \alpha$,

[19] An alternative proof of this lemma can be constructed along the lines of Fujinaka and Wakayama (2012, Theorem 3).

1. $(x_\alpha-(n-1)\delta,\alpha)\, I_i^\delta\, (x_\beta+\delta,\beta)$.

2. $(x_\alpha,\alpha)\, I_i^\delta\, (x_\beta+m,\beta)$.

Let $R^\delta \equiv (R_T, R^\delta_{N\setminus T})$ and $z\equiv(x,\mu)\in f(R^\delta)$. We claim that for each $i\in N\setminus T$, $\mu(i)=\hat{\mu}(i)$. Suppose by contradiction that $\mu(i)\neq\hat{\mu}(i)$. Since F satisfies the identical preferences lower bound, then $x_{\mu(i)}\geq \hat{x}_{\mu(i)}+\delta$. Since $\{\hat{z},z\}\subseteq F(R^\delta)$, then by the Decomposition Lemma, $z_i\, P_i^\delta\, \hat{z}_i$. By (2) in the definition of R_i^δ, $x_{\mu(i)}>\hat{x}_{\mu(i)}+m$. Let $j\in T$. Since $z\in F(R^\delta)$, then $z_j\, R_j\, z_i$. By definition of m, $z_i\, P_j\, \hat{z}_j$. Thus, by the Decomposition Lemma, $x_{\mu(j)}>\hat{x}_{\mu(j)}$ and $x_{\hat{\mu}(j)}>\hat{x}_{\hat{\mu}(j)}$. Let $k\in N\setminus T$. Since F satisfies the IPLB, then by (1) in the definition of R_i^δ, $x_{\mu(k)}\geq x_{\mu(k)}-(n-1)\delta$. Then,

$$\sum_{\alpha\in A} x_\alpha \geq \hat{x}_{\mu(i)}+M+\sum_{\alpha\in A\setminus\{\mu(i)\}}[\hat{x}_\alpha-(n-1)\delta] > \sum_{\alpha\in A}\hat{x}_\alpha.$$

This is a contradiction.

Let $i\in N\setminus T$ and $\nu>0$. We prove that there is $\delta>0$ such that for each $\tilde{R}_i\in\mathscr{R}$ and each $\tilde{z}\equiv(\tilde{x},\tilde{\mu})\in F(R^\delta_{-i},\tilde{R}_i)$ such that $\tilde{x}_{\hat{\mu}(i)}\geq\hat{x}_{\hat{\mu}(i)}$ we have that for each $j\neq i$, $\tilde{x}_{\hat{\mu}(j)}\geq\hat{x}_{\hat{\mu}(j)}-\nu$. Suppose by contradiction that there is a sequence $\{\delta_t\}_{t\in\mathbb{N}}$ such that as $t\to\infty$, $\delta_t\to 0$ and for each $t\in\mathbb{N}$ there is $R_i^t\in\mathscr{R}$, $z^t\equiv(x^t,\mu^t)\in F(R^{\delta_t}_{-i},R_i^t)$ such that $x^t_{\mu^t(i)}\geq\hat{x}_{\mu^t(i)}$ and $j^t\neq i$ such that $x^t_{\mu^t(j^t)}<\hat{x}_{\mu^t(j^t)}-\nu$. Since N and A are finite, we can assume without loss of generality that the sequences $\{\mu^t\}_{t\in\mathbb{N}}$ and $\{j^t\}_{t\in\mathbb{N}}$ are constant. Denote the constant bijection by σ and the agent in the sequence by j^*. Since F satisfies the IPLB, $j^*\in T$, for otherwise $x^t_{\sigma(j^*)}\geq\hat{x}_{\sigma(j^*)}-(n-1)\delta^t\to\hat{x}_{\sigma(j^*)}$. Since for each $t\in\mathbb{N}$, $x^t_{\sigma(i)}\geq\hat{x}_{\sigma(i)}$, the sequence $\{x^t\}_{t\in\mathbb{N}}$ is bounded, and thus, we can suppose without loss of generality that it is convergent. Let $\tilde{x}\equiv\lim_{t\to\infty}x^t$. Then, $\tilde{x}_{\sigma(j^*)}<\hat{x}_{\sigma(j^*)}$. There are two cases:

1. Suppose that $\sigma(j^*)\in\hat{\mu}(N\setminus T)$. Let $k\in N$ be such that $\sigma(k)=\hat{\mu}(j^*)$. Let $t\in\mathbb{N}$. We claim that $x^t_{\sigma(k)}\geq\hat{x}_{\sigma(k)}$. If $k\in N\setminus(T+j^*)$, then $x^t_{\sigma(k)}\geq\hat{x}_{\sigma(k)}+\delta_t$. If $k\in T$ is such that there is $k'\in N\setminus(T+j^*+k)$ such that $\sigma(k')=\hat{\mu}(k)$, then $x^t_{\sigma(k')}\geq\hat{x}_{\sigma(k')}+\delta^t$. Since $\sigma(k')=\hat{\mu}(k)$, then $x^t_{\hat{\mu}(k)}\geq\hat{x}_{\hat{\mu}(k)}$. Since $z^t\in F(R^{\delta_t}_{-i},R^t_i)$ and $k\in T$, then $x^t_{\sigma(k)}\geq\hat{x}_{\sigma(k)}$. The recursive argument shows that $x^t_{\sigma(k)}\geq\hat{x}_{\sigma(k)}$. Since $\sigma(k)=\hat{\mu}(j^*)$, then $x^t_{\hat{\mu}(j^*)}\geq\hat{x}_{\hat{\mu}(j^*)}$. Since $z^t\in F(R^{\delta_t}_{-i},R^t_i)$, $j^*\in T$, and $x^t_{\hat{\mu}(j^*)}\geq\hat{x}_{\hat{\mu}(j^*)}$, we have that $x^t_{\sigma(j^*)}\geq\hat{x}_{\sigma(j^*)}$. Thus, $\tilde{x}_{\sigma(j^*)}\geq\hat{x}_{\sigma(j^*)}$. This is a contradiction.

2. Suppose that $\sigma(j^*)\in\hat{\mu}(T)$. Let $j\in N\setminus T$ be such that $j^*\trianglerighteq(R,\hat{z})j$. Assume that the indifference chain in $j^*\trianglerighteq(R,\hat{z})j$ is of length one (has two agents

forming a cycle). Let $t \in \mathbb{N}$. Since $(x^t, \sigma) \in F(R^{\delta_t}_{-i}, R^t_i)$, then $(x^t_{\sigma(j^*)}, \sigma(j^*))\, R_{j^*}\, (x^t_{\hat{\mu}(j)}, \hat{\mu}(j))$. Since preferences are continuous, then $(\tilde{x}_{\sigma(j^*)}, \sigma(j^*))\, R_j\, (\tilde{x}_{\hat{\mu}(j)}, \hat{\mu}(j))$. Since $\tilde{x}_{\sigma(j^*)} < \hat{x}_{\sigma(j^*)} - \nu$ and $\hat{z} \in F(R)$, then $\hat{z}_{j^*}\, P_{j^*}\, (\tilde{x}_{\sigma(j^*)}, \sigma(j^*))$. Thus, $\hat{z}_{j^*}\, P_{j^*}\, (\tilde{x}_{\hat{\mu}(j)}, \hat{\mu}(j))$ Let $k \equiv \sigma^{-1}(\hat{\mu}(j))$. If $k \in T$, then by the same argument in Case 1, we have that $\tilde{x}_{\sigma(k)} \geq \hat{x}_{\sigma(k)}$. If $k \in N \setminus T$, then for each $t \in \mathbb{N}$, $x^t_{\sigma(k)} \geq \hat{x}_{\hat{\mu}(j)}$. Thus, $\tilde{x}_{\sigma(k)} \geq \hat{x}_{\sigma(k)}$, or equivalently $\tilde{x}_{\hat{\mu}(j)} \geq \hat{x}_{\hat{\mu}(j)}$. Thus, $\hat{z}_{j^*}\, P_{j^*}\, (\hat{x}_{\hat{\mu}(j)}, \hat{\mu}(j)) = \hat{z}_j$. This contradicts $j^* \trianglerighteq (R, \hat{z})\, j$. The recursive argument shows that this is true for indifference chains of any length.

Let $i \in N \setminus T$. Let $\delta > 0$ be such that for each $\tilde{R}_i \in \mathscr{R}$ and each $\tilde{z} \equiv (\tilde{x}, \tilde{\mu}) \in F(R^{\delta}_{-i}, \tilde{R}_i)$ such that $\tilde{x}_{\tilde{\mu}(i)} \geq \hat{x}_{\tilde{\mu}(i)}$ we have that for each $j \neq i$, $\tilde{x}_{\tilde{\mu}(j)} \geq \hat{x}_{\tilde{\mu}(j)} - \frac{\varepsilon}{2(n-1)}$. Let $\tilde{R}_i \in \mathscr{R}$ and $\tilde{z} \equiv (\tilde{x}, \tilde{\mu}) = f(R^{\delta}_{-i}, \tilde{R}_i)$. Thus, $\tilde{x}_{\tilde{\mu}(i)} \leq \hat{x}_{\tilde{\mu}(i)} + \frac{\varepsilon}{2}$. Thus, R^{δ} is an ε-equilibrium. Let $z^{\delta} \equiv (x^{\delta}, \mu^{\delta})$. Assume without loss of generality that μ^{δ} is independent of δ. Let $i \in N \setminus T$. By construction, $\mu^{\delta}(i) = \hat{\mu}(i)$ and as $\delta \to 0$, $x^{\delta}_{\hat{\mu}(i)} \to \hat{x}_{\hat{\mu}(i)}$. Let $i \in T$. Since there is $j \in N \setminus T$ such that $i \trianglerighteq (R, \hat{z})\, j$, an argument as that in Case 2 above shows that as $\delta \to 0$, $x^{\delta}_{\hat{\mu}(i)} \to \hat{x}_{\hat{\mu}(i)}$. Thus, as $\delta \to 0$, $x^{\delta} \to \hat{x}$. Thus the limit of the sequence $\{z^{\delta}\}$ defines an allocation, say z. Since z is the limit of ε-equilibria, then it is a limit Nash equilibrium. Since preferences are continuous, then $z \in F(R)$. By the Decomposition Lemma, for each $i \in T$, $\hat{z}_i\, I_i\, z_i$. □

Proof of Theorem 1. (ii) Let $z \in \mathscr{O}(N \setminus T, \mathscr{R}^N \setminus T, f(R_T, \cdot), R_{N\setminus T})$. By Lemma 1, $z \in F(R)$. We prove that there is no $z' \in F(R)$ such that for each $i \in N \setminus T$, $z'_i\, R_i\, z_i$ and for some $j \in N \setminus T$, $z'_j\, P_j\, z_j$. Let $z' \in F(R)$ be such that for each $i \in N \setminus T$, $z'_i\, R_i\, z_i$. By the decomposition lemma, for each $i \in N \setminus T$, $x'_{\mu(i)} \geq x_{\mu(i)}$. Let $i \in T$. By Lemma 2, there is $j \in N \setminus T$ such that $i \trianglerighteq (R, z)\, j$. Since $x'_{\mu(j)} \geq x_{\mu(j)}$, then $z'_i\, R_i\, z_i$. By the Decomposition Lemma, $x'_{\mu(i)} \geq x_{\mu(i)}$. Thus, $x' = x$. Since $z \in F(R)$, then for each $i \in N \setminus T$, $z_i\, R_i\, z'_i$.

(iii) Now, suppose that $\hat{z} \in F(R)$ and there is no $z' \in F(R)$ such that for each $i \in N \setminus T$, $z'_i\, R_i\, \hat{z}_i$ and for some $j \in N \setminus T$, $z'_j\, P_j\, \hat{z}_j$. We prove that there is $z \in \mathscr{O}(N \setminus T, \mathscr{R}^N \setminus T, f(R_T, \cdot), R_{N\setminus T})$ such that for each $i \in T$, $\hat{z}_i\, I_i\, z_i$ and for each $j \in N \setminus T$, $\hat{z}_j = z_j$. We will prove that for each $i \in T$ there is $j \in N \setminus T$ such that $i \trianglerighteq (R, z)\, j$, which by Lemma 3 completes the proof of the Theorem. Suppose by contradiction that there is $i \in T$ such that for each $j \in N \setminus T$ it is not true that $i \trianglerighteq (R, z)\, j$. Let $M \equiv \{j \in N : i \trianglerighteq (R, z)\, j\}$. If $M \subseteq T$, then by a perturbation argument, one can easily construct an allocation $z' \in F(R)$ that is preferred by all agents in $N \setminus M$.

(i) Let $i \in N \setminus T$. If $z \in F(R)$ is the best allocation in $F(R)$ for agent i, then for each $j \in N \setminus \{i\}$, $j \trianglerighteq (R,z)\, i$ (Velez, 2011; Fujinaka and Wakayama, 2012) . By Lemma 3, $z \in \mathcal{O}(N \setminus T, \mathcal{R}^N \setminus T, f(R_T,\cdot), R_{N\setminus T})$. □

Proof of Corollary 2. Let $i \in T$ and $z \in \mathcal{O}(N \setminus T, \mathcal{R}^N \setminus T, f(R_T,\cdot), R_{N\setminus T})$. Let $j \in T \setminus \{i\}$. By Lemma 2, there is $k \in N \setminus T$ such that $j \trianglerighteq (R,z)\, k$. By Lemma 3, there is a limit Nash Equilibrium outcome of the game associated with f at R when $T \setminus \{i\}$ are sincere, z', such that $z'_i = z_i$. □

Proof of Lemma 4. Let f be an envy-free solution and $R \in \mathcal{R}^N$. Assume that agents in $T \subsetneq N$ are truthful and the set of admissible preferences is $\mathcal{R}$. Let $i \in N \setminus T$ and $z \in \mathcal{O}(N \setminus T, \mathcal{R}^N \setminus T, f(R_T,\cdot), R_{N\setminus T})$. Let $S \equiv N \setminus (T \cup \{i\})$. If there is $j \in S$ such that $i \trianglerighteq (R,z)\, j$, then by Lemma 3, z is the desired allocation. Suppose now that for no $j \in S$, $i \trianglerighteq (R,z)\, j$. Let $u \equiv (u_j)_{j\in N}$ be a continuous utility representation of R such that: for each $j \in S$, u_j is bounded below by 0 and $u_j(z_j) = 1$, and for each $j \in N \setminus S$, u_j is bounded above by 0. Let $V \equiv \arg\max\{\min_{j\in S} u_j(z'_j) : z' \in F(R)\}$. Since preferences are continuous and $F(R)$ is compact, then V is non-empty. Let $\hat{z} \equiv (\hat{x}, \hat{\mu}) \in V$. We claim that $\hat{z}$ is a limit Nash equilibrium of the game associated with f at R when the set of sincere agents is $T \cup \{i\}$ such that $z_i\, P_i\, \hat{z}_i$. For each $j \in N \setminus S$, there is $k \in S$ such that $j \trianglerighteq (R,z)\, k$, for otherwise a perturbation argument as in Velez (2011, Lemmas 1 and 2) shows that $\hat{z} \notin V$. By Lemma 3, $\hat{z}$ is a limit Nash equilibrium of the game associated with f at R when the set of sincere agents is $T \cup \{i\}$. By definition of V, for each $j \in S$, $\hat{z}_j\, P_j\, z_j$. Let $k \in N \setminus S$ be such that there is $j \in S$ such that $z_k\, I_k\, z_j$. By the Decomposition Lemma, $\hat{z}_k\, P_k\, z_k$ and $\hat{x}_{\mu(k)} > x_{\mu(k)}$. Thus, if $k' \in N \setminus S$ is such that $z_{k'}\, I_{k'}\, z_k$, then $\hat{z}_{k'}\, P_{k'}\, z_{k'}$ and $\hat{x}_{\mu(k')} > x_{\mu(k')}$. The inductive argument shows that if $k \in N \setminus S$ is such that there is $j \in S$ such that $k \trianglerighteq (R,z)\, j$, then $\hat{z}_k\, P_k\, z_k$. Since z is Pareto efficient for R, then there is $j \in N \setminus S$ such that $z_j\, P_j\, \hat{z}_j$. By Lemma 2, there is $k \in S \cup \{i\}$ such that $j \trianglerighteq (R,z)\, k$. Thus, $j \trianglerighteq (R,z)\, i$. An inductive argument as described above shows that $z_i\, P_i\, \hat{z}_i$. □

Proof of Proposition 4. Suppose that $z \notin \mathcal{O}(N \setminus T, \mathcal{R}^N \setminus T, f(R_T,\cdot), R_{N\setminus T})$. By Lemma 3, there is an agent $i \in T$ such that for each agent $j \in N \setminus T$, it is not the case that $i \trianglerighteq (R,z)\, j$. Let $N(i) \equiv \{j \in N : i \trianglerighteq (R,z)\, j\} \subseteq T$. Let $m < \sum_{j\in N\setminus(i)} x_j$. There is a vector $y \in R^{N(i)}$, $y \leq x_{N(i)}$ and a bijection $\sigma : N(i) \to \mu(N(i))$ such that for each pair $\{j,k\} \in N(i)$, $(y_j, \sigma(j))\, R_j\, (y_k, \sigma(k))$ (Alkan et al., 1991; Velez, 2012b). Let μ' be the bijection defined by: for each $j \in N(i)$, $\mu'(i) = \sigma(i)$, and for each $j \in N \setminus N(i)$, $\mu'(i) = \mu(i)$. Since preferences are continuous, one can select m such that the allocation $z' = (x_{N\setminus N(i)}, y, \mu')$ is envy-free for the economy with budget $\sum_{j\in N(i)} y_i + \sum_{j\in N\setminus N(i)} x_j$ and there is $j \in N(i)$ and $k \in N \setminus N(i)$ such that $j \trianglerighteq(R,z')\,k$. Since T is finite, one can repeat this process and find an envy-free allocation (y,σ) such that $M' \equiv \sum_{i\in N} y_i < M$, for each $j \in N \setminus T$, $y_j = x_j$, and for each $i \in T$ there is $j \in N\setminus T$, such that $i \trianglerighteq(R,z')\,j$. Let $u \equiv (u_i)_{i\in N}$ be a representation of R such that: (i) for each $\{i,j\} \in N\setminus T$, $u_i(y_i,\sigma(i)) = u_j(y_j,\sigma(j))$ and (ii) there is $b \in \mathbb{R}$ such that for each $i \in T$ and each $j \in N \setminus T$, u_j is bounded above by b and u_i is bounded below by b. For each $r \in \mathbb{R}$, let

$$S(r) \equiv \arg\max\left\{\min_{i\in N} u_i(z_i) : \begin{array}{l} z \equiv (z_i)_{i\in N} \text{ is envy-free for } R \\ \text{and has budget } r \end{array}\right\}.$$

Since for each $i \in T$ there is $j \in N \setminus T$ such that $i \trianglerighteq(R,z')\,j$, then $(y,\sigma) \in S(M')$ (Velez, 2012b). Let $i \in N$. For each $r \in \mathbb{R}$, agent i is indifferent among all allocations in $S(r)$, and her welfare in $S(r)$ is an increasing function of r (Velez, 2012b). Let $\hat{z} \in S(M)$. Then, for each $i \in N$ there is $j \in N\setminus T$ such that $i \trianglerighteq(R,\hat{z})\,j$. By Lemma 3, $\hat{z} \in \mathscr{O}(N\setminus T, \mathscr{R}^N \setminus T, f(R_T,\cdot), R_{N\setminus T})$. Since $M' > M$ then for each $j \in N \setminus T$, $\hat{z}_j \, P_j \, z_j$. Since $z \in F(R)$ and each envy-free allocation for R is Pareto efficient for R, then there is $i \in N$ such that $z_j \, P_j \, \hat{z}_j$. Then, $j \in T$. □

Proof of Corollary 3. Let $z \in \arg\max\{u[R_i](z_i) : z \in F(R)\}$. Then, for each $j \in N$, $j \trianglerighteq(R,z)\,i$. Since $i \in N\setminus T$, then by Lemma 3, there is $\hat{z} \in \mathscr{O}(N\setminus T, \mathscr{R}^{N\setminus T}, f(R_T,\cdot), R_{N\setminus T})$ such that $\hat{z}_i = z_i$. Thus, $\hat{z} \in \arg\max\{u[R_i](z_i) : z \in F(R)\}$. □

Proof of Proposition 3. Let f be an *envy-free* solution, $R \in \mathscr{R}^N$, $i \in N$, and $T \subseteq N\setminus\{i\}$. (1) follows from Corollary 2. We prove (2). Let $R \in \mathscr{R}^N$ be such that $PSO_i[f,\mathscr{R},T,R] = 1$. Then, there is $z \equiv (x,\mu) \in \mathscr{O}(N\setminus(T\cup\{i\}), \mathscr{R}^N\setminus(T\cup\{i\}), f(R_{T\cup\{i\}},\cdot), R_{N\setminus(T\cup\{i\})})$ such that $z \in \arg\max\{u[R_i] : z \in F(R)\}$.

Let $\varepsilon > 0$. We construct $R_i^\varepsilon \in \mathscr{R}$ such that for each $j \in N\setminus\{i\}$, $z_i \, P_i \, z_j$ and $|u[R_i] - u[R_i^\varepsilon]|_\infty < \varepsilon$. Let $M \in \mathbb{R}_{++}$, $\delta \in (0,1)$. Let for each $\alpha \in A$, let $\{l_\alpha, L_\alpha, a_\alpha\} \subseteq \mathbb{R}$ be such that $(l_\alpha,\alpha)\, I_i\, (x_{\mu(i)} - M, \mu(i))$, $(a_\alpha,\alpha)\, I_i\, (x_{\mu(i)}, \mu(i))$, and $(L_\alpha,\alpha)\, I_i\, (x_{\mu(i)} + M, \mu(i))$. Let $R_i^\delta \in \mathscr{R}$ be the preference such that:

(i) for each $\alpha \in A$ such that $z_i \, P_i \, (x_\alpha,\alpha)$, we have that for each pair $\{a,b\} \subseteq \mathbb{R}$, $(a,\alpha)\, I_i^\delta\, (b,\mu(i))$ if and only if $(a,\alpha)\, I_i\, (b,\mu(i))$.

(ii) for each $\alpha \in A$ such that $z_i \, I_i \, (x_\alpha,\alpha)$, each $b \in \mathbb{R}\setminus[x_{\mu(i)} - M, x_{\mu(i)} - M]$, and each $a \in \mathbb{R}$, $(a,\alpha)\, I_i^\delta\, (b,\mu(i))$ if and only if $(a,\alpha)\, I_i\, (b,\mu(i))$.

(iii) for each $\alpha \in A$ such that $z_i\, I_i\, (x_\alpha, \alpha)$, each $b \in \backslash[x_{\mu(i)} - M, x_{\mu(i)}]$, if we denote by $a \in \mathbb{R}$, the amount of money such that $(a, \alpha)\, I_i\, (b, \mu(i))$ and

$$a^\delta \equiv \frac{a - l_\alpha}{a_\alpha - l_\alpha}(\delta L_\alpha + (1-\delta) a_\alpha) + \left(1 - \frac{a - l_\alpha}{a_\alpha - l_\alpha}\right) l_\alpha. \quad (1)$$

we have that $(a^\delta, \alpha)\, I_i\, (b, \mu(i))$.

(iv) for each $\alpha \in A$ such that $z_i\, I_i\, (x_\alpha, \alpha)$, each $b \in \backslash[x_{\mu(i)}, x_{\mu(i)} + M]$, if we denote by $a \in \mathbb{R}$, the amount of money such that $(a, \alpha)\, I_i\, (b, \mu(i))$ and

$$b^\delta \equiv \frac{a - a_\alpha}{L_\alpha - a_\alpha} L_\alpha + \left(1 - \frac{a - a_\alpha}{L_\alpha - a_\alpha}\right)(\delta L_\alpha + (1-\delta) a_\alpha). \quad (2)$$

we have that $(b^\delta, \alpha)\, I_i\, (b, \mu(i))$.

Let $m^\delta \equiv \max_{\alpha \in A, a \in [l_\alpha, L_\alpha]} |u[R_i](a, \alpha) - u[R_i^\delta](a, \alpha)|$. By construction, $|u[R_i] - u[R_i^\delta]|_\infty = m^\delta$. We claim that there is $\delta > 0$ for which $m^\delta < \varepsilon$. Suppose by contradiction that for each $\delta > 0$, $m^\delta \geq \varepsilon$. Let $\{\delta_t\}_{t \in \mathbb{N}} \in (0,1)^\infty$ such that as $t \to \infty$, $\delta_t \to 0$. Since for each $\alpha \in A$, each $[l_\alpha, L_\alpha]$ is compact, then there is $\alpha^* \in A \setminus \{\mu(i)\}$ such that $(x_{\alpha^*}, \alpha^*)\, I_i\, (x_{\mu(i)}, \mu(i))$ so cases (iii) and (iv) in the definition of R_i^δ are not trivial, and a convergent sequence $\{a_t\}_{t \in \mathbb{N}} \in [l_{\alpha^*}, L_{\alpha^*}]^\infty$ such that for each $t \in \mathbb{N}$, $|u[R_i^{\delta_t}](a_t, \alpha^*) - u[R_i](a_t, \alpha^*)| \geq \varepsilon$. Let $c \equiv \lim_{t \to \infty} a_t$. For each t, $u[R_i^{\delta_t}](a_t, \alpha^*) \in \{u[R_i](a_t^\delta, \alpha^*), u[R_i](b_t^\delta, \alpha^*)\}$, where a_t^δ and b_t^δ are obtained by substituting δ_t for δ and a_t for a in (1) and (2), respectively. Since u is continuous and as $\delta \to 0$, $a_t^\delta \to a_t$ and $b_t^\delta \to a_t$, then as $\delta \to 0$, $u[R_i^{\delta_t}](a_t, \alpha^*) \to u[R_i](a_t, \alpha^*)$. This is a contradiction. Denote R_i^ε a preference R_i^δ such that $m^\delta < \varepsilon$. By construction, for each $\alpha \in A \setminus \{\mu(i)\}$, $z_i\, P_i^\varepsilon\, (x_\alpha, \alpha)$. Since $z \in \arg\max\{u[R_i] : z \in F(R)\}$, then for each $j \in N \setminus \{i\}$, $j \trianglerighteq (R, z)\, i$ (Fujinaka and Wakayama, 2012). Then, $z \in \arg\max\{u[R_i^\varepsilon] : z \in F(R_{-i}, R_i^\varepsilon)\}$ and $z \notin \mathcal{O}(N \setminus (T \cup \{i\}), \mathcal{R}^N \setminus (T \cup \{i\}), f(R_T, R_i^\varepsilon, \cdot), R_{N \setminus (T \cup \{i\})})$. Thus, $PSO_i[f, \mathcal{R}, T, R_{-i}, R_i^\varepsilon] > 1$. Moreover, the set of utility profiles $u[R]$ with $R \in \mathcal{R}^N$ such that $PSO_i[f, \mathcal{R}, T, R_{-i}, R_i^\varepsilon] > 1$ is dense in $\mathcal{U}^N$.

We now prove that the set of profiles $u[R]$ for which $PSO_i[f, \mathcal{R}, T, R] > 1$ is open in the product topology of $\mathcal{U}^N$. Let $u[R]$ be such that $PSO_i[f, \mathcal{R}, T, R] > 1$. Let $z \in \arg\max\{u[R_i] : z \in F(R)\}$. Then, for each $j \in N \setminus \{i\}$, $j \trianglerighteq (R, z)\, i$ (Fujinaka and Wakayama, 2012). Since $PSO_i[f, \mathcal{R}, T, R] > 1$, then there is no $j \in N \setminus T$ such that $i \trianglerighteq (R, z)\, j$. Let $\delta > 0$ and consider the neighborhood of R, $V^\delta \equiv \Pi_{i \in N}\{u[R_i'] \in \mathcal{U} : |u[R_i] - u[R_i']| < \delta\}$. We claim that there is $\delta > 0$ such that for each $R \in V^\delta$, $PSO_i[f, \mathcal{R}, T, R] > 1$. Suppose by contradiction that for each

$\delta > 0$ there is $R^\delta \in V^\delta$ such that $PSO_i[f, \mathscr{R}, T, R] = 1$. Let $\{\delta_t\}_{t\in\mathbb{N}} \in (0,1)^\infty$ be such that as $t \to \infty$, $\delta_t \to 0$. For each $t \in \mathbb{N}$, let $R^t \in \mathscr{R}$ be such that $u[R^t] \in V^{\delta_t}$ and $PSO_i[f, \mathscr{R}, T, R^t] = 1$. Let $z^t \equiv (x^t, \mu^t) \in \arg\max\{u[R^t_i] : z \in F(R^t)\}$ be such that $z^t \in \mathscr{O}(N \setminus (T \cup \{i\}), \mathscr{R}^N \setminus (T \cup \{i\}), f(R^t_T \cup \{i\}, \cdot), R^t_{N\setminus(T\cup\{i\})})$. Let $t \in \mathbb{N}$. There is $j \in N \setminus T$ such that $i \trianglerighteq (R, z^t)\, j$. Moreover, for each $j \in N \setminus \{i\}$, $j \trianglerighteq (R, z^t)\, i$ (Fujinaka and Wakayama, 2012). By passing to subsequences if necessary, we can assume that $\{\mu^t\}_{t\in\mathbb{N}}$ is a constant sequence with common element $\bar{\mu}$, and that there is $j \in N \setminus \{i\}$, such that for each $t \in \mathbb{N}$, $i \trianglerighteq (R, z^t)\, j$. We claim that $\{x^t\}_{t\in\mathbb{N}}$ is a bounded sequence. For each $j \in N$ there is $b \in \mathbb{R}$ such that for each $\beta \in A$, $\left(\frac{1}{n}M, \beta\right) P_j\, (b, \bar{\mu}(j))$. Thus, after some finite t, $\left(\frac{1}{n}M, \beta\right) P^t_j\, (b, \bar{\mu}(j))$. Since there is at least one agent whose consumption of money is at least $\frac{1}{n}M$, then after some finite t, $x^t_{\bar{\mu}(j)} \geq b$. Thus, $\{x^t\}_{t\in\mathbb{N}}$ is bounded below. Since for each $t \in N$, $\sum_{\alpha\in A} x_\alpha = M$, then $\{x^t\}_{t\in\mathbb{N}}$ is bounded. We can then assume without loss of generality that this sequence is convergent. Let $\bar{x} \equiv \lim_{t\to\infty} x^t$ and $\bar{z} \equiv (\bar{x}, \mu)$. Let $\{j, k\} \subseteq N$. Then, for each $t \in \mathbb{N}$,

$$\begin{aligned} |u[R^t_j](x^t_{\mu(k)}, \bar{\mu}(k)) - u[R_j](\bar{x}_{\mu(k)}, \bar{\mu}(k))| &\leq |u[R^t_j](x^t_{\mu(k)}, \bar{\mu}(k)) - u[R_j](x^t_{\mu(k)}, \bar{\mu}(k))| \\ &\quad + |u[R_j](x^t_{\mu(k)}, \bar{\mu}(k)) - u[R_j](\bar{x}_{\mu(k)}, \bar{\mu}(k))| \\ &\leq \delta_t \\ &\quad + |u[R_j](x^t_{\mu(k)}, \bar{\mu}(k)) - u[R_j](\bar{x}_{\mu(k)}, \bar{\mu}(k))|. \end{aligned}$$

Since $u[R_i]$ is continuous and as $t \to \infty$, $\delta_t \to 0$, then $\lim_{t\to\infty} u[R^t_j](x^t_{\mu(k)}, \bar{\mu}(k)) = u[R_j](\bar{x}_{\mu(k)}, \mu(k))$. Then, if for each $t \in \mathbb{N}$, $z^t_j\, I^t_j\, z^t_k$, then $\bar{z}_j\, I_j\, \bar{z}_k$. Let $j \in N \setminus \{i\}$. Since for each $t \in \mathbb{N}$, $j \trianglerighteq (R, z^t)\, i$, then $j \trianglerighteq (R, \bar{z})\, i$. Then, $\bar{z} \in \arg\max\{u[R_i] : z \in F(R)\}$ and $\bar{x} = x$ (Alkan et al., 1991; Fujinaka and Wakayama, 2012). Thus, $\bar{\mu}(i) = \mu(i)$. We know that there is $j \in N \setminus \{i\}$ such that for each $t \in \mathbb{N}$, $i \trianglerighteq (R, z^t)\, j$. Then, $i \trianglerighteq (R, \bar{z})\, j$. We prove that $i \trianglerighteq (R, z)\, j$. Let $k \in N$ be such that $i \trianglerighteq (R, \bar{z})\, k$ and let $k' \equiv \mu^{-1}(\bar{\mu}(k))$. We claim that $i \trianglerighteq (R, z)\, k'$. We know that there is a list of different agents $\{i_1, \ldots, i_m\}$ such that for each $r = 1, \ldots, m$, $\bar{z}_r\, I_r\, \bar{z}_{r+1}$, $i_1 = i$, and $i_m = k$. If $m = 2$, the claim clearly holds. Suppose now that it holds for lists with $m - 1 < n$ agents. We claim it holds for lists with m agents. Suppose that for some $l \in N$, $i \trianglerighteq (R, z)\, l$ and $\bar{z}_{\bar{\mu}^{-1}(\mu(l))}\, I_{\bar{\mu}^{-1}(\mu(l))}\, \bar{z}_k$. We claim that $l \trianglerighteq (R, z)\, \bar{\mu}^{-1}(\mu(l))$. Suppose w.l.o.g that $l \neq \bar{\mu}^{-1}(\mu(l))$, for otherwise the claim is trivial. Let $i_1 \equiv l$, $i_2 = \mu^{-1}(\bar{\mu}(i_1))$, $i_3 = \mu^{-1}(\bar{\mu}(i_2))$,..., and so on. Since $i_1 \neq i_2$, then, for each r, $i_r \neq i_{r+1}$. Thus, there has to be $m' \leq n$ such that $i_1 = \mu^{-1}(\bar{\mu}(i_{m'}))$. That is, $i_{m'} = \bar{\mu}^{-1}(\mu(l))$. Since, $\bar{x} = x$, then for each agent all allocations in $\arg\max\{u[R_i] : z \in F(R)\}$ are welfare equivalent (Alkan et al., 1991). Let $r = 1, \ldots, m'$. Since z and $\bar{z}$ are welfare equivalent and $\mu(i_{r+1}) = \bar{\mu}(i_r)$, then $z_{i_r}\, I_{i_r}\, z_{i_{r+1}}$. Thus, $l \trianglerighteq$

$(R,z)\bar{\mu}^{-1}(\mu(l))$. Since z and $\bar{z}$ are welfare equivalent, $z_{\bar{\mu}^{-1}(\mu(l))}\, I_{\bar{\mu}^{-1}(\mu(l))}\, \bar{z}_{\bar{\mu}^{-1}(\mu(l))}$. Thus, $z_{\bar{\mu}^{-1}(\mu(l))}\, I_{\bar{\mu}^{-1}(\mu(l))}\, \bar{z}_k = z_{k'}$. Thus, $i \trianglerighteq (R,z)\, k'$. Let $j' \equiv \mu^{-1}(\bar{\mu}(j))$. Thus, $i \trianglerighteq (R,z)\, j'$. Suppose without loss of generality that $j' \neq j$, for otherwise our claim is proved. Let $i_1 \equiv j'$, $i_2 = \mu^{-1}(\bar{\mu}(i_1))$, $i_3 = \mu^{-1}(\bar{\mu}(i_2))$,..., and so on. Since $i_1 \neq i_2$, then, for each r, $i_r \neq i_{r+1}$. Thus, there has to be $m' \leq n$ such that $i_1 = \mu^{-1}(\bar{\mu}(i_{m'}))$. That is, $i_{m'} = \bar{\mu}^{-1}(\mu(j')) = j$. Let $r = 1, \ldots, m'$. Since z and $\bar{z}$ are welfare equivalent and $\mu(i_{r+1}) = \bar{\mu}(i_r)$, then $z_{i_r}\, I_{i_r}\, z_{i_{r+1}}$. Thus, $j' \trianglerighteq (R,z)\, j$. Thus, $i \trianglerighteq (R,z)\, j$. This is a contradiction. □

Proof of Proposition 5. The first part of the Proposition follows from Proposition 1. We prove that if $R'_i \in \mathcal{Q}$ is different from R^0_i, then there is $R'_{-i} \in Q^{N\setminus\{i\}}$ and $z_i \in \mathbb{R} \times A$ such that for each $R_{-i} \in Q^{N\setminus\{i\}}$, $f_i(R^0_i, R_{-i})\, R^0_i\, z_i\, P^0_i\, f_i(R')$. Let x be the vector defined in the proof of *(i)*. Let $z_i \equiv (x_\gamma, \gamma)$ be the best bundle for R'_i in $\{(x_\alpha, \alpha) : \alpha \in A\}$. Since envy-free allocations satisfy the IPLB, then for each $R_{-i} \in Q^{N\setminus\{i\}}$, $f_i(R^0_i, R_i)\, R^0_i\, z_i$. Let $x \equiv (x'_\alpha)_{\alpha \in A} \in \mathbb{R}^A$ be such that for each $\alpha \in A$, $(x'_\alpha, \alpha)\, I'_i\, (x_\gamma, \gamma)$. Since preferences are money monotone, then $x' \geq x$. Let $B \equiv \{\alpha \in A : x'_\beta > x_\beta\}$ and $C \equiv A \setminus (B \setminus \{\gamma\})$. Since $R'_i \in \mathcal{Q}$ and $R^0_i \neq R'_i$, then $B \neq \emptyset$. Let $y \equiv (y_\alpha)_{\alpha \in A}$ be the vector defined by: $y_\gamma = x'_\gamma$; for each $\alpha \in B$, $y_\alpha = \frac{x_\alpha + x'_\alpha}{2}$; and for each $\alpha \in C$, $y_\alpha = x'_\alpha - \delta$, where $\delta > 0$ is such that $\sum_{\alpha \in A} y_\alpha > M$. For each $j \in N \setminus \{i\}$, let $R'_j \in \mathcal{Q}$ be the such that for each pair $\{\alpha, \beta\} \subseteq A$, $(y_\alpha, \alpha)\, I'_j\, (y_\beta, \beta)$. Since $R' \in \mathcal{Q}^N$ and envy-free allocations are Pareto efficient, then agent i receives object γ at $f_i(R')$. Since envy-free allocations satisfy the IPLB, then the consumption of money of agent i at $f_i(R')$ is at most $y_\gamma - \frac{1}{n}\left(\sum_{\alpha \in A} y_\alpha - M\right) < y_\gamma$. Thus, $z_i\, P^0_i\, f_i(R')$. □

Proof of Proposition 6. Let $T \subsetneq N$ and $i \in T$. Suppose that agents in T are sincere. Let $\mathcal{D} \subseteq \mathcal{Q}^N$ be the subdomain of quasi-linear profiles R the best allocation among all envy-free allocations for R is an outcome of the game associated with f at R. We prove that D is the complement of an open dense set in $\mathbb{R}^{(n-1)\times n}$. Let $v \equiv (v^i_\alpha) \in \mathbb{R}^n$ be the linear additive representation of R_i. Let $z \equiv (x, \mu) \in F(R)$ be agent i's best allocation in $F(R)$. Then, for each $j \in N$, $j \trianglerighteq (R,z)\, i$ (Fujinaka and Wakayama, 2012). By Lemma 2, there is $j \in N \setminus T$ such that $i \trianglerighteq (R,z)\, j$. Thus, there is a list of different agents, $i_1, i_2, \ldots, i_k$ such that $k > 1$, $z_{i_1}\, I_{i_1}\, z_{i_2}, \ldots, z_{i_{k-1}}\, I_{i_{k-1}}\, z_{i_k}$, and $z_{i_k}\, I_{i_k}\, z_{i_1}$. For simplicity, denote $i_1 \equiv 1$, $i_2 \equiv 2$ and so on. Then, $x_1 + v^1_{\mu(1)} = x_2 + v^1_{\mu(2)}, \ldots, x_{k-1} + v^{k-1}_{\mu(k-1)} = x_k + v^{k-1}_{\mu(k)}$ and $x_k + v^k_{\mu(k)} = x_1 + v^k_{\mu(1)}$. Adding up these k equations we obtain that

$$v^1_{\mu(1)} + \cdots + v^k_{\mu(k)} = v^1_{\mu(2)} + \cdots + v^k_{\mu(1)}.$$

Then, $\mathscr{D}$ is a subset of a finite family of hyperplanes in $\mathbb{R}^{(n-1)\times n}$. Thus it belongs to the complement of an open dense set in $\mathbb{R}^{(n-1)\times n}$. Clearly, its Lebesgue measure its zero. $\square$

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