ON RISK-SHARING EQUILIBRIA UNDER STRATEGIC BEHAVIOR

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Abstract. The paper studies an oligopolistic equilibrium model of financial agents who aim to share their random endowments. The structure of the risk sharing securities and their prices are endogenously determined as the outcome of a strategic game played among the participating agents. In the complete market setting, each agent's set of strategic choices consists of the security payoffs that are consistent with the optimal sharing rules; while in the incomplete setting, agents respond demand schedules on a vector of given tradeable securities. It is shown that at the (Nash) risk sharing equilibrium, the sharing securities are suboptimal and reflect that agents prefer to share risk exposures different than their true ones. On the other hand, the Nash equilibrium prices stay unaffected by the game only in the special case of agents with the same risk aversion. In addition, the oligopolistic structure always benefits the relatively low risk averse agents, who reduce the efficiency of risk sharing and absorb utility gains from the high risk averse ones.

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1. Introduction

The concept of risk sharing is central in a large variety of financial fields ranging from investment management and structured finance to insurance and derivative markets. Its importance stems from

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the fact that investors, financial institutions and insurers find that sharing of investment random payoffs, defaultable incomes and insurance liabilities is often mutually beneficial in terms of reduction of risk exposures. The search for the best way to share risk is connected to financial innovation, in the sense that such sharing is fulfilled by designing and trading new financial securities.

The large majority of the growing body of research in risk sharing assumes that agents act as *perfect competitors* and the induced risk allocation is Pareto optimal. However, the transactions through which financial institutions share risk usually consist of few participants, each of which possesses the power to affect the equilibrium allocations and prices. In such situations (where the participation in trading of securities is limited or when some of participants have strong market power), the actions of each individual one clearly influence the structure of the transaction. Market models that do not take into account this fact lead to equilibria that *overestimate* the market efficiency (and as stated in [26] and [36], they require agents to behave "schizophrenically" by ignoring their ability to affect the market). If we assume that participating agents do exploit their impact on the way the risk is shared, the equilibrium is different and the efficiency is most likely reduced, since we go from competitive to Nash-type equilibria.

We follow the standard setting in the risk sharing literature and assume that agents are endowed with some risky portfolios, which shall be called *random endowments*. The mutually beneficial sharing of these endowments can be achieved in complete or in incomplete market setting. In the former, agents design and price new securities that optimally share their risky exposures and bringing them in a Pareto optimal situation.[1] On the other hand, if the financialization of the agents' endowment is not possible (due to exogenous constrains, such as transaction costs and strict regulation), an *incomplete* risk sharing can be done through the trading of a *given* vector of financial securities. Although there is an extended literature dealing with both of these settings of risk sharing transactions, the effect of the use of the market power has not been sufficiently addressed. The main objective of this paper is to fill this gap by (i) modelling the strategic behavior of agents with heterogenous risk aversions when they share risk, (ii) establishing and analyzing the Nash-type equilibria that occur as the outcome of these strategies and (iii) investigating which are the agents (if any) that benefit from the oligopolistic structure of a risk sharing transaction.

1.1. **Model description and the main findings.** We consider the *one-shot* risk sharing transaction among agents who use their ability to influence the equilibrium allocations and prices. We first take the position of an individual risk averse agent who wishes to share her risky endowment

[1] The securities that share risk in a Pareto optimal way were introduced by the seminal works of [7, 8] and [52] (see also [20] and the surveys of [3] and [21]). These models, however, impose a competitive structure of the market and do not allow agents to apply any kind of strategic behavior when they negotiate the designing of risk sharing securities.

with the rest of the participating agents. Given that the agents' submitted endowments are shared through the welfare optimal sharing rules, we ask whether she has a motive to share a risk exposure different than her true one. Since the risk sharing securities are functions of the reported endowments, the endowment that she chooses to share directly affects the structure of the designed securities. According to the proposed model, she should respond as her shared-to-be endowment the random quantity (called *best endowment response*) that maximizes her expected utility when the agreed sharing rules are applied.[2,3] Under mean-variance preferences (hereafter M-V preferences), it is shown that it is *never optimal* for an agent to share her true risk exposure.

A similar best response strategy is imposed when agents negotiate the trading of a given bundle of tradable securities (incomplete market setting). In this case, we model the agents' strategic behavior following way: we consider an individual agent who knows the aggregate demand schedule of the rest of the agents[4] and we ask: *Which is her preferable equilibrium price of the tradeable securities given the demand of the rest of the agents?* Naturally, her preferable price is one that clears out the market and at the same time maximizes her utility. She can then drive the market to this price by submitting an appropriate demand schedule (which shall be called best demand response). It is shown that under M-V preferences, the best demand response is *different* than her true demand in any non-trivial case (i.e., in any case where some risk is to be shared). In fact, the best demand response is equal to the demand that corresponds to the best endowment response.[5]

The aforementioned strategic behavior, when applied by all agents, forms a negotiation scheme on the risk sharing transaction. Being consistent with the optimal sharing rules, the risk sharing securities are in fact functions of the agents' reported endowments. Hence, responding on the other agents' reported endowment is in effect equivalent to responding to the securities proposed by the other agents. In other words, the risk sharing equilibrium is the outcome of a Nash-type game played among the participating agents, where their set of strategic choices are the risks they choose to share (or equivalently the securities they are willing to get). In Section 4, it is proved that under M-V preferences the complete market setting admits a pure strategy Nash equilibrium.

[2]In contrast to the relevant literature on market manipulation (see e.g., [51] and the references therein), here the strategic set of choices is of infinite dimension; namely it is the set of all random variables that can be considered as agent's random endowment. As discussed in the sequel, this set of choices is in fact equivalent with the set of sharing securities that are consistent with the optimal sharing rules.

[3]The generality of the agents' random endowments in our model allows its application in a number of (oligopolistic) financial transactions such as the trading of innovated derivative products, the reinsurance contracts or the transactions among inter-dealers (e.g., see also the related discussion in [49]).

[4]Under the M-V preferences, the demand schedules are linear and hence a demand function can be inferred by knowing only two of its values (orders).

[5]In contrast to the strategy market game concept on security trading (see among others [42]), in our model the pricing and the allocation mechanism is endogenously determined as the outcome of the game.

Furthermore, although it is rare to have Nash equilibrium uniqueness in models with uncertainty (see among others, [25] and [32]), the equilibria are indeed *unique* and fully characterized. Similarly, in the incomplete market case, the game is played on the agents' reported demand schedules or equivalently on the prices and the allocations of the tradeable assets.

In the Nash risk sharing equilibrium is inefficient in all non-trivial cases. This implies a loss in aggregate utility, but interestingly enough, for agents with sufficiently low risk aversion the Nash equilibrium gives higher expected utility, when compared to the competitive equilibrium transaction. The main message of this theoretical result is that *relatively low risk averse agents are not preferable counterparties when the risk sharing has an oligopolistic structure and the security design is an outcome of a negotiation game.* Furthermore, the securities that are designed to share risk are the optimal ones only under the special case of agents with the same risk aversion. Even in this case, however the equilibrium volume is reduced and the financial innovation losses its efficiency.

In the incomplete market game, the induced Nash equilibrium can be considered as an *oligopoly variation of the CAPM.* In Section 4, it is shown that *the Nash equilibrium prices are equal to the perfect competition ones if and only if agents are homogeneous with respect to their risk preferences.* Even in this case however, the allocation in Nash equilibrium is not efficient and the volume is reduced (in particular, the volume percentage reduction is $1/n$, where n is the number of participating agents). In the more general situations of heterogeneous agents, we establish an exact measure of the *price impact* that is caused as a result of game played among them.[6] As in the complete market, sufficiently low risk averse agents always profit from the market inefficiency.

1.2. **On risk sharing inefficiency.** For the definition of market inefficiency, we follow the related literature (see among others [2]). In particular, we define as *risk sharing inefficiency* the difference between the agents' aggregate utility gain at the optimal (Pareto) risk sharing and the agents' aggregate utility at the (Nash) equilibrium that imposes the use of market power. As stated above, inefficiency is positive and its size is mainly determined by the number of participating agents and the level of their risk aversion.

In Section 5, we take the position of a market regulator (considered as social planner) and discuss possible ways to increase the efficiency of risk sharing. A standard way is by facilitating the *participation of more agents* into the sharing. Indeed, both in complete and incomplete market settings, the differences between the Nash and competitive equilibria vanish and the market becomes efficient as the number of agents increases to infinity.

[6]For example, according to this price impact, an intense upward price pressure occurs not only when the agents with intense hedging demand are high risk averse too, but also when agents with low risk aversion participate in the trading.

Another way to improve the market efficiency is by *restricting the agents' set of strategic choices*. According to the aforementioned best response strategy, each agent has motive to report exposure to the risk that the other agents possess. Reporting exposure to these risks may have unpleasant consequences to the agent for reasons beyond the modelling at hand.[7] In these situations, agents can alternatively restrict their choice only to the size of their true endowments that they are going to submit for sharing. A (restricted) pure Nash equilibrium is proved to hold and since the set of the agents' strategic choices is smaller, the difference between the Nash equilibrium and the competitive one is reduced.[8]

In addition, a market is more efficient when is closer to *completion*. It is shown that even when the market is small and manipulateable, each agent gets more utility when the risk sharing setting is complete (i.e., when securities consist of reported endowments' payoffs). This implies that even in thin market, regulator still has a motive to facilitate the financial innovation.

1.3. **Connection with the relevant literature.** Motivated by the growing financial innovation and the ensuing increased establishment of new financial assets, many researchers have studied several theoretical aspects of optimal (complete) risk sharing. The challenging mathematical problem of the existence of the efficient allocations and the determination of the factors that characterize them have been recently studied by many authors under quite general preferences and market models (see among others [6, 18, 29] and the references therein). However, this large body of research has paid little attention to the more realistic situation where agents *do not act as price-takers*, but rather they exploit the fact that they could impact the market equilibrium by their actions.

Although, the large participation in the majority of financial markets does not leave room to large agents for market manipulation, several empirical studied argued that in many financial transaction among institutional investors a form of market power is used (see among others [13, 30, 31, 35][9]). In OTC risk sharing markets, the assumption of perfect competition structure is even further away from being realistic. When financial agents negotiate the sharing of their (otherwise unhedgeable) risky exposures by designing new or by trading given securities, the participation is normally limited (at least at the primary market level) among some financial institutions and/or some of their clients (who submit specific hedging demands or are willing to play the role of speculator).

[7] Such strategy implies speculation attitude that may be against the general hedging character of the institution and may also be disapproved by the other participants since it decreases the efficiency of the sharing.

[8] For example, this percentage game eliminates the speculators' market power. Speculators are the agents with no risk to share (or with endowments of very low risk), which means that at this transaction they are left with zero (or negligible) market power.

[9] Strategic behavior is also empirically proved to be applied by financial market makers (see e.g., [14, 15, 24] and the references therein). These studies have shown that the market power did exist, was exploited and did cause inefficient prices/allocations.

The majority of the studies on market inefficiency argues that the agents' asymmetric information is the main reason for inefficient transactions. In contrast to this approach, we examine in this paper how the inefficiency of the risk sharing follows from a negotiation game among agents in a thin market. Also, other part of the literature on market power modelling (see e.g., [9] and [50]) distinguishes agents between price-takers and those with market power (or large strategic investors and noisy traders, informed and uniformed agents as in [36]; or arbitrageurs and investors, as in [43, 46][10]). Participants' difference in the structure of the risk sharing model is also imposed in the recent papers that study the adverse selection problem (see for instance [27, 28, 39, 40]). The results of this literature do not cover our model which doesn't distinguish participating agents; all of them have and use their market power (the agent's characterization is solely based on her risk aversion coefficient and her random endowment).

In [12] (see also [51]), a non-competitive market model is developed where the price impact is also determined endogenously as part of the equilibrium concept (see [11] for a broader discussion). This model however does not cover the one we present here, since the imposed price impact is strongly based on their assumption of finite probability spaces.[11] Non-competitive security market models, that are based on the concept of the strategic game market of [45], have been developed by [42] (see also [33, 34]). The equilibrium on trading given vector of securities is also of Nash-type, where each individual agent responds to the other agents' orders. However, the pre-specified allocation and pricing mechanism is different than our equilibrium setting (for instance, in [34] the prices are calculated as the ratio of the aggregate amount of money put up for each security over the aggregate asked units of the security, the allocation is proportional and there are strict budget constrains imposed).

In [44, 49], market power models in discrete time dynamic settings have also been introduced. In both papers, the strategic behavior of homogenous (with respect to their risk preferences) agents in some risk sharing environments is considered. Agents submit (linear) demand functions on a given asset and the trade is organized as a Walrasian auction. The settings of the sharing is however restricted, in the sense that the agents' endowments have only a specific structure which is some

[10]In [43] a security design game played among arbitrageurs is introduced and analyzed. The induced Nash equilibrium is optimal for the arbitrageurs regarding their profits from the mispricings across different market segments. Although the arbitrageurs' strategic behavior has some common features with the model presented here (e.g., it refers to maximization of quadratic utility functions), the game is played on different field (arbitrageurs imperfectly compete on the profits that come as a result of segmented markets and they do not share any risky exposure).

[11]Although in [12] the model preferences are more general, the tractability of their model is based on the assumption of finite Ω. In addition, our model of the risk sharing equilibrium in complete markets follows a different approach, namely by addressing the question of how much risk an individual agent wants to share with the other participating agents.

exogenously given number of shares of the same stocks (in [49] each agent also gets independently random shares in every time period).[12]

The paper proceeds as follows: Section 2 introduces the market model and sets up the optimal risk sharing rules in both complete and incomplete risk sharing settings. Section 3 establishes the model for an individual agent's strategy, regarding the risk that she shares and the demand schedule she asks. The Nash risk sharing equilibria are defined and analyzed in Section 4. Possible ways for the regulators to increase the efficiency are discussed in Section 5. We summarize the contributions of the paper in Section 6. For the reader's convenience, the proofs are omitted from the main body of the paper and provided in Appendix.

2. Risk Sharing Equilibria without Use of Market Power

We consider a static market model of n agents aiming to reduce their risk exposures by executing a transaction. Throughout this manuscript it is assumed that there exists an exogenously priced numéraire in units of which all the mentioned financial quantities are denominated. Each agent has already been exposed to a risk (called *random endowment*) that incorporates the net discounted payoffs of all the unhedgeable financial positions that she has taken with maturity up to a given future time horizon T (the risk that cannot be hedged out by trading in any market outside this setting). These endowments shall be denoted through random variables $\mathcal{E}_i$, $i \in \{1, ..., n\}$ which are defined in a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$[13]. The sum of these random endowments is called the *aggregate endowment* and is denoted by $\mathcal{E} = \sum_{i=1}^{n} \mathcal{E}_i$. The preferences of agent i are modelled by M-V utility[14]

$$\mathbb{U}_i(X) = \mathbb{E}[X] - \gamma_i \, \mathbb{V}\text{ar}[X], \tag{2.1}$$

[12] Although, in the present manuscript we study only a static model, the results are more general, in the sense that agents' random endowments can be of any arbitrary form and agents' set of strategic choices is much larger (for example, in complete market risk sharing setting, the securities' payoffs are themselves part of the equilibrium). Besides that, an important part of the following analysis is the imposed heterogeneity on agents risk aversion (especially when we examine which agents benefit from the market power).

[13] $\mathbb{P}$ is the so-called "*subjective*" probability measure, and is assumed to be common for each agent. Removal of the homogeneous beliefs assumption doesn't change the general concept of this paper, although it adds a great deal of complexity in calculations. In case where difference of beliefs changes the general concept of a result, a related remark is provided (see e.g., Remark 4.12). For a broader study on the effect of ambiguity in risk sharing, we refer the reader to [17] and [47].

[14] The quadratic utility has be widely used in the risk sharing literature in a variety of topics from adverse selection problem (see e.g., [10, 27, 28]) to games on financial innovation (as in [43]). It is also important to note that M-V preferences can generally be considered as a second order approximation of other utility functionals (see for instance Section 4 of [4]).

with X being any random payoff that belongs in $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ (hereafter denoted simply by $\mathbb{L}^2$), constant $\gamma_i > 0$ being her *risk aversion coefficient* and where $\mathbb{E}[\cdot]$ and $\mathbb{V}\mathrm{ar}[\cdot]$ stands for the expectation and variance maps under probability measure $\mathbb{P}$. Given her random endowment $\mathcal{E}_i$, agent i evaluates any financial position $X \in \mathbb{L}^2$ using her (adapted to endowment) utility $\mathbb{U}_i(\mathcal{E}_i + X)$. A well studied static utility maximization example is the case of k given financial securities available for trading, $\mathbf{C} = (C_1, ..., C_k) \in (\mathbb{L}^2)^k$, whose prices are given by a vector $\mathbf{p} \in \mathbb{R}^k$. Hence, the utility maximization problem takes the following form

$$\sup_{\mathbf{a}\in\mathbb{R}^k} \{\mathbb{U}_i(\mathcal{E}_i + \mathbf{a}\cdot\mathbf{C} - \mathbf{a}\cdot\mathbf{p})\} = \sup_{\mathbf{a}\in\mathbb{R}^k} \{\mathbb{U}_i(\mathcal{E}_i + \mathbf{a}\cdot\mathbf{C}) - \mathbf{a}\cdot\mathbf{p}\}. \tag{2.2}$$

The set of vectors $\mathbf{a} \in \mathbb{R}^k$ that maximize (2.2) for a given price vector $\mathbf{p}$ is the *demand* of agent i on $\mathbf{C}$ at price $\mathbf{p}$ (hereafter denoted by $Z_i(\mathbf{p})$), i.e.

$$Z_i(\mathbf{p}) = \operatorname*{argmax}_{\mathbf{a}\in\mathbb{R}^k} \{\mathbb{U}_i(\mathcal{E}_i + \mathbf{a}\cdot\mathbf{C}) - \mathbf{a}\cdot\mathbf{p}\}.\text{[15]} \tag{2.3}$$

For the set $Z_i(\mathbf{p})$ to be a singleton for any price vector $\mathbf{p}$, it is sufficient to impose the following standing assumption.

Standing Assumption. *For every considered vector of securities* $\mathbf{C}$*, the matrix* $\mathbb{V}\mathrm{ar}[\mathbf{C}]$ *is non-singular.*

Under this assumption, $Z_i(\mathbf{p})$ is a function from $\mathbb{R}^k$ to $\mathbb{R}^k$ which shall be called the *demand function* (or demand schedule) of agent i of securities $\mathbf{C}$ and is given by

$$Z_i(\mathbf{p}) = \left(\frac{\mathbb{E}[\mathbf{C}] - \mathbf{p}}{2\gamma_i} - \mathbb{C}\mathrm{ov}(\mathbf{C}, \mathcal{E}_i)\right) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}], \tag{2.4}$$

where $\mathbb{E}[\mathbf{C}]$ stands for the vector $(\mathbb{E}[C_1], ..., \mathbb{E}[C_k])$ and for any payoff $X \in \mathbb{L}^2$, $\mathbb{C}\mathrm{ov}(\mathbf{C}, X)$ denotes the vector $(\mathbb{C}\mathrm{ov}(C_1, X), ..., \mathbb{C}\mathrm{ov}(C_k, X))$. Note that the demand has two distinguished sources; the risk premium (scaled by agent's risk aversion): $\frac{\mathbb{E}[\mathbf{C}]-\mathbf{p}}{2\gamma_i} \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}]$, and the correlation between the tradeable securities and the agent's endowment: $\mathbb{C}\mathrm{ov}(\mathbf{C}, \mathcal{E}_i) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}]$ (see also [20] for analogous and more detailed discussion). The demand for a particular security C_j (for $j \in \{1, ..., k\}$), is a decreasing function of the covariance between C_j and agent's endowment, a fact that supports the use of the M-V evaluation for risk management purposes. Indeed, we expect that when the covariance between a particular security and an agent's endowment is large and negative, the agent

[15] Optimization problem (2.3) is a fundamental portfolio choice problem that has been analyzed by many authors (see among others [16] for an overview in the case of quadratic utility and budget constrains and [5] for more general risk preference modelling). Note that the decision criterion (2.3) imposes no short selling constrains on securities $\mathbf{C}$. One may think that the absence of such constrains implies the possibility of unbounded supply; however, this can well be avoided by imposing regularity constrains on the set of admissible positions $\mathcal{X}$. Unbounded supply is indeed the case if the price is sufficiently high (or low), however the equilibrium arguments that follow will endogenously exclude such extreme situations.

is willing to take long position on this security, since this will decrease the risk exposure created by her random endowment. Also, $|Z_{i,j}(\mathbf{p})|$ is decreasing with respect to $\mathbb{V}\mathrm{ar}[C_j]$, i.e., the riskier the security becomes, the smaller the desired position on it is.

We also define as the *utility level* of agent i at price $\mathbf{p}$ of vector of securities $\mathbf{C}$, her utility when she gets her optimal demand at price $\mathbf{p}$, i.e.,

$$v_i(\mathbf{p}) = \mathbb{U}_i(\mathcal{E}_i + Z_i(\mathbf{p}) \cdot \mathbf{C}) - Z_i(\mathbf{p}) \cdot \mathbf{p}. \tag{2.5}$$

Hence, the difference $v_i(\mathbf{p}) - \mathbb{U}_i(\mathcal{E}_i)$ measures the gain (in terms of utility) that agent i gets by buying her optimal demand on $\mathbf{C}$ at $\mathbf{p}$. The advantage of using $v_i(\mathbf{p})$ as a measure of agent's gain from a transaction is that it is measured in numéraire units and therefore can be used for comparisons among different equilibria and agents.

2.1. **Risk sharing in incomplete markets.** As discussed in the introductory section, the construction of new securities that optimally share the risky endowments among the agents and complete the market is rarely possible.[16] In such situation, agents can reduce their unhedgeable risk exposures by writing *standardized* securities whose payoffs are correlated with their endowments. These securities could be any structured financial derivative products, such as credit derivatives, asset backed securities, reinsurance contracts etc. Although, trading a given vector of securities is not a Pareto optimal transaction and leaves the market incomplete, an equilibrium allocation of these securities without the use of any market power will be mutually beneficial for each individual agent in terms of utility improvement.

We assume that there is a finite number of these securities and their payoffs shall be denoted by the random vector $\mathbf{C} = (C_1, ..., C_k) \in (\mathbb{L}^2)^k$. Before we give the exact equilibrium definition, we need to introduce some further notation. Let $\mathbb{A}_{n\times k} \subset \mathbb{R}^n \times \mathbb{R}^k$ denotes the set of matrices that represent the allocation of the vector $\mathbf{C}$. More precisely, the element a_{ij}, $i \in \{1, ..., n\}$ and $j \in \{1, ..., k\}$ of an allocation $A \in \mathbb{A}_{n\times k}$, stands for the units of security C_j that agent i buys (negative a_{ij} means short position) and the zero supply of tradeable securities implies that $\sum_{i=1}^n a_{ij} = 0$, for each $j \in \{1, ..., k\}$. Also, for an allocation matrix $A \in \mathbb{A}_{n\times k}$, $\mathbf{a}_i$ shall denote its i-th row, for each $i \in \{1, ..., n\}$.

The risk sharing using the vector of securities $\mathbf{C}$ is then given as a price-allocation equilibrium, which coincides with the CAPM (see among others [37]).

[16] Examples of the exogenous reasons that restrict agents from trading the optimal risk sharing securities are: (i) The stricter regulation, regarding OTC transactions, that imposes restrictions on credit levels, (ii) the indivisibility of some types of random incomes, e.g., real estate investments, revenues of a running business, dividends from illiquid shares etc, (iii) further liquidity concerns and transaction costs that make the optimal risk sharing trading disadvantageous (an issue addressed among others in [3]).

Definition 2.1. *For a given vector of securities* $\mathbf{C} \in (\mathbb{L}^2)^k$*, the pair* $(\mathbf{p}^*, A^*) \in \mathbb{R}^k \times \mathbb{A}_{n\times k}$ *is called a price-allocation equilibrium if* $Z_i(\mathbf{p}^*) = \mathbf{a}_i^*$ *for each* $i \in \{1, ..., n\}$.

When none of the agents uses any kind of strategic behavior, this equilibrium is an optimal pricing and allocation of $\mathbf{C}$, it maximizes the aggregate utility after trading $\mathbf{C}$ and hence it shall be called *Pareto equilibrium price* (or competitive price) of $\mathbf{C}$. Taking into account equation (2.4), we get the following representation of the CAPM in an incomplete market, where γ stands for the aggregate risk aversion coefficient, i.e., $\gamma = (\sum_{i=1}^n 1/\gamma_i)^{-1}$.

Proposition 2.2 (CAPM)**.** *The equilibrium price of a vector of securities* $\mathbf{C}$ *is given by*

$$\mathbf{p}^* = \mathbb{E}[\mathbf{C}] - 2\gamma\,\mathbb{C}\mathrm{ov}(\mathbf{C}, \mathcal{E}) \tag{2.6}$$

and the equilibrium allocation A^* *is the matrix whose rows are*

$$\mathbf{a}_i^* = \mathbb{C}\mathrm{ov}\left(\mathbf{C}, C_i^o\right) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \tag{2.7}$$

for each $i \in \{1, ..., n\}$*, where*

$$C_i^o := \frac{\gamma}{\gamma_i}\mathcal{E}_{-i} - \frac{\gamma_i - \gamma}{\gamma_i}\mathcal{E}_i. \tag{2.8}$$

Proposition 2.2 states that the equilibrium prices do not depend on the covariance matrix of securities, but only on their expectations and their covariances with the aggregate endowment (this covariance is usually called the *covariance value* of the securities). Under no use of market power, the equilibrium price of a security C_j increases as its covariance with the aggregate endowment decreases, a fact that reflects the higher demand of this particular security.

2.2. **Optimal risk sharing.** In the case where there is no exogenous constrain or additional transaction costs, agents freely negotiate the design of securities that *Pareto-optimally* share their risk exposures and complete the market. More precisely, we define the set $\mathcal{A} = \{\mathbf{K} = (K_1, ..., K_n) \in (\mathbb{L}^2)^n : \sum_{i=1}^n K_i = 0\}$ which contains all the possible sharing rules of agents' endowments, where K_i is the payoff of the contract that agent i receives. The requirement that the sum of these securities is zero implies that risk sharing transaction is cleared out and there is not further risk added in the market. The formal definition of the optimal risk sharing follows.

Definition 2.3. *A vector of securities* $\mathbf{K}^o \in \mathcal{A}$ *is a Pareto optimal risk sharing if for all other* $\mathbf{K} \in \mathcal{A}$ *the following implication holds true:*
If for some i, $\mathbb{U}_i(\mathcal{E}_i + K_i) > \mathbb{U}_i(\mathcal{E}_i + K_i^o)$*, then* $\exists j \neq i$ *s.t.* $\mathbb{U}_j(\mathcal{E}_j + K_j) < \mathbb{U}_j(\mathcal{E}_j + K_j^o)$.

Note that this equilibrium doesn't take into account any use of market power from the participating agents. An optimal sharing occurs when all agents share all of their random endowments and their allocation is optimal in aggregate sense (this risk allocation is consistent with the social welfare optimization).

It is well known that under M-V preferences the optimal risk sharing securities are linear functions of the agent's endowments (see among others [20, 52]). This implies that we can restrict the problem of finding the optimal sharing to finding the price-allocation of the agents' vector of endowments $\mathbf{E} := (\mathcal{E}_1, ..., \mathcal{E}_n)$. Straight-forward calculations and Proposition 2.2 yield that the price-allocation equilibrium of vector of securities $\mathbf{E}$ is the pair $(\mathbf{p}^o, A^o) \in \mathbb{R}^n \times \mathbb{A}_{n\times n}$, where

$$\mathbf{p}^o = \mathbb{E}[\mathbf{E}] - 2\gamma \mathbf{1}_n \cdot \mathbb{V}\mathrm{ar}[\mathbf{E}], \tag{2.9}$$

$\mathbf{1}_n := (1, ..., 1)$ and the elements of A^o are given by

$$a^o_{ii} = \frac{\gamma - \gamma_i}{\gamma_i} \quad \text{and} \quad a^o_{ij} = \frac{\gamma}{\gamma_i}, \text{for } j \neq i. \tag{2.10}$$

The formal statement is given in the following proposition (its proof is placed in Appendix).

Proposition 2.4. *Let $(\mathbf{p}^o, A^o) \in \mathbb{R}^n \times \mathbb{A}_{n\times n}$ be the price-allocation equilibrium of the vector of securities $\mathbf{E}$. Then, the vector of securities $A^o \cdot (\mathbf{E} - \mathbf{p}^o) \in \mathcal{A}$ is the unique Pareto optimal risk sharing.*

Therefore, at the optimal risk sharing transaction, agent i gets the total payoff $\mathbf{a}^o_i \cdot \mathbf{E} - \mathbf{a}^o_i \cdot \mathbf{p}^o$, where $\mathbf{a}^o_i$ denotes the vector $(a^o_{i1}, ..., a^o_{in}) \in \mathbb{R}^n$. Note that

$$\mathbf{a}^o_i \cdot \mathbf{E} = C^o_i,$$

with C^o_i being given in (2.8). This means that according to optimal sharing rules, agent i gets the contract C^o_i (hereafter called *optimal sharing contract*) and pays the price

$$\begin{aligned} \pi^o_i := \mathbf{a}^o_i \cdot \mathbf{p}^o &= \frac{\gamma}{\gamma_i}\left(\mathbb{E}[\mathcal{E}_{-i}] - 2\gamma\,\mathbb{C}\mathrm{ov}(\mathcal{E}_{-i}, \mathcal{E})\right) - \frac{\gamma_i - \gamma}{\gamma_i}\left(\mathbb{E}[\mathcal{E}_i] - 2\gamma\,\mathbb{C}\mathrm{ov}(\mathcal{E}_i, \mathcal{E})\right) \\ &= \mathbb{E}[C^o_i] - 2\gamma\,\mathbb{C}\mathrm{ov}(C^o_i, \mathcal{E}), \quad \text{for each } i \in \{1, ..., n\}, \end{aligned} \tag{2.11}$$

where $\mathcal{E}_{-i} := \sum_{j\neq i} \mathcal{E}_j$, is the aggregate endowment of the rest of the agents. In other words, in the case of no use of market power, agent i is going to short a part of her true random endowment and long equal parts of the other agents' random endowments.

The aggregate maximized utility of the optimal risk transaction is given by

$$\sum_{i=1}^n \mathbb{U}_i(\mathcal{E}_i + C^o_i - \pi^o_i) = \sum_{i=1}^n \mathbb{U}_i(\mathcal{E}_i + C^o_i) = \mathbb{E}[\mathcal{E}] - \gamma\,\mathbb{V}\mathrm{ar}[\mathcal{E}],$$

and hence the difference $\sum_{i=1}^n \mathbb{U}_i(\mathcal{E}_i + C^o_i - \pi^o_i) - \sum_{i=1}^n \mathbb{U}_i(\mathcal{E}_i) = \sum_{i=1}^n \gamma_i\,\mathbb{V}\mathrm{ar}[\mathcal{E}_i] - \gamma\,\mathbb{V}\mathrm{ar}[\mathcal{E}]$ is the *maximum aggregate utility gain* that the market (or the representative agent) gets by the optimal risk sharing transaction.

2.3. **Utility gains and the inefficiency measure.** When none of the agents use any kind of market power, the utility level of each agent at equilibrium (see (2.5)) is higher or at least equal to her initial utility level, i.e., $v_i(\mathbf{p}^*) \geq \mathbb{U}_i(\mathcal{E}_i)$ (the equality holds if and only if the equilibrium price coincides with agent's reservation price). The following proposition can be proved by simple calculations.

Proposition 2.5. *The utility level at equilibrium of agent i at the price-allocation equilibrium of the security vector $\mathbf{C}$ is given by*

$$v_i(\mathbf{p}^*) = \mathbf{a}_i^* \cdot \mathbb{V}\mathrm{ar}[\mathbf{C}] \cdot \mathbf{a}_i^* + \mathbb{U}_i(\mathcal{E}_i) = \gamma_i \, \mathbb{C}\mathrm{ov}(\mathbf{C}, C_i^o) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \cdot \mathbb{C}\mathrm{ov}(\mathbf{C}, C_i^o) + \mathbb{U}_i(\mathcal{E}_i). \tag{2.12}$$

Note that the gain of equilibrium transaction of agent i is increasing with respect to the absolute size of her positions on $\mathbf{C}$.

Remark 2.6. *The term $\mathbb{C}\mathrm{ov}(\mathbf{C}, C_i^o) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \cdot \mathbb{C}\mathrm{ov}(\mathbf{C}, C_i^o)$ has a nice interpretation in the case where both $\mathbf{C}$ and $\mathbf{E}$ follow multivariate normal distribution. Under this assumption, it follows (see e.g., Chapter 3 of* [38]*) that*

$$\mathbb{C}\mathrm{ov}(\mathbf{C}, C_i^o) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \cdot \mathbb{C}\mathrm{ov}(\mathbf{C}, C_i^o) = \mathbb{V}\mathrm{ar}[C_i^o] - \mathbb{V}\mathrm{ar}[C_i^o | \mathbf{C} = c], \quad \textit{for every vector } c \in \mathbb{R}^k.$$

This implies that $v_i(\mathbf{p}^*) = \gamma_i \, \mathbb{V}\mathrm{ar}[\mathbb{E}[C_i^o|\mathbf{C}]] + \mathbb{U}_i(\mathbb{E}_i)$. This representation is a way to see how agent i benefits from trading $\mathbf{C}$, given the endowments and the risk aversion level of the rest of the agents. Namely, the more uncertainty of C_i^o is revealed by $\mathbf{C}$, the higher the gain of utility of agent i is. Note also that when C_i^o is measurable with respect to the information generated by $\mathbf{C}$, $v_i(\mathbf{p}^*)$ gets its maximum value, and when C_i^o is independent with $\mathbf{C}$, agents is indifferent between trading and not trading $\mathbf{C}$.

Similarly in complete market case, based on Proposition 2.4, we readily calculate that the utility level of agent i on the Pareto equilibrium risk sharing is given by

$$v_i^o(\mathbf{p}^o) = \mathbb{U}_i(\mathcal{E}_i + \mathbf{a}_i^o \cdot \mathbf{E} - \mathbf{a}_i^o \cdot \mathbf{p}^o) = \gamma_i \, \mathbb{V}\mathrm{ar}[C_i^o] + \mathbb{U}_i(\mathcal{E}_i), \tag{2.13}$$

where, v_i^o denotes the utility level of agent i when the tradable security vector is $\mathbf{E}$. It follows from (2.12) and (2.13) that the utility *"loss"* of each agent when the risk sharing is in incomplete market and not in complete one, i.e., the difference

$$v_i^o(\mathbf{p}^o) - v_i(\mathbf{p}^*) = \gamma_i \left(\mathbb{V}\mathrm{ar}[C_i^o] - \mathbb{C}\mathrm{ov}(\mathbf{C}, C_i^o) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \cdot \mathbb{C}\mathrm{ov}(\mathbf{C}, C_i^o) \right), \tag{2.14}$$

is always non-negative and equal to zero if and only if the optimal security of agent i can be written as a linear combination of elements of $\mathbf{C}$. We state this fact in the following proposition the proof of which follows from standard arguments.

Proposition 2.7. *Each individual agent suffers a loss of utility from market incompleteness. This loss is zero for agent i if and only if there exist $b \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^k$ such that $C_i^o = b + \mathbf{a} \cdot \mathbf{C}$, where C_i^o is given in* (2.8).

A similar utility gain comparison we may use in the aggregate utility. In particular, we define as the measure of the *risk sharing inefficiency* the difference between the optimal aggregate utility gain and any realized (suboptimal) risk sharing:

$$\text{Risk Sharing Inefficiency} = \text{Optimal Aggregate Utility} - \text{Realized Aggregate Utility.}^{17} \tag{2.15}$$

Note that Pareto optimality and the M-V preferences guarantees that $\sup_{\mathbf{K} \in \mathcal{A}} \sum_{i=1}^{n} \mathbb{U}_i(\mathbb{E}_i + K_i) = \sum_{i=1}^{n} \mathbb{U}_i(\mathbb{E}_i + C_i^o)$ (see also the proof of Proposition 2.4).

3. Best Responses in Risk Sharing Transactions

This section establishes a novel agent's strategic behavior in risk sharing transactions. As mentioned in the introductory section, participation in financial risk sharing is limited to a specific number of agents. Modelling the equilibrium of such *oligopolies* should incorporate agents' best responses to other agents' actions. In order to establish such strategic behavior, we take the position of an individual agent who knows the aggregate endowment or the aggregate demand of the rest of the agents on a given security vector and we examine how she is going to respond aiming the maximization of her own utility.

3.1. **Best response in complete risk sharing markets.** In the complete market risk sharing, agents design the securities that are going to trade. As we have seen in Proposition 2.4, the optimal sharing securities are (linear) functions of the agents' submitted random endowments. Given this transaction set up, we are asking whether it is preferable for an individual agent to share her true risk exposure or to report as her endowment a different exposure.

Let's consider agent 1, who knows the aggregate risk exposure of the rest of the agents. According to optimal risk sharing rules, if she reports endowment $\mathcal{E}_1$, her position at the equilibrium will be $C_1^o - \pi_1^o$ (see (2.8) and (2.11)). However, she may exploit the other agents' hedging needs that stem from their random endowments and drive the security designing to a more preferable security that the other agents are still willing to offset. Since the optimal sharing rules are given functions of the submitted endowments, she can get her preferable equilibrium transaction by choosing the endowment she is going to report for sharing. Following the game theory terminology, we shall call the optimal choice for reported endowment *best endowment response*.

[17] This measure of aggregate loss of utility is also used in [2]. In [49] the aggregate utilities are compared in terms of certainty equivalent, which under M-V preferences coincides with our measure.

More formally, given that the rest agents have reported aggregate endowment $\mathcal{E}_{-1}$, if agent 1 reports as her random endowment some random variable $B \in \mathbb{L}^2$, according to optimal sharing rules (see (2.8)), she gets contract with payoff

$$C_1^o(B) := \frac{\gamma}{\gamma_1}\mathcal{E}_{-1} - \frac{\gamma_1 - \gamma}{\gamma_1}B, \tag{3.1}$$

and her accumulate cash transfer is (see (2.11))

$$\pi_1^o(B) := \frac{\gamma}{\gamma_1}\left(\mathbb{E}[\mathcal{E}_{-1}] - 2\gamma\,\mathbb{C}\mathrm{ov}(\mathcal{E}_{-1}, \mathcal{E}_{-1} + B)\right) - \frac{\gamma_1 - \gamma}{\gamma_1}\left(\mathbb{E}[B] - 2\gamma\,\mathbb{C}\mathrm{ov}(B, \mathcal{E}_{-1} + B)\right). \tag{3.2}$$

Hence, her utility after the transaction would be

$$G_1(B; \mathcal{E}_{-1}) := \mathbb{U}_1(\mathcal{E}_1 + C_1^o(B) - \pi_1^o(B)) = \mathbb{E}[\mathcal{E}_1 + C_1^o(B)] - \gamma_1\,\mathbb{V}\mathrm{ar}[\mathcal{E}_1 + C_1^o(B)] - \pi_1^o(B). \tag{3.3}$$

Similarly, we define the utility level of agent i when she reports B and the rest of the agents have reported $\mathcal{E}_{-i}$ and denote it by $G_i(B; \mathcal{E}_{-i})$. Therefore, the best endowment response of agent i is the solution of the following maximization problem

$$\max_{B \in \mathbb{L}^2} \{G_i(B; \mathcal{E}_{-i})\}. \tag{3.4}$$

The optimizer yields the best endowment response of agent i, while $\mathbb{L}^2$ is the set of her strategic choices. It is important to notice at this point that the optimizer B_i^* of (3.4) directly determines the payoff of the security that she buys at the equilibrium and the price she has to pay. In other words, submitting endowment B_i^* for sharing is equivalent to proposing the security $C_i^o(B_i^*)$, determined by equation (3.1), and the inducing cash compensation $\pi_i^o(B_i^*)$ given in (3.2).[18] The solution of problem (3.4) is stated in the following proposition.

Proposition 3.1. *For each $i \in \{1, ..., n\}$, the unique (up to constants) best endowment response of agent i, when the rest of the agents have reported aggregate endowment $\mathcal{E}_{-i}$, is given by*

$$B_i^* = \frac{\gamma_i}{\gamma_i + \gamma}\mathcal{E}_i + \frac{\gamma^2}{\gamma_i^2 - \gamma^2}\mathcal{E}_{-i}. \tag{3.5}$$

Hence, if agent i submits endowment B_i^*, the risk sharing contract that she buys has payoff $C_i^o(B_i^*)$ instead of C_i^o, where we readily calculate that $C_i^o(B_i^*) = \frac{\gamma_i}{\gamma_i+\gamma}C_i^o$. Thus, agent i prefers only a fraction of the Pareto optimal risk sharing contract, whereas the price she pays is less than $\frac{\gamma_i}{\gamma_i+\gamma}p_i^*$ and particular it equals to $\frac{\gamma_i}{\gamma_i+\gamma}p_i^* - 2\frac{\gamma^2\gamma_i^2}{(\gamma_i-\gamma)(\gamma_i+\gamma)^2}\,\mathbb{V}\mathrm{ar}[C_i^o]$. A number of further observations worth some attention. First, the best response B_i^* is invariant on probability distribution of the random endowments. Also, given that agent i knows the endowments of the rest of the agents, the best endowment response strategy indicates that she should share only a fraction of her risk exposure and also report exposure to the risk that the other agents face. In this way, she increases

[18] Note that letting the set of strategic choices be equal to $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ implies that agent's reported endowment is measurable with respect to the information that is generated by the true endowments.

(decreases) the demand of the securities she sells (buys) at the risk sharing transaction. The utility level after reporting the best endowment response is $G_i(B_i^*; \mathcal{E}_{-i}) = \frac{\gamma_i^3}{\gamma_i^2-\gamma^2} \mathbb{V}\mathrm{ar}\,[C_i^o] + \mathbb{U}_i(\mathcal{E}_i)$, which means that the gain of utility of agent i from following this strategic behavior is equal to $\gamma_i \mathbb{V}\mathrm{ar}[C_i^o] \frac{\gamma^2}{\gamma_i^2-\gamma^2}$ or in other words, compared to (2.13), the percentage increase of utility is $\frac{\gamma^2}{\gamma_i^2-\gamma^2}$. Therefore, adapting the best endowment response is more beneficial for agents with relatively low risk aversion, while for homogenous agents case, the percentage increase of utility equals to $\frac{1}{n^2-1}$. It is also clear that the difference between B_i^* and $\mathcal{E}_i$ reduces as the number of participating agents increases (*the larger the market, the less effective is the use of market power*).

3.2. **What percentage of her true endowment should an agent report?** According to the best endowment response (3.5), the preferable security of agent i given the other agents endowments, i.e., contract $C_i^o(B_i^*)$, corresponds to an endowment that she does not possess. An agent may be concerned that revealing exposure to the risks that only the other agents have, will deteriorate the conditions of future businesses with them. In such situation, she can alternatively exploit her market power by choosing (not the endowment she will report but rather) the size of her true risk exposure that she is willing to share. In other words, the agent's set of strategic choices is *restricted* to a positive percentage of the true endowment.

To emphasize the difference with problem (3.4), we hereafter use the notation $g_i(b; \mathcal{E}_{-i}) := G_i(b\mathcal{E}_i; \mathcal{E}_{-i})$, where the set of strategic choices is the interval $[0, \infty)$. Hence, problem (3.4) becomes

$$\max_{b \in [0,\infty)} \{g_i(b; \mathcal{E}_{-i})\} \tag{3.6}$$

and the maximizer shall be called *best percentage response*. We deal with this restriction in the proposition below, the proof of which follows the lines of Proposition 3.1.

Proposition 3.2. *Assume that $\mathcal{E}_i$ is not constant and let the rest of the agents' reported aggregate endowment be $\mathcal{E}_{-i}$. The best percentage response of agent i is*

$$b_i^* = 0 \vee \left(\frac{\gamma_i}{\gamma_i + \gamma} + \frac{\gamma^2}{\gamma_i^2 - \gamma^2} \rho(\mathcal{E}_i, \mathcal{E}_{-i}) \sqrt{\frac{\mathbb{V}\mathrm{ar}[\mathcal{E}_{-i}]}{\mathbb{V}\mathrm{ar}[\mathcal{E}_i]}} \right) \tag{3.7}$$

for each $i \in \{1, ..., n\}$, where $\rho(.,.)$ is the correlation coefficient map.

According to the sharing rules (2.8) and (2.11), reporting endowment $b_i^* \mathcal{E}_i$ implies that she gets the security with payoff $C_i^o(b_i^*) = \frac{\gamma}{\gamma_i} \mathcal{E}_{-i} - \frac{\gamma_i - \gamma}{\gamma_i} b_i^* \mathcal{E}_i$ and pays the cash amount $\pi_i(b_i^*) = \mathbb{E}[C_i^o(b_i^*)] - 2\gamma\, \mathbb{C}\mathrm{ov}(C_i^o(b_i^*), \mathcal{E})$. Note that the best percentage response of agent i is an increasing function of $\rho\left(\mathcal{E}_{-i}, \mathcal{E}_i\right)$ and agent i is going to take a speculation only position (not selling any of her endowment) if the correlation $\rho\left(\mathcal{E}_{-i}, \mathcal{E}_i\right)$ is sufficiently negative. This behavior is at first glance surprising, since negative endowments' correlation implies good hedging. However, this strategy is meaningful if we take into account that in this strategic behavior of agent i intends to exploit the knowledge of the risk exposure of her counterparties. If this exposure is negatively correlated to her endowment, it is preferable to share less of her risk exposure in order to achieve a better cash compensation for buying some of the other agents' risk and at the same time exploit the negative correlation for her own hedging needs. The situation differs when $\rho\left(\mathcal{E}_{-i}, \mathcal{E}_i\right)$ is positive. If the correlation is close to one, agent i may report overexposure on $\mathcal{E}_i$ which means that after the transaction she will be left with much more less exposure in $\mathcal{E}_i$, which together with the simultaneous long position on $\frac{\gamma}{\gamma_i}\mathcal{E}_{-i}$, results in total position of less variance.

Regarding the agent's risk aversion, we observe that as it increases, the best percentage response approaches one, i.e., $\lim_{\gamma_i\to\infty} b_i^* = 1$, independently on the correlations of the endowments or on their variances. This is consistent with the fact that the speculation attitude that characterizes the strategic behavior vanishes as the risk aversion increases (similarly as in the best endowment response). On the other hand, this attitude is more intense when γ_i goes to zero (i.e., when agent becomes risk neutral). In the limit, it holds

$$\lim_{\gamma_i\to 0} b_i^* = \begin{cases} 0, & \rho\left(\mathcal{E}_i, \mathcal{E}_{-i}\right) < 0; \\ \frac{1}{2}, & \rho\left(\mathcal{E}_i, \mathcal{E}_{-i}\right) = 0; \\ +\infty, & \rho\left(\mathcal{E}_i, \mathcal{E}_{-i}\right) > 0. \end{cases}$$

3.3. **Best demand response in incomplete risk sharing markets.** In the case of incomplete markets, where agents share their risk only through the transaction of a given vector of securities $\mathbf{C}$, each agent i reports her demand function $Z_i(\mathbf{p})$ and the allocation of $\mathbf{C}$ is the one at which the price vector sums the demands to zero. More precisely, we consider agent i and suppose that she knows (or can exact) the aggregate demand function of the rest of the agents. Note that under M-V preferences each agent's demand function is linear with slope equal to $-\frac{\mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}]}{2\gamma_i}$ (see equation (2.4)). This implies that the aggregate demand function of all the other $n-1$ agents is also linear and is given by $\sum_{j\neq i} Z_j(\mathbf{p}) = \left(\frac{\mathbb{E}[\mathbf{C}]-\mathbf{p}}{2\gamma_{-i}} - \mathbb{C}\mathrm{ov}(\mathbf{C}, \mathcal{E}_{-i})\right)\cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}]$, where we use the notation $1/\gamma_{-i} := \sum_{j\neq i} 1/\gamma_j$. The linearity of the demand function means that agent i can extract the aggregate demand functions of the rest of the agents by getting only two aggregate orders (note that $\mathbb{E}[\mathbf{C}]$ and $\mathbb{V}\mathrm{ar}[\mathbf{C}]$ are common for all agents, reflecting the standing common belief assumption). Given this knowledge, agent i could submit a demand function that clears out the market at the price that maximizes her own utility level. In particular, we ask: *Which is the preferable equilibrium price for agent i given the aggregate demand of the other agents?* This problem is written as

$$\max_{\mathbf{p}\in\mathbb{R}^k} \phi_i(\mathbf{p}; \sum_{j\neq i} Z_j(\mathbf{p})), \tag{3.8}$$

where $\phi_i(\mathbf{p};\mathbf{a}) := \mathbb{U}_i(\mathcal{E}_i - \mathbf{a}\cdot\mathbf{C}) + \mathbf{a}\cdot\mathbf{p}$, for $\mathbf{p}, \mathbf{a} \in \mathbb{R}^k$. Under this formulation, we are able to see how the competitive equilibrium prices change when an individual agent's market power stems from the asymmetric information on the other agents' demands. This is the goal of the following proposition.

Proposition 3.3. *The equilibrium price of* $\mathbf{C}$ *that maximizes the utility of agent* i *given the aggregate demand function of the other agents is*

$$\hat{\mathbf{p}}_i = \mathbb{E}[\mathbf{C}] - 2\gamma\, \mathbb{C}\mathrm{ov}\left(\mathbf{C}, \frac{\gamma_i}{\gamma_i+\gamma}\mathcal{E}_i + \frac{\gamma_i^2}{\gamma_i^2-\gamma^2}\mathcal{E}_{-i}\right). \tag{3.9}$$

Therefore, the *effective* aggregate endowment in the covariance part of the CAPM is equal to $\frac{\gamma_i}{\gamma_i+\gamma}\mathcal{E}_i + \frac{\gamma_i^2}{\gamma_i^2-\gamma^2}\mathcal{E}_{-i}$ instead of $\mathcal{E}$. Note that the equality of these endowments holds only in the trivial case where agent i does not share any risk with the rest of the agents (that is, when $\gamma_{-i}\mathcal{E}_{-i} = \gamma_i\mathcal{E}_i$ up to constants). In the effective aggregate endowment, the size of $\mathcal{E}_i$ is reduced by a percentage equal to $\frac{\gamma}{\gamma_i+\gamma}$, whereas the size of $\mathcal{E}_{-i}$ is increased by a percentage equal to $\frac{\gamma^2}{\gamma_i^2-\gamma^2}$. It is implied therefore that agent i could drive the market to price $\hat{\mathbf{p}}_i$ if her submitted demand reflects less exposure to her true endowment and some exposure to the other agents' aggregate endowment (exactly like the case of the best endowment response). The exact demand function of agent i that clears out the market at price $\hat{\mathbf{p}}_i$ shall be called *best demand response.*

To formulate this best response problem, we should rewrite problem (3.8) in terms of demand functions. For this, we denote by $\mathcal{Z}_i$ the set of all demand functions of an agent with M-V preferences and risk aversion γ_i on a given and fixed security vector $\mathbf{C}$. Since, all these demand functions have the specific form given in (2.4), we can parameterize $\mathcal{Z}_i$ by the covariance vectors: A function $z_i : \mathbb{R}^k \to \mathbb{R}^k$ belongs in $\mathcal{Z}_i$ if and only if there exists a vector $\mathbf{c} \in \mathbb{R}^k$, such that

$$z_i(\mathbf{p}) = \left(\frac{\mathbb{E}[\mathbf{C}] - \mathbf{p}}{2\gamma_i} - \mathbf{c}\right) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}].$$

Hence, the best response of agent i is to report the demand $\hat{Z}_i \in \mathcal{Z}_i$, such that $\hat{Z}_i(\hat{\mathbf{p}}_i) + \sum_{j\neq i} Z_j(\hat{\mathbf{p}}_i) = \mathbf{0}$, or equivalently to choose the vector $\hat{\mathbf{c}}_i$ such that $\left(\frac{\mathbb{E}[\mathbf{C}]-\hat{\mathbf{p}}_i}{2\gamma} - (\mathbb{C}\mathrm{ov}(\mathbf{C}, \mathcal{E}_{-i}) + \hat{\mathbf{c}}_i)\right) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] = 0$. It follows that $\hat{Z}_i(\mathbf{p}) = \left(\frac{\mathbb{E}[\mathbf{C}]-\mathbf{p}}{2\gamma_i} - \mathbb{C}\mathrm{ov}(\mathbf{C}, B_i^*)\right) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}]$, where the random variable B_i^* is the best endowment response given in (3.5). We summarize the above analysis in the following proposition, which states the clear connection of the agent's best responses in complete and incomplete market settings.

Proposition 3.4. *For every vector of tradeable securities* $\mathbf{C} \in (\mathbb{L}^2)^k$*, the best demand response of an agent* i *coincides with the demand function that is induced by her best endowment response.*

By submitting the demand $\hat{Z}_i$, agent i increases (decreases) the demand of the securities she is going to short (long) at the equilibrium, yielding a beneficial impact on the equilibrium prices. A measure of this price impact can simply be given by the difference $\hat{\mathbf{p}}_i - \mathbf{p}^*$. For any tradeable security vector $\mathbf{C}$, we readily calculate that the price impact caused by the strategic behavior of agent i

equals to $\frac{2\gamma^2}{(\gamma_i+\gamma_{-i})}\,\mathbb{C}\mathrm{ov}(\mathbf{C},\mathcal{E}_i-\frac{\gamma}{\gamma_i-\gamma}\mathcal{E}_{-i})$, which is positive (negative) and increasing (decreasing) with respect to γ_i for securities that agent i sells (buys).

An interesting example is the case of a speculator (i.e., an agent with constant endowment). If the payoff of the security C_j is negatively correlated with $\mathcal{E}_{-i}$, she is supposed to take short positions and hence satisfy the hedging demand of her counterparties. According to her best demand response however, she is going to submit a demand function which reveals some hedging needs from her behalf that are of the same direction as the other agents' aggregate demand. Although at equilibrium she still takes a short position on C_j, her strategy drives its price at higher levels.

4. Equilibria when Agents Use their Market Power

After setting up the agents' strategic behavior in oligopolistic risk sharing environments, the next step is to examine whether and how these markets reach equilibrium points. This can be considered as modelling the *negotiation game on security designing or on trading given vector of securities*. Each agent responds to the other agents' choices forming a type of pure Nash game, where the strategic sets of choices are the set of reported endowments (equivalently the suggested securities) or the reported demands, depending on the market completion.

The questions addressed in this section are whether this game reaches a (Nash) equilibrium, how this equilibrium differs from the Pareto optimal one and which are the agents (if any) that benefit (in expected utility terms) by the game. We first analyze the complete market setting and then the case where agents negotiate the price-allocation of a given vector of securities $\mathbf{C}$.

4.1. **Nash equilibrium in complete risk sharing market.** The way that agents use their market power in the complete risk sharing market is indicated by problem (3.4). Namely, each agent declares the random endowment she chooses to share (the one indicated by (3.5)) or equivalently proposes securities that are in line with the optimal sharing rules (see equations (3.1) and (3.2)). This procedure sets the conditions of the Nash game in the complete risk sharing transaction, and the equilibrium is defined as the security payoffs that all the agents agree on. We shall call this equilibrium *Nash risk sharing equilibrium*.[19] Before giving the exact definition, we recall from (3.3) that the utility level of an agent i, when she reports endowment B and the rest of the agents have reported aggregate endowment $B_{-i}:=\sum_{j\neq i}B_j$ is written in the following form

$$G_i(B;B_{-i})=\mathbb{U}_i(\mathcal{E}_i+C_i^o(B)-\pi_i^o(B))=\mathbb{E}[\mathcal{E}_i+C_i^o(B)]-\gamma_i\,\mathbb{V}\mathrm{ar}[\mathcal{E}_i+C_i^o(B)]-\pi_i^o(B). \tag{4.1}$$

[19] This is a pure Nash equilibrium where the set of strategic choices is in fact $\mathbb{L}^2$. The definition presupposes that the additional costs and the time of this negotiation are negligible, which are standard assumptions on the literature of the Nash equilibria in financial transactions.

Definition 4.1. *We call a vector of random variables $(\hat{B}_1, ..., \hat{B}_n) \in (\mathbb{L}^2)^n$ Nash equilibrium endowments if for each $i \in \{1, ..., n\}$*

$$G_i(\hat{B}_i; \hat{B}_{-i}) \geq G_i(B; \hat{B}_{-i}), \qquad \text{for all } B \in \mathbb{L}^2. \tag{4.2}$$

The induced risk sharing securities $(\hat{C}_1, ..., \hat{C}_n)$ given by

$$\hat{C}_i := C_i^o(\hat{B}_i) = \frac{\gamma}{\gamma_i}\hat{B}_{-i} + \frac{\gamma_i - \gamma}{\gamma_i}\hat{B}_i, \qquad \text{for each } i \in \{1, ..., n\}, \tag{4.3}$$

are called Nash risk sharing securities.

If a vector of Nash equilibrium endowments exists, the prices of the induced risk sharing securities $(\hat{C}_1, .., \hat{C}_n)$ are determined by the pricing rule (2.11), where the considered aggregate endowment is the Nash equilibrium one, hereafter denoted by $\hat{\mathcal{B}} := \sum_{i=1}^n \hat{B}_i$. Hence, for each $i \in \{1, ..., n\}$ the Nash equilibrium price of security $\hat{C}_i$ is $\hat{\pi}_i := \pi_i^o(\hat{B}_i) = \mathbb{E}[\hat{C}_i] - 2\gamma\, \mathbb{C}\text{ov}(\hat{C}_i, \hat{\mathcal{B}})$.

The following theorem states the explicit form of the *unique* Nash equilibrium.

Theorem 4.2. *There exists a unique (up to constants) Nash risk sharing equilibrium characterized as follows. For each $i \in \{1, ..., n\}$*

(i) *The Nash equilibrium endowment of agent i is*

$$\hat{B}_i = \frac{\gamma_i}{\gamma_i + \gamma_{-i}}\mathcal{E}_i + \left(\frac{\gamma_{-i}}{\gamma_i + \gamma_{-i}}\right)^2 \hat{\mathcal{B}}, \tag{4.4}$$

where

$$\hat{\mathcal{B}} = \frac{1}{c}\left(\mathcal{E} - \sum_{j=1}^n \frac{\gamma}{\gamma_j}\mathcal{E}_j\right) \tag{4.5}$$

and constant $c \in \mathbb{R}_+$ is defined by $c = 1 - \sum_{j=1}^n \frac{\gamma^2}{\gamma_j^2}$.

(ii) *The Nash risk sharing security that agent i gets is*

$$\hat{C}_i = -\frac{\gamma_i - \gamma}{\gamma_i}\left(1 - \frac{\gamma(\gamma_i - \gamma)}{c\gamma_i^2}\right)\mathcal{E}_i + \frac{\gamma_i - \gamma}{c\gamma_i^2}\sum_{j \neq i}\frac{\gamma_j - \gamma}{\gamma_j}\mathcal{E}_j. \tag{4.6}$$

Indeed, the use of the market power by the agents heavily changes the risk sharing transactions. Not only the security that each agent gets at equilibrium deviates from the optimal sharing security, but also the effective aggregate endowment generally differs from the true one (they are equal only when agents are homogeneous, see Corollary 4.5 below). In particular, one can readily show that $0 < 1 - \frac{\gamma(\gamma_i - \gamma)}{c\gamma_i^2} < 1$, which means that at Nash equilibrium each agent still shares some of her true endowment, but the size is always smaller than in Pareto optimal transaction (compared to what is revealed by security C_i^o in (2.8)). Therefore, the aggregate shared risk is lower resulting (among other things) in an inefficient risk sharing. Thanks to the explicit formulas of the equilibrium securities, we are able to measure the exact inefficiency.

Corollary 4.3. *The risk sharing inefficiency caused by agents' use of market power is given by*

$$\sum_{i=1}^{n} \mathbb{U}_i(\mathcal{E}_i + C_i^o - \pi_i^o) - \sum_{i=1}^{n} \mathbb{U}_i(\mathcal{E}_i + \hat{C}_i - \hat{\pi}_i) = \sum_{i=1}^{n} \gamma_i \,\mathbb{V}\mathrm{ar}[\mathcal{E}_i - \hat{B}_i] - \gamma \,\mathbb{V}\mathrm{ar}[\mathcal{E} - \hat{\mathcal{B}}]. \tag{4.7}$$

The inefficient is non-negative and vanishes if and only if the Pareto optimal securities are constants.

Recall that Pareto optimal securities are constants (i.e., there is no risk transfer) if and only if $\gamma_i \mathcal{E}_i = \gamma_j \mathcal{E}_j$ (modulo constants), for each $i, j \in \{1, ..., n\}$. Hence, when agents do share some risk, Nash equilibrium risk sharing implies loss of efficiency.

In order to interpret the insights of Theorem 4.2, we see closer the simplified case of two agents, which is summarized in the following table (the considered position is the one of agent 1). The discussion that follows however can well be generalized when $n > 2$.

	Pareto Sharing	Nash Sharing
Aggregate endowment	$\mathcal{E}$	$\hat{\mathcal{B}} = \frac{\gamma_1 \mathcal{E}_1 + \gamma_2 \mathcal{E}_2}{2\gamma}$
Reported endowment	$\mathcal{E}_1$	$\hat{B}_1 = \frac{2\gamma_1 + \gamma_2}{2(\gamma_1 + \gamma_2)} \mathcal{E}_1 + \frac{\gamma_2^2}{2\gamma_1(\gamma_1 + \gamma_2)} \mathcal{E}_2$
Purchased security	$C_1^o = \frac{\gamma_2 \mathcal{E}_2 - \gamma_1 \mathcal{E}_1}{\gamma_1 + \gamma_2}$	$\hat{C}_1 = \frac{C_1^o}{2}$
Utility gain from sharing	$\gamma_1 \,\mathbb{V}\mathrm{ar}\left[\frac{\gamma_1 \mathcal{E}_1 - \gamma_2 \mathcal{E}_2}{\gamma_1 + \gamma_2}\right]$	$\frac{\gamma_1 + 2\gamma_2}{4} \,\mathbb{V}\mathrm{ar}\left[\frac{\gamma_1 \mathcal{E}_1 - \gamma_2 \mathcal{E}_2}{\gamma_1 + \gamma_2}\right]$
Inefficiency	0	$\frac{1}{\gamma_1 + \gamma_2} \,\mathbb{V}\mathrm{ar}\left[\frac{\gamma_1 \mathcal{E}_1 - \gamma_2 \mathcal{E}_2}{2}\right]$

TABLE 1. Comparison of Pareto and Nash risk sharing equilibria when $n = 2$.

The Nash equilibrium sharing security reveals that each agent chooses to share only a fraction of her risk exposure and declares some exposure to the other agent's endowment. Note also that for agents of relatively low risk aversion, this strategic behavior is more intense. In any case, the Nash risk sharing of lower risk is exactly the half of the Pareto optimal one, independently on agents' risk aversion coefficients. Furthermore, although the aggregate utility is lower in the case of Nash equilibrium, there exist situations where some of the agents enjoy higher expected utility when the game is played. In order to check the total utility gain of each agent at Nash equilibrium, we have to take into account the cash transfer. The equilibrium price of security $\hat{C}_i$ is given by $\mathbb{E}[\hat{C}_i] - 2\gamma \,\mathbb{C}\mathrm{ov}(\hat{C}_i, \hat{\mathcal{B}})$. In the case of $n = 2$, it is easily calculated that the price change is in favor of the agent with lower risk aversion.

Proposition 4.4. *Let $n = 2$ and assume that $\gamma_1 < \gamma_2$. Then, besides that $\hat{C}_1 = C_1^o/2$, when the price of C_1^o is positive (negative) the cash compensation $\hat{\pi}_1$ is lower (higher) than $\pi_1^o/2$.*

Therefore, market power and the induced Nash equilibrium have two effects for agent 1: one negative, since she shares some of the risk sharing inefficiency, and one positive which stems from the price impact. The total outcome is positive when her aversion coefficient is sufficiently lower than the one of her counterparty (in particular when $\gamma_1 < 2\gamma_2/3$). Thus, *agents with sufficiently low risk aversion benefits by the oligopolistic structure of a risk sharing transaction.* On the other hand, agent with higher risk aversion not only shares some of the risk sharing inefficiency, but also suffers from the induced price impact (heavily when her counterparty's preferences are close to risk neutrality).[20]

Another aspect that highlights the importance of agents' risk aversion is the asymptotic interaction of agents' behaviors, namely it holds that $\lim_{\gamma_i \to \infty} \hat{B}_i = \mathcal{E}_i$, for every $\gamma_{-i} \in \mathbb{R}_+$; and $\lim_{\gamma_{-i} \to 0} \hat{B}_i = \mathcal{E}_i$, for every $\gamma_i \in \mathbb{R}_+$. Intuitively, agents with risk preferences close to risk neutrality *dominates* the Nash risk sharing game, in the sense that the other agents do not act strategically and do share their true endowments.

It is therefore apparent that the agents heterogeneity in risk aversion is a crucial factor on the outcome of the game. The large majority of literature on Nash equilibria in risk sharing transactions assume homogeneous with respect to risk aversion agents (see for example [43, 44, 49]), which is only a special case in the present setting ($\gamma_i = \gamma_j$, for all $i, j \in \{1, ..., n\}$). Homogeneity of agents' risk aversion is in fact the only non-trivial case where the aggregate Nash endowment is equal to the true one, which implies that the game has no price impact.

Corollary 4.5. *Let $\gamma_i \mathcal{E}_i \neq \gamma_j \mathcal{E}_j$, for some pair $i, j \in \{1, ..., n\}$ with $i \neq j$. Then, $\hat{\mathcal{B}} = \mathcal{E}$ if and only if agents are homogenous.*

The induced Nash equilibrium risk sharing securities are fractions of the optimal ones ($\hat{C}_i = \frac{n-1}{n} C_i^o$ for each $i \in \{1, ..., n\}$) and their prices are analogously reduced. This implies that the risk sharing inefficiency, which is equal to $\frac{1}{n^2}\left(\sum_{i=1}^n \mathbb{V}\mathrm{ar}[\mathcal{E}_i] - \frac{\mathbb{V}\mathrm{ar}[\mathcal{E}]}{n}\right)$, is shared *equally* among agents.

4.2. **Restricted Nash risk sharing equilibrium.** The arguments that lead to the Nash equilibrium risk sharing can also be applied when agents' set of strategic choices is restricted (as in problem (3.6)). Each agent proposes sharing securities consistent with optimal sharing rules by choosing the percentage (possibly higher than one) of her true endowment. We now examine whether this restricted game equilibrates. Recall (see also (4.1)) that for every $b \in [0, \infty)$

$$g_i(b; \sum_{j \neq i} b_j \mathcal{E}_j) = \mathbb{U}_i(\mathcal{E}_i + C_i^o(b\mathcal{E}_i) - \pi_i^o(b\mathcal{E}_i)) = \mathbb{E}[\mathcal{E}_i + C_i^o(b\mathcal{E}_i)] - \gamma_i \,\mathbb{V}\mathrm{ar}[\mathcal{E}_i + C_i^o(b\mathcal{E}_i)] - \pi_i^o(b\mathcal{E}_i),$$

where sharing rules C_i^o and π_i^o are given in (3.1) and (3.2), provided that the aggregate endowment from the rest of the agents is $\sum_{j \neq i} b_j \mathcal{E}_j$. We now state the adjustment of the Nash risk sharing equilibrium when the agents' set of strategic choices is restricted.

[20]Comparison of utility gain has also been studied by [41], where it is shown that there are cases where noise traders enjoy higher utility than the sophisticated ones under non-competitive (Nash) equilibrium prices.

Definition 4.6. *We call a vector* $(\hat{b}_1, ..., \hat{b}_n) \in [0, \infty)^n$ *restricted Nash equilibrium if for each* $i \in \{1, ..., n\}$

$$g_i(\hat{b}_i; \sum_{j \neq i} \hat{b}_j \mathcal{E}_j) \geq g_i(b; \sum_{j \neq i} \hat{b}_j \mathcal{E}_j), \qquad \text{for all } b \in [0, \infty). \tag{4.8}$$

The induced risk sharing securities $(\hat{C}^r_1, ..., \hat{C}^r_n)$ *given by*

$$\hat{C}^r_i = \frac{\gamma}{\gamma_i} \sum_{j \neq i} \hat{b}_j \mathcal{E}_j - \frac{\gamma_i - \gamma}{\gamma_i} \hat{b}_i \mathcal{E}_i, \qquad \text{for each } i \in \{1, ..., n\} \tag{4.9}$$

are called restricted Nash risk sharing securities.

Thanks to Glicksburg-Fan-Debreu Theorem (see among others Chapter 1 of [23]) and the linearity of the best response function for positive values (see (3.7)), we are able to establish the existence and the uniqueness of this equilibrium, provided there is an exogenously given bound on agents' percentage choices.[21]

Proposition 4.7. *Suppose that the set of choices for each agent is bounded from above by an upper bound* $\kappa > 0$. *Then, there exists a unique restricted Nash risk sharing equilibrium.*

In the simplified case of $n = 2$, Nash risk sharing percentage equilibrium, $(\hat{b}_1, \hat{b}_2)$, solves the equations

$$\hat{b}_i = \left(0 \vee \left(\frac{\gamma_i}{\gamma_i + \gamma} + \frac{\gamma^2 \hat{b}_{-i}}{\gamma_i^2 - \gamma^2} \rho(\mathcal{E}_i, \mathcal{E}_{-i}) \sqrt{\frac{\mathbb{V}\mathrm{ar}[\mathcal{E}_{-i}]}{\mathbb{V}\mathrm{ar}[\mathcal{E}_i]}}\right)\right) \wedge \kappa, \qquad \text{for } i = 1, 2.^{22} \tag{4.10}$$

Let first see closer the special situation of homogeneous agents, where the dependence on the endowments' correlation can be isolated. Three different choices of the fraction $\mathbb{V}\mathrm{ar}[\mathcal{E}_2]/\mathbb{V}\mathrm{ar}[\mathcal{E}_1]$ are illustrated in Figures 1 and 2.

As expected from the discussion in subsection 3.2, the agent with riskier endowment (agent 2 in this particular example) applies this strategic behavior less intensely, since her increased hedging needs count more than the cash transfer (all else equal).

Regarding the risk aversion coefficients, $\hat{b}_i$ is a decreasing (increasing) function of γ_i when $\rho(\mathcal{E}_1, \mathcal{E}_2)$ is positive (negative), implying that high risk averse agents are more reluctant in applying this strategic behavior; in extreme case it holds that $\lim_{\gamma_i \to \infty} \hat{b}_i = 1$. Furthermore, as in the unrestricted game, there is a similar asymptotic interaction between agents' behavior and their risk aversions. Namely,

[21] This bound in the set of choices mitigates the magnitude of the reported endowments, however it is by no means restrictive when we consider real-world situations.

[22] In the case where $\mathcal{E}_i$ is constant, $\hat{b}_i = \frac{\gamma_i}{\gamma_i + \gamma} \wedge \kappa$.

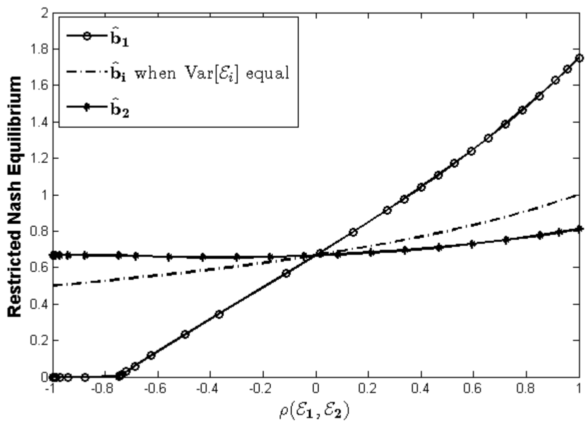

FIGURE 1. $\gamma_1 = \gamma_2$ and $\mathbb{V}\mathrm{ar}[\mathcal{E}_2] = 16\,\mathbb{V}\mathrm{ar}[\mathcal{E}_1]$

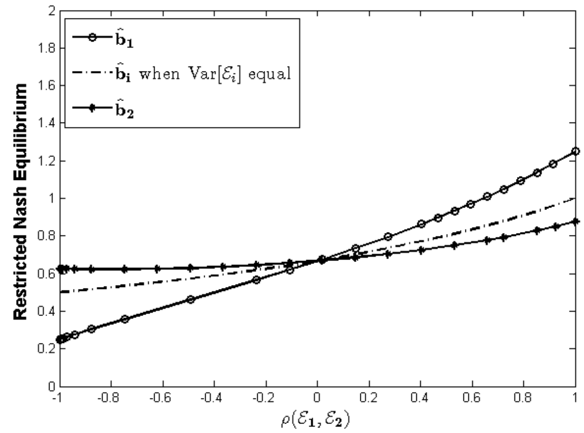

FIGURE 2. $\gamma_1 = \gamma_2$ and $\mathbb{V}\mathrm{ar}[\mathcal{E}_2] = 4\,\mathbb{V}\mathrm{ar}[\mathcal{E}_1]$

$$\lim_{\gamma_i \to \infty} \hat{b}_{-i} = \lim_{\gamma_{-i} \to 0} \hat{b}_{-i} = \begin{cases} 0, & \rho\left(\mathcal{E}_i, \mathcal{E}_{-i}\right) < 0; \\ \frac{1}{2}, & \rho\left(\mathcal{E}_i, \mathcal{E}_{-i}\right) = 0; \\ \kappa, & \rho\left(\mathcal{E}_i, \mathcal{E}_{-i}\right) > 0, \end{cases} \qquad \text{and} \qquad \lim_{\gamma_i \to \infty} \hat{b}_i = \lim_{\gamma_i \to 0} \hat{b}_{-i} = 1.$$

The above limits verify that even in the restricted game, an agent with risk preferences close to risk neutrality dominates the risk sharing transaction.

Finally, there are cases where the restricted Nash is preferable by particular agents when compared to the Pareto optimal risk sharing. In Figures 3 and 4, we compare the utility gain of agent 1 in the restricted Nash equilibrium and in Pareto optimal sharing in different situations.

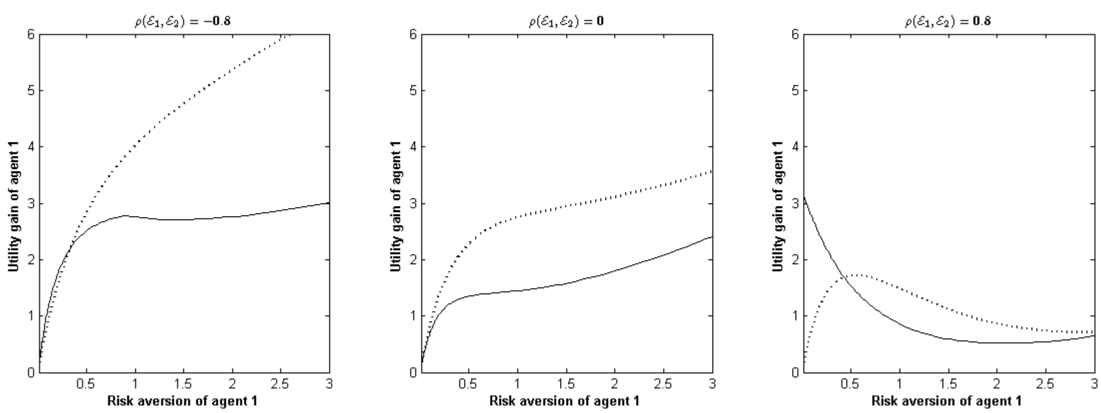

FIGURE 3. Utility gains of agent 1 when $\mathbb{E}[\mathcal{E}_i] = 0$, $\mathbb{V}\mathrm{ar}[\mathcal{E}_1] = 1$, $\mathbb{V}\mathrm{ar}[\mathcal{E}_2] = 10$, and $\gamma_2 = 1$. Solid lines are the gain at restricted Nash game and the dotted lines are the gains at the Pareto optimal transaction.

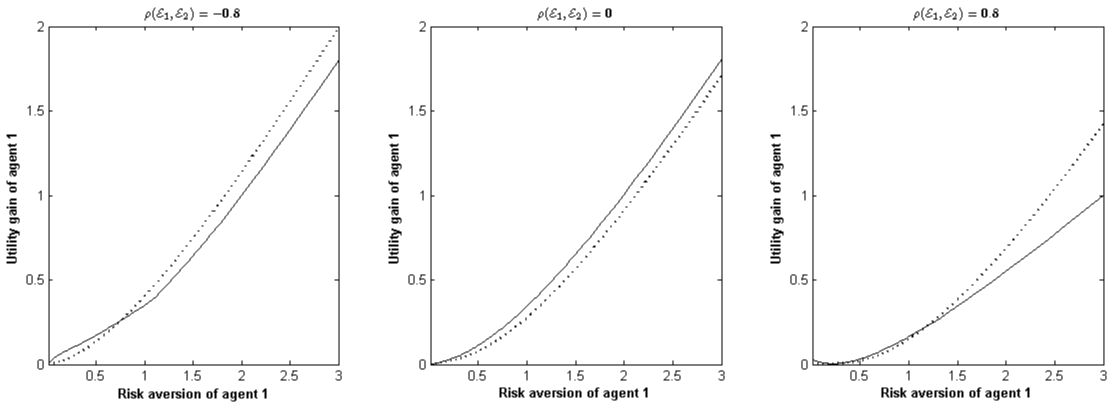

FIGURE 4. Utility gains of agent 1 when $\mathbb{E}[\mathcal{E}_i] = 0$, $\mathbb{V}\text{ar}[\mathcal{E}_1] = 1$, $\mathbb{V}\text{ar}[\mathcal{E}_2] = 0.1$ and $\gamma_2 = 1$. Solid lines are the gain at restricted Nash game and the dotted lines are the gains at the Pareto optimal transaction.

The utility gain for agents with preferences close to risk neutrality is higher even when the Nash equilibrium is restricted. The highest possible gain occurs when a low risk averse agent shares a low risky endowment with a positively correlated and but riskier endowment (see the right graph in Figure 3). In this case, an agent uses the negotiation game to exploit the increased hedging needs of her counterparty and improves her cash transfer. Figure 4 also implies that there is a negligible gain of utility for agents with risky endowments, when the correlation is negative (or zero).

4.3. **The CAPM in oligopoly financial markets.** In the incomplete market setting, the risk sharing game is of similar spirit but of different structure. Following the arguments of subsection 3.3, we let the set of strategic choices to be the demand functions on the given vector of securities, i.e., the sets $\mathcal{Z}_i$ for each $i \in \{1, ..., n\}$. We then suppose that each agent responds to the other agents' aggregate demands aiming to drive the market to her preferable security price vector (recall problem (3.8)). Naturally, the (Nash) equilibrium is the point at which the submitted demand functions sum up to zero at a price which is preferable for all the agents. We set $\mathcal{Z} := \mathcal{Z}_1 \times ... \times \mathcal{Z}_n$ and give the exact definition below.

Definition 4.8. *A pair* $(\hat{\mathbf{p}}, \hat{\mathbf{Z}}) \in \mathbb{R}^k \times \mathcal{Z}$ *is called Nash price-demand equilibrium of a vector of securities* $\mathbf{C} \in (\mathbb{L}^2)^k$ *if the following two conditions hold:*

(i) $\sum_{i=1}^{n} \hat{Z}_i(\hat{\mathbf{p}}) = 0$.

(ii) *For each* $i \in \{1, ..., n\}$, $\phi_i(\hat{\mathbf{p}}; \sum_{j \neq i} \hat{Z}_j) \geq \phi_i(\mathbf{p}; \sum_{j \neq i} \hat{Z}_j)$, *for all* $\mathbf{p} \in \mathbb{R}^k$.

For every Nash price $\hat{\mathbf{p}}$*, the vector* $(\hat{Z}_1(\hat{\mathbf{p}}), ..., \hat{Z}_n(\hat{\mathbf{p}}))$ *is called Nash allocation of* $\mathbf{C}$.

This equilibrium can be seen as a *variation of the CAPM* in markets where one of the main assumptions of this model, that all participants in the market are price-takers, is withdrawn.

Although games on demand functions generally equilibrate in a continuum of Nash equilibria, in case of M-V preferences the Nash equilibrium is unique.

Theorem 4.9 (CAPM in Oligopoly Markets)**.** *For every vector of securities* $\mathbf{C} \in (\mathbb{L}^2)^k$*, there is a unique Nash price-demand equilibrium* $(\hat{\mathbf{p}}, \hat{\mathbf{Z}})$ *given by*

$$\hat{\mathbf{p}} = \mathbb{E}[\mathbf{C}] - 2\gamma\, \mathbb{C}\mathrm{ov}(\mathbf{C}, \hat{\mathcal{B}}), \tag{4.11}$$

and for each $i \in \{1, ..., n\}$

$$\hat{Z}_i(\mathbf{p}) = \left(\frac{\mathbb{E}[\mathbf{C}] - \mathbf{p}}{2\gamma_i} - \mathbb{C}\mathrm{ov}(\mathbf{C}, \hat{B}_i) \right) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}], \tag{4.12}$$

where $\hat{B}_i$ *and* $\hat{\mathcal{B}}$ *are given in* (4.4) *and* (4.5)*. The induced Nash allocation is*

$$\hat{Z}_i(\hat{\mathbf{p}}) = \mathbb{C}\mathrm{ov}(\mathbf{C}, \hat{C}_i) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \tag{4.13}$$

where $\hat{C}_i$ *is given in* (4.3)*.*

Theorem 4.9 states a clear connection between the Nash equilibria in complete and incomplete markets. The effective aggregate endowment in pricing $\mathbf{C}$ is equal to $\hat{\mathcal{B}}$, which is the aggregate Nash endowment in the complete market game and each agent's equilibrium demand is the one corresponds to her Nash equilibrium endowment. A bottom line is that in any risk sharing setting (complete or incomplete) the considered agents' endowments should be $\hat{B}_i$ rather than $\mathcal{E}_i$.

Taking into account Corollary 4.5 and equation (4.11), we also get that the oligopolistic structure of the market leaves the prices unaffected only when participating agents have the same risk aversion.

Corollary 4.10. *Let* $\gamma_i \mathcal{E}_i \neq \gamma_j \mathcal{E}_j$*, for some pair* $i, j \in \{1, ..., n\}$ *with* $i \neq j$*. Then, the Nash and the Pareto equilibrium price of a vector of securities* $\mathbf{C} \in (\mathbb{L}^2)^k$ *coincides if and only if agents are homogeneous.*

Although the prices are equal under agents' risk aversion homogeneity, the equilibrium allocations do not coincide. One can calculate that the units traded in Nash equilibrium are reduced for each security and in particular, $\hat{Z}_i(\hat{\mathbf{p}}) = \frac{n-1}{n} Z_i(\mathbf{p}^*)$, for each $i \in \{1, ..., n\}$. This implies (see also Proposition 2.5) that the negotiation game on trading any given vector of securities causes a decrease in utility gain by a percentage equal to $\frac{2n-1}{n^2}$ for each homogenous agent (*equal percentage loss of utility for each agent, even though the prices are equal to the competitive ones*).

The situation differs when agents have different risk aversion. In fact, as in the complete market setting, there are cases where some agents gain more utility on the Nash equilibrium transaction of $\mathbf{C}$. To see when this occurs, consider for instance the simplified case of $n = 2$, $k = 1$ and $\mathbb{V}\mathrm{ar}[C] = 1$. Then, $\mathbb{U}_1(\mathcal{E}_1 + \hat{Z}_1(\hat{\mathbf{p}})(C - \hat{\mathbf{p}})) = \mathbb{U}_1(\mathcal{E}_1 + Z_1(\mathbf{p}^*)(C - \mathbf{p}^*)) - \left(\frac{3\gamma_1 - 2\gamma_2}{4}\right) \mathbb{C}\mathrm{ov}^2(C, C_1^o)$, which means that only the level of agents' risk aversion determines for which agent the use of the market power is beneficial. Similar calculations in the more general cases allow the following conclusion (which is consistent with corresponding result in the complete market setting).

Corollary 4.11. *For every given vector of securities* $\mathbf{C}$*, the use of the market power in equilibrium trading of* $\mathbf{C}$ *is beneficial for agents with sufficiently low risk aversion.*

Furthermore, comparison of prices (2.6) and (4.11) gives the exact measure and the direction of the *price impact* that agents' strategic behavior causes on $\mathbf{C}$. For each $j \in \{1, ..., k\}$,

$$p_j^* < \hat{p}_j \quad \text{if and only if} \quad \mathbb{C}\text{ov}(C_j, \hat{\mathcal{B}}) < \mathbb{C}\text{ov}(C_j, \mathcal{E}).$$

Hence, $\mathbb{C}\text{ov}(C_j, \mathcal{E} - \hat{\mathcal{B}})$ can be considered as the measure of the price impact that is caused on security C_j when the participating agents use their market power. In particular, the game causes upward price impact if $\mathbb{C}\text{ov}(C_j, \mathcal{E} - \hat{\mathcal{B}}) > 0$ and negative price impact if $\mathbb{C}\text{ov}(C_j, \mathcal{E} - \hat{\mathcal{B}}) < 0$. For example in the case of two agents, $\mathbb{C}\text{ov}(C_j, \mathcal{E} - \hat{\mathcal{B}}) = \frac{\gamma_1 - \gamma_2}{2\gamma_1\gamma_2} \mathbb{C}\text{ov}(C_j, \gamma_2\mathcal{E}_2 - \gamma_1\mathcal{E}_1)$. Then, assuming that agent 1 is more risk averse than agent 2, the price impact is positive (negative) if agent 1 buys (sells) security C_j in the equilibrium transaction.[23] We may conclude that under the use of market power, there is a price impact in favor of low risk averse agents (as Corollary 4.11 insinuates).

Remark 4.12. *The equality of Nash and Pareto equilibrium prices under homogeneity of agents' preference does not hold when agents have different beliefs regarding the moments of the tradeable assets. Although the effect of belief difference is beyond the scope of this manuscript, it is a matter of calculations to check that Nash and Pareto equilibrium prices are generally different when agents have the same risk aversion but different subjective probability measures. In fact,* $\mathbf{p}^* = \hat{\mathbf{p}}$ *only in the special case of equal subjective variance matrix of* $\mathbf{C}$.[24]

5. On the Increase of Efficiency

Theorems 4.2 and 4.9 state that in all non-trivial cases, the use of market power results in non-optimal risk sharing. Moreover, the analytic formulas of the Nash equilibria allow us to examine closer how the model parameters affect the inefficiency of the risk sharing caused by agents' game. One application of this analysis is its possible use by a market regulator (or a social planner), whose

[23] Recall from equation (2.7) that the allocation of agent 1 at Pareto equilibrium is $\mathbb{C}\text{ov}(\mathbf{C}, \frac{\gamma_2\mathcal{E}_2 - \gamma_1\mathcal{E}_1}{\gamma_1 + \gamma_2}) \cdot \mathbb{V}\text{ar}^{-1}[\mathbf{C}]$.

[24] Similarly to the discussion on gains' comparison, one can calculate that given that homogeneous agents have different estimations on the moments of C, the agents who benefit from the game are the ones with sufficiently lower estimated variance of the tradeable assets. For example, in case of two agents with the same risk aversion, agent 1 gets more utility on Nash equilibrium if and only if $\mathbb{V}\text{ar}_1[C] < 2\,\mathbb{V}\text{ar}_2[C]/3$, where $\mathbb{V}\text{ar}_i[C]$ is the variance of under the subjective probability measure of agent i.

goal is to increase the welfare of the risk sharing, as the latter is measured by the efficiency of the transaction.

Intuitively, the inefficiency of the risk sharing should vanish as the number of agents increases, meaning that the market power of each individual agent is getting less important. Upon some mild assumptions, the following proposition verifies this standard fact in the case of the complete market setting.

Proposition 5.1. *Consider a sequence of agents with random endowments* $(\mathcal{E}_i)_{i\in\mathbb{N}}$ *and risk aversion coefficients* $(\gamma_i)_{i\in\mathbb{N}}$*. If* $(\mathcal{E}_i)_{i\in\mathbb{N}}$ *is uniformly bounded in* $\mathbb{L}^2$ *and there exist positive constants* c_l *and* c_u *such that* $c_l \leq \gamma_i \leq c_u$ *for each* $i \in \mathbb{N}$*, the risk sharing inefficiency given in* (4.7) *goes to zero as* $n \to \infty$*.*

Thus, a market regulator could increase the efficiency of the risk sharing transactions if he facilitates the enlargement of the market. It should also be pointed out that the enlargement of the market increases the efficiency even if the new agents are speculators (that is, they do not have any risk exposure). In fact, although agents with less risk aversion coefficient get more utility when market is small, the total efficiency is increased even when the extra agents are low risk averse or/and without any hedging needs.

Similar result holds in any incomplete markets too. More precisely, when the number of agents goes to infinity, the differences between the Pareto and the Nash equilibrium prices/allocations of any vector of securities $\mathbf{C}$ vanish. This is stated in the following proposition, where the indication (n) refers to the market of n agents.

Proposition 5.2. *Consider a sequence of agents with random endowments* $(\mathcal{E}_i)_{i\in\mathbb{N}}$ *and risk aversion coefficients* $(\gamma_i)_{i\in\mathbb{N}}$*. If* $(\mathcal{E}_i)_{i\in\mathbb{N}}$ *is uniformly bounded in* $\mathbb{L}^2$ *and there exist positive constants* c_l *and* c_u *such that* $c_l \leq \gamma_i \leq c_u$ *for each* $i \in \mathbb{N}$*, then for any vector of securities* $\mathbf{C}$*, as* $n \to \infty$ *it holds that*

(i) $||\mathbf{p}^*(n) - \hat{\mathbf{p}}(n)|| \to 0$ *and*
(ii) $||\mathbf{a}_i^*(n) - \hat{\mathbf{a}}_i(n)|| \to 0$*, for each* $i \in \mathbb{N}$*.*

Another factor that certainly influences the risk sharing efficiency is the market completion. By definition, the efficiency is higher when the tradeable securities are more correlated with the agents' endowments. According to Proposition 2.7, under no use of market power, each agent suffers a loss of utility when the market is incomplete (i.e., *every agent has a motive to complete the market*). It turns out that a similar result holds also in the case of oligopolistic risk sharing transactions. This is established in the following proposition (the proof of which is a matter of simple calculations).

Proposition 5.3. *In Nash equilibrium risk sharing, each individual agent suffers a loss of utility when the market is incomplete. This loss is zero for agent i if and only if there exist $b \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^k$ such that $\hat{C}_i = b + \mathbf{a} \cdot \mathbf{C}$, where $\hat{C}_i$ is given in* (4.3).

In other words, even when the market consists of few agents and all use their market power in the negotiation of the risk sharing transaction, each one gets more expected utility if the sharing is done in a complete market setting. Completion needs the creation of new financial securities that consists of the reported endowments' payoffs. Hence, even in thin markets, a regulator could increase the efficiency of the risk sharing if he facilitates the financial innovation of the agents (for instance, by minimizing the involved transaction costs).

On another perspective, more efficiency can be achieved by the restriction of the set of the agents' strategic choices, since shortened set of choices implies that the best responses are closer to the true agents' conditions. Such an example has been developed in subsection 4.2, where agents choose only the size of their true endowments that will be considered for sharing. Proposition 4.7 guarantees the existence of a unique equilibrium under these restricted strategic sets, which then results in higher aggregate utility than the equilibrium of Theorem 4.2.

Finally, throughout the preceding analysis we have pointed out the importance of the agents' risk aversion coefficients in risk sharing transactions. For intance, the best endowment (demand) response of an individual agent is closer to her true endowment (demand) when she is more risk averse. Hence, more risk averse agents mean that the market equilibrium is closer to the Pareto one and hence the efficiency is increased. Although agents' risk aversion is a subjective issue, a market regulator has motive to increase the risk aversion level of all participating agents. The ways that a regulator can achieve this are not standard. In financial literature for example, agents' risk tolerance coefficient (i.e., the reciprocal of the risk aversion) is sometimes proxied by measures of the severity of the capital constraints required to participate in a market (e.g., a broker desk or some other fixed costs, see for instance [19]). Hence, a regulator could increase the risk averse of the participating agents by simply strengthening the capital constraints. However, this policy may have undesired side effects, since additional requirements retard the market enlargement and hinder the financial innovation. The marker regulator has to find the optimal balance between the required capital constraints and the facilitation of the market entrance. This problem has widely been studied for transactions in exchanges (see for example [22, 48]) and in the case of risk sharing under the use of market power is left as a topic of future research.

6. Conclusion

This article establishes a novel theoretical model which studies the effect of agents' market power in risk sharing transactions. In contrast to the large majority of related existing literature, it is supposed that agents apply certain type of strategic behavior in order to exploit their ability to influence the market equilibrium. Our analysis includes two market settings, the complete where agents design new securities, and the incomplete where they negotiate the transaction of a given vector of tradeable securities. In both settings, the risk sharing equilibrium transaction is an outcome of a pure strategy Nash game.

In the complete case, agents negotiate the designing of the risk sharing by proposing securities that are consistent with the optimal sharing rules. Since the sharing rules are functions of the submitted endowments, proposing sharing securities is equivalent to reporting specific random endowments in the sharing mechanism. It is proved that each agent has motive to report less risk exposure than her true one and a risky position proportional to the aggregate endowment reported by the other agents. It is furthermore shown that under M-V preferences, the Nash equilibrium does exist, is unique and in any non-trivial case is different than the optimal risk sharing transaction. Although the Nash game implies a loss in the aggregate utility, for the sufficiently low risk averse agents the gain of utility is higher at Nash equilibrium than in the Pareto optimal one. Therefore, when the market is small, agents with high risk aversion have motive to avoid sharing their risks with low risk averse agents, since not only they will bear all the market inefficiency, but they will also pay the extra utility gain of the low risk averse counterparties.

The situation is similar in the incomplete market setting, where agents negotiate the price and the allocation of a given vector of securities. Each agent optimally responds to the other agents' orders by submitting the appropriate demand function that clears out the market at the price that maximizes her utility. The fixed point of this negotiation is defined as the Nash price/allocation of the given securities. In principle, this market structure can be considered as an oligopolistic version of the CAPM. Again, the existence and uniqueness of the Nash equilibrium is proved to hold and a clear connection between the complete and incomplete market setting is established. Namely, submitted demands at the Nash equilibrium are the demand functions that are induced when agents' endowments are the Nash equilibrium ones. Moreover, the Nash and the competitive prices coincide if and only if all participating agents have the same risk aversion (even in this case however the volume is lower and the allocation is inefficient). As in the complete case, when agents are not homogeneous, the price impact caused by the Nash game is beneficial for relatively low risk averse agents.

Summarizing both settings, we point out that when the sharing environment is oligopolistic, the considered endowments should be the Nash equilibrium endowments and not the true ones. This implies (among other things) that the covariance part of CAPM changes when a non-competitive market consists of agents with different risk aversions. This price's change always benefit the relatively low risk averse agents who get more utility in any transaction that admits the use of market power.

The last part of the paper exploits the analytic findings of the previous sections and examines how the market inefficiency can be reduced. Besides the enlarged participation and the market completion, efficiency is also increased if the agents' strategic sets are restricted to the proportion of their true endowments or when risk aversion of each participating agent is getting higher.

Appendix A

Proof of Proposition 2.4. Thanks to cash invariant property of the M-V preferences and the assumed zero supply, the optimal sharing securities are the ones that that maximizes the sum of agents utility (see among others [1, 29]). We have that for every $\mathbf{C}=(C_1,...,C_n)\in\mathcal{A}$

$$\sum_{i=1}^{n}\mathbb{U}_i(\mathcal{E}_i+C_i)=\sum_{i=1}^{n}\mathbb{E}[\mathcal{E}_i]-\sum_{i=1}^{n}\gamma_i\,\mathbb{V}\mathrm{ar}[C_i+\mathcal{E}_i].$$

Hence, it is enough to find $\mathbf{C}^o\in\left(\mathbb{L}^2\right)^n$ that minimizes the sum $\sum\limits_{i=1}^{n}\gamma_i\,\mathbb{V}\mathrm{ar}[C_i+\mathcal{E}_i]$. Note the for each $i\in\{1,...,n\}$,

$$C_i^o+\mathcal{E}_i=\frac{\gamma}{\gamma_i}\mathcal{E},$$

where C_i^o is defined as the payoff $\mathbf{a}_i^o\cdot\mathbf{E}$, and $\gamma_i\,\mathbb{V}\mathrm{ar}[C_i^o+\mathcal{E}_i]=\frac{\gamma^2}{\gamma_i}\,\mathbb{V}\mathrm{ar}[\mathcal{E}]$, where we recall that $\mathcal{E}=\sum_{i=1}^{n}\mathcal{E}_i$. Therefore,

$$\sum_{i=1}^{n}\gamma_i\,\mathbb{V}\mathrm{ar}[C_i^o+\mathcal{E}_i]=\sum_{i=1}^{n}\frac{\gamma^2}{\gamma_i}\,\mathbb{V}\mathrm{ar}[\mathcal{E}_i]=\gamma\,\mathbb{V}\mathrm{ar}[\mathcal{E}].$$

It is then enough to show that $\sum\limits_{i=1}^{n}\gamma_i\,\mathbb{V}\mathrm{ar}[C_i+\mathcal{E}_i]\geq\gamma\,\mathbb{V}\mathrm{ar}[\mathcal{E}]$, for all $\mathbf{C}\in\mathcal{A}$, or equivalently

$$\sum_{i=1}^{n}\gamma\gamma_i\,\mathbb{V}\mathrm{ar}[C_i+\mathcal{E}_i]\geq\gamma^2\,\mathbb{V}\mathrm{ar}[\mathcal{E}],\quad\text{for all }\mathbf{C}\in\mathcal{A}.\tag{A.1}$$

The left hand side of (A.1) equals to

$$\sum_{i=1}^{n}\frac{\gamma}{\gamma_i}\gamma_i^2\,\mathbb{V}\mathrm{ar}[C_i+\mathcal{E}_i]=\sum_{i=1}^{n}\frac{\gamma}{\gamma_i}\,\mathbb{V}\mathrm{ar}[\gamma_i(C_i+\mathcal{E}_i)]$$

and inequality (A.1) follows by the convexity of $\mathbb{V}\mathrm{ar}[\cdot]$ and the fact that $\sum_{i=1}^{n}C_i=0$.

The fact that the equilibrium price of endowments are given by $\pi^*=\mathbb{E}[\mathbf{E}]-2\gamma\mathbf{1}_n\cdot\mathbb{V}\mathrm{ar}[\mathbf{E}]$ follows from Proposition 2.2.

Proof of Proposition 3.1. We fix vector $(\mathcal{E}_1, ..., \mathcal{E}_{i-1}, \mathcal{E}_{i+1}, ..., \mathcal{E}_n) \in (\mathbb{L}^2)^{n-1}$ and from (3.3) (with a slight abuse of notation) we get that

$$\begin{aligned} G_i(B) &= \mathbb{E}[\mathcal{E}_i] - \frac{\gamma^2}{\gamma_i}\mathbb{V}\mathrm{ar}[\mathcal{E}] - \frac{(\gamma_i-\gamma)^2}{\gamma_i}\mathbb{V}\mathrm{ar}[\mathcal{E}_i - B] - \frac{2\gamma(\gamma_i-\gamma)}{\gamma_i}\left(\mathbb{C}\mathrm{ov}(\mathcal{E}, \mathcal{E}_i - B) + \mathbb{V}\mathrm{ar}[B]\right) \\ &+ \frac{2\gamma(2\gamma-\gamma_i)}{\gamma_i}\mathbb{C}\mathrm{ov}(B, \mathcal{E} - \mathcal{E}_i). \end{aligned}$$

Hence, in order to solve problem (3.4), it is enough to find $B \in \mathbb{L}^2$ that minimizes the quantity

$$\frac{\gamma_i-\gamma}{2\gamma}\mathbb{V}\mathrm{ar}[\mathcal{E}_i - B] + \mathbb{C}\mathrm{ov}(\mathcal{E}, \mathcal{E}_i - B) + \mathbb{V}\mathrm{ar}[B] + \mathbb{C}\mathrm{ov}(B, \mathcal{E} - \mathcal{E}_i)\left(1 - \frac{\gamma}{\gamma_i - \gamma}\right).$$

Since the above quantity is constant invariant, we may consider for now that $\mathbb{E}[B] = 0$. Thus, we have to minimize function $W_i : \mathbb{L}^2 \to \mathbb{R}$ defined by

$$W_i(B) = \mathbb{E}\left[\frac{\gamma_i-\gamma}{2\gamma}(\mathcal{E}_i - B)^2 + \mathcal{E}(\mathcal{E}_i - B) + B^2 + \left(1 - \frac{\gamma}{\gamma_i-\gamma}\right)B(\mathcal{E} - \mathcal{E}_i)\right].$$

The minimum of $W_i(\cdot)$ can be obtained through its *Fréchet derivative*. For this, we first show that the Fréchet derivative of $W_i(\cdot)$ at $B \in \mathbb{L}^2$ (denoted by $D_{W_i(B)}$) is given by

$$D_{W_i(B)}[X] = \mathbb{E}\left[\left(\frac{\gamma_i+\gamma}{\gamma}B - \frac{\gamma_i-\gamma}{\gamma}\mathcal{E}_i - \mathcal{E} + \left(1 - \frac{\gamma}{\gamma_i-\gamma}\right)(\mathcal{E} - \mathcal{E}_i)\right)X\right],$$

for every $X \in \mathbb{L}^2$. In order to prove this statement, it is enough to get the following limit

$$\frac{|W_i(B+X) - W_i(B) - D_{W_i(B)}[X]|}{||X||_{\mathbb{L}^2(\mathcal{F})}} \longrightarrow 0 \tag{A.2}$$

as $||X||_{\mathbb{L}^2} \to 0$. Indeed,

$$\frac{|W_i(B+X) - W_i(B) - D_{W_i(B)}[X]|}{||X||_{\mathbb{L}^2}} = \left(\frac{\gamma_i+\gamma}{2\gamma}\right)\frac{|\mathbb{E}\left[X^2\right]|}{||X||_{\mathbb{L}^2}} = \left(\frac{\gamma_i+\gamma}{2\gamma}\right)||X||_{\mathbb{L}^2} \longrightarrow 0.$$

Note that function $W_i(\cdot)$ is strictly convex in $\mathbb{L}^2$ and hence, if there exists $B^* \in \mathbb{L}^2$ such that $D_{W_i(B^*)}[X] = 0$ for each $X \in \mathbb{L}^2$, then B^* is the unique (in the class of $\mathbb{L}^2$ with expectation equal to zero) minimizer of $W_i(\cdot)$. Clearly, for B_i^* given in (3.5), $D_{W_i(B_i^*)}[X] = 0$ for every $X \in \mathbb{L}^2$ which makes B_i^* the best response of agent i.

Proof of Theorem 4.2. From Proposition 3.1 and the FOCs for the Nash equilibrium, we get that Nash equilibria are the solutions $(\hat{B}_1, ..., \hat{B}_n) \in (\mathbb{L}^2)^n$ of the following system of n linear equations

$$\begin{aligned} \hat{B}_1 &= \frac{\gamma_1}{\gamma_1+\gamma}\mathcal{E}_1 + \frac{\gamma^2}{\gamma_1^2-\gamma^2}\sum_{j\neq 1}^{n}\hat{B}_j \\ \hat{B}_2 &= \frac{\gamma_2}{\gamma_2+\gamma}\mathcal{E}_2 + \frac{\gamma^2}{\gamma_2^2-\gamma^2}\sum_{j\neq 2}^{n}\hat{B}_j \\ &\vdots \\ \hat{B}_n &= \frac{\gamma_n}{\gamma_n+\gamma}\mathcal{E}_n + \frac{\gamma^2}{\gamma_n^2-\gamma^2}\sum_{j=1}^{n-1}\hat{B}_j \end{aligned} \tag{A.3}$$

We fix $\omega \in \Omega$ and we are looking for $\mathbf{B} = (B_1, ..., B_n)$ which solves the above linear system. For each $i \in \{1, ..., n\}$, we have that $\hat{B}_i = \frac{\gamma_i}{\gamma_i+\gamma}\mathcal{E}_i + \frac{\gamma^2}{\gamma_i^2-\gamma^2}\sum_{j=1}^{n}\hat{B}_j - \frac{\gamma^2}{\gamma_i^2-\gamma^2}\hat{B}_i$ which gives (4.4) and (4.5). Since $\hat{B}_i(\omega)$ is a linear combination of $\mathcal{E}_1(\omega), ..., \mathcal{E}_n(\omega)$, $\hat{B}_i : \Omega \longrightarrow \mathbb{R}$ is an $\mathcal{F}$-measurable and in particular it belongs in $\mathbb{L}^2$.

Finally, it is left to observe that if $\hat{\mathcal{B}} = (\hat{B}_1, ..., \hat{B}_n)$ is a Nash equilibrium, every vector of the form $\hat{\mathcal{B}} + \mathbf{c}$, for $\mathbf{c} \in \mathbb{R}^n$ is also a Nash equilibrium, since it satisfies (4.2). The fact that this form of Nash equilibria is unique follows from the strict convexity of $G_i(\cdot)$. Equation (4.6) follows by straightforward calculations.

Proof of Theorem 4.9. Taking Proposition 3.3 into account, we conclude that the market equilibrates when the covariances $\mathbb{C}\text{ov}(\mathbf{C}, \hat{B}_i)$ equilibrate. As we have seen in Theorem 4.2, this may happen for endowments $\hat{B}_i$ given in (4.4), which gives price (4.11). The uniqueness of the Nash equilibrium price follows by the Standing Assumption.

The equivalence of Nash equilibrium price and the Pareto equilibrium price when agents have common risk aversion is then induced by Corollary 4.5 (see also equation (2.6)). Finally, (4.13) follows by Proposition 2.2.

Proof of Proposition 5.1. We assume without loss of generality that $\mathbb{E}[\mathcal{E}_i] = 0$ for each $i \in \mathbb{N}$. The uniform integrability assumption guarantees that there exists a positive constant M, such that $||\mathcal{E}_i||_{\mathbb{L}^2} \leq M$, for all $i \in \mathbb{N}$. It is enough to show that the first term in (4.7), i.e., the sum $\sum_{i=1}^{n} \gamma_i \, \mathbb{V}\text{ar}[\mathcal{E}_i - \hat{B}_i]$ vanishes, as n goes to infinity. We fix $n \in \mathbb{N}$ and for an arbitrarily chosen $i \in \{1, ..., n\}$ we have

$$||\mathcal{E}_i - \hat{B}_i||^2 = \left(\frac{\gamma_{-i}}{\gamma_i+\gamma_{-i}}\right)^2 ||\mathcal{E}_i||^2 + \left(\frac{\gamma_{-i}}{\gamma_i+\gamma_{-i}}\right)^4 ||\hat{\mathcal{B}}(n)|| - 2\left(\frac{\gamma_{-i}}{\gamma_i+\gamma_{-i}}\right)^3 \langle \mathcal{E}_i, \hat{\mathcal{B}}(n)\rangle, \tag{A.4}$$

where $(\gamma_{-i})^{-1} = \sum_{j\neq i}^{n} \frac{1}{\gamma_j}$, $\hat{\mathcal{B}}(n)$ is the aggregated reported endowment of the n first agents, all mentioned norms are in $\mathbb{L}^2$ and $\langle \cdot, \cdot \rangle$ is the associated inner product.

$$\begin{aligned}\sum_{i=1}^{n} \gamma_i \left(\frac{\gamma_{-i}}{\gamma_i+\gamma_{-i}}\right)^2 ||\mathcal{E}_i||^2 &\leq Mc_u \sum_{i=1}^{n}\left(\frac{\gamma_{-i}}{\gamma_i+\gamma_{-i}}\right)^2 \\ &\leq \frac{Mc_u^3}{(n-1)^2}\sum_{i=1}^{n}\frac{1}{(\gamma_i+\gamma_{-i})^2} \\ &\leq \frac{Mc_u^3}{(n-1)^2}\frac{n(n-1)^2}{((n-1)c_l+c_u)^2}\end{aligned}$$

which goes to zero as $n \to \infty$. Also,

$$\begin{aligned}||\hat{\mathcal{B}}(n)||^2 &= \frac{1}{1-\sum_{i=1}^{n}\left(\frac{\gamma(n)}{\gamma_i}\right)^2}\left(||\mathcal{E}(n)||^2 + \gamma(n)^2\left|\left|\sum_{i=1}^{n}\frac{\mathcal{E}_i}{\gamma_i}\right|\right|^2 - 2\gamma(n)\left\langle \mathcal{E}(n), \sum_{i=1}^{n}\frac{\mathcal{E}_i}{\gamma_i}\right\rangle\right) \\ &\leq \frac{1}{1-\frac{c_l^2}{nc_u^2}}\left(n^2M + \frac{c_u^2}{n^2}\left|\left|\sum_{i=1}^{n}\frac{\mathcal{E}_i}{\gamma_i}\right|\right|^2 + 2\frac{c_u}{n}\left|\left\langle \mathcal{E}(n), \sum_{i=1}^{n}\frac{\mathcal{E}_i}{\gamma_i}\right\rangle\right|\right)\end{aligned}$$

where $\gamma(n)$ and $\mathcal{E}(n)$ stands for the aggregate risk aversion and endowment of the n first agents. But, $\left|\left|\sum_{i=1}^{n}\frac{\mathcal{E}_i}{\gamma_i}\right|\right|^2 \leq \frac{1}{c_l}||\sum_{i=1}^{n}\mathcal{E}_i||^2 \leq \frac{n^2M}{c_l}$ and

$$\left|\left\langle \mathcal{E}(n), \sum_{i=1}^{n}\frac{\mathcal{E}_i}{\gamma_i}\right\rangle\right| \leq \frac{1}{c_l}\sum_{i=1}^{n}|\langle \mathcal{E}(n), \mathcal{E}_i\rangle| \leq \frac{1}{c_l}\sum_{i=1}^{n}||\mathcal{E}(n)||||\mathcal{E}_i|| \leq \frac{n^2M}{c_l}.$$

Therefore, we have the following estimation for the aggregated reported endowment of the n first agents

$$||\hat{\mathcal{B}}(n)||^2 \leq \frac{M}{c}\left(\frac{cn^2+C^2+2nC}{1-\frac{c^2}{nC^2}}\right) \tag{A.5}$$

Also, $\sum_{i=1}^{n}\gamma_i\left(\frac{\gamma_{-i}}{\gamma_i+\gamma_{-i}}\right)^4 \leq \frac{c_u^5}{c_l^4n^3}$, which together with (A.5) implies that

$$\sum_{i=1}^{n}\gamma_i\left(\frac{\gamma_{-i}}{\gamma_i+\gamma_{-i}}\right)^4||\hat{\mathcal{B}}(n)||^2 \leq \frac{Mc_u^5}{c_l^5}\left(\frac{c_ln^2+c_u^2+2nc_u}{n^3-\frac{n^2c_l^2}{c_u^2}}\right) \to 0 \tag{A.6}$$

as $n \to \infty$. Continuing with the terms in (A.4), we have

$$\begin{aligned}\sum_{i=1}^{n}\gamma_i\left(\frac{\gamma_{-i}}{\gamma_i+\gamma_{-i}}\right)^3\langle \mathcal{E}_i, \hat{\mathcal{B}}(n)\rangle &\leq \frac{c_u^4}{c_l^3n^3}||\hat{\mathcal{B}}(n)||\sum_{i=1}^{n}||\mathcal{E}_i|| \\ &\leq \frac{c_u^4}{c_l^3n^2}\sqrt{M}||\hat{\mathcal{B}}(n)|| \\ &\leq \frac{c_u^4}{c_l^{7/2}n^2}M\sqrt{\frac{c_ln^2+c_u^2+2nc_u}{1-\frac{c_l^2}{nc_u^2}}} \to 0 \qquad \text{(A.7)}\end{aligned}$$

as $n \to \infty$. From (A.4), (A.6) and (A.7) we get that the first term of (4.7) goes to zero as the number of agents approaches infinity.

Proof of Proposition 5.2. We assume as usual that expectations are equal to zero and from (2.6) and (4.11) we have that

$$||\mathbf{p}^*(n) - \hat{\mathbf{p}}(n)|| = 2\gamma(n)||\,\mathbb{C}\mathrm{ov}(\mathbf{C}, \hat{\mathcal{B}}(n) - \mathcal{E}(n))|| \leq 2\gamma(n)||\hat{\mathcal{B}}(n) - \mathcal{E}(n)||\sqrt{\sum_{j=1}^{k} ||C_j||^2},$$

where the notation is the one introduced in the proof of Proposition 5.1. Therefore, it is enough to show that $\gamma(n)||\hat{\mathcal{B}}(n) - \mathcal{E}(n)||$ vanishes as $n \to \infty$. For sufficient large n, we have

$$\begin{aligned}
\gamma(n)||\hat{\mathcal{B}}(n) - \mathcal{E}(n)|| &= \frac{\gamma^2(n)}{1 - \sum_{i=1}^n \left(\frac{\gamma(n)}{\gamma_i}\right)^2} \left|\left|\sum_{i=1}^{n} \frac{\gamma_i \mathcal{E}_i - \gamma(n)\mathcal{E}(n)}{\gamma_i^2}\right|\right| \\
&\leq \frac{\gamma^2(n)}{c_l^2 - \frac{c_u^2}{n}} \left|\left| n\gamma(n)\mathcal{E}(n) - \sum_{i=1}^{n} \gamma_i \mathcal{E}_i \right|\right| \leq \frac{c_u^2}{n^2 c_l^2 - n c_u^2} \left(n\gamma(n)\,||\mathcal{E}(n)|| + \left|\left| \sum_{i=1}^{n} \gamma_i \mathcal{E}_i \right|\right| \right) \\
&\leq \frac{\sqrt{M} c_u^2}{n^2 c_l^2 - n c_u^2} \left(n^2 \gamma(n) + n c_u \right) \leq 2 \frac{n\sqrt{M} c_u^3}{n^2 c_l^2 - n c_u^2} \to 0
\end{aligned}$$

as n goes to ∞. For item (ii), we fix an agent i and from (2.7) and (4.13), it is enough to show that $||\,\mathbb{C}\mathrm{ov}(\mathbf{C}, C_i^o(\hat{B}_i) - C_i^o)||$ goes to zero as $n \to \infty$. Following the steps above it suffices to show that $||C_i^o(\hat{B}_i) - C_i^o|| \to 0$.

$$\begin{aligned}
||C_i^o(\hat{B}_i) - C_i^o|| &= \frac{\gamma(n)}{\gamma_i} \left|\left| \mathcal{E}_i + \frac{\gamma_i - \gamma(n)}{\gamma_i} \hat{\mathcal{B}}(n) - \mathcal{E}(n) \right|\right| \\
&\leq \frac{\gamma(n)}{\gamma_i} ||\mathcal{E}_i|| + \frac{\gamma(n)}{\gamma_i} ||\hat{\mathcal{B}}(n) - \mathcal{E}(n)|| + \frac{\gamma^2(n)}{\gamma_i^2} ||\hat{\mathcal{B}}(n)||
\end{aligned}$$

We have seen in the proof of item (i) that $\gamma(n)||\hat{\mathcal{B}}(n) - \mathcal{E}(n)|| \to 0$. Also, since $\gamma(n) \leq \frac{c_u}{n}$, $\frac{\gamma(n)}{\gamma_i}||\mathcal{E}_i|| \to 0$. Finally, taking into account (A.5), we get that $\gamma^2(n)||\hat{\mathcal{B}}(n)|| \to 0$, which completes the proof.

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