On the Existence of Nash Equilibrium in General Imperfectly Competitive Insurance Markets with Asymmetric Information

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Abstract

We model general imperfectly competitive insurance markets with multidimensional heterogeneity. We formally characterise competition in menus of insurance contracts among insurance companies. We define the set of *Least Costly Incentive Compatible* (LCIC) allocations and we show that, in every insurance environment, if the LCIC payoff profile is strongly Pareto optimal, an equilibrium exists. We then provide primitive sufficient conditions such that the equilibrium set is empty. In general, equilibrium need not exist in environments in which companies are undifferentiated Bertrand competitors, two commonly-imposed conditions are satisfied, monotonicity and sorting, and the the LCIC payoff profile is not strictly Pareto optimal.

KEYWORDS: Insurance Markets, Competition, Asymmetric Information, Nash Equilibrium, Existence.

I. Introduction

■ *Motivation.* The question of (non) existence of equilibrium in competitive insurance markets with asymmetric information has attracted considerable attention in the literature ever since the seminal contributions of Rothschild and Stiglitz (1976) (henceforthe RS) and Wilson (1977).[1] Most studies have focused on the canonical insurance paradigm of RSW, either trying to establish existence by departing from the definition of Nash equilibrium, or by employing Nash equilibrium but changing the way insurance companies compete with each other. In particular, very little effort has been devoted towards

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[1] We sometimes use the acronym RSW for Rothschild-Stiglitz-Wilson. Rothschild and Stiglitz (1976) examined a model of competition in insurance markets where companies supply a unique insurance contract to the market and highlighted the difficulties in establishing existence. Wilson (1977) confronted the same existence problem and proposed a new definition of equilibrium. See also Discussion at the end of this paper.

analysing existence in general insurance environments that go beyond the canonical insurance paradigm. More recently, theoretical and empirical work has attempted to identify selection in insurance markets and estimate the welfare cost.

The motivation of this paper is twofold. First, it provides a formal model of a general imperfectly competitive insurance markets with multidimensional heterogeneity. Second, it rigorously studies existence of Nash equilibrium is such general markets. With regards to the former, competition is defined in menus of insurance contracts among a finite number of insurance companies. We define a menu simply as a finite set of insurance contracts. No restrictions are placed in the type of insurance contracts companies can offer. In general we believe that insurance markets are highly concentrated with barriers of entry resulting from high capital requirements. Consumers purchase an insurance contract from one insurance company at a time (exclusivity). The main idea is to characterise primitive conditions such that equilibrium exists (or does not). With regards to the latter, our first result (Proposition 1) is a generalisation of the existence result of RS. In particular, in RS an equilibrium exists if and only if the *Least Costly Incentive Compatible* (LCIC) allocation is strongly Pareto optimal (SPO). We first formally define the set of LCIC allocations in our general insurance environment and we show that when the LCIC payoff profile belongs to the set of the SPO payoff profiles then an equilibrium indeed exists. Next, we provide primitive sufficient conditions for the non-existence of equilibrium. It turns out (Proposition 2) that in environments in which companies must earn zero profits in equilibrium, an equilibrium exists does not exist if two mainly-imposed assumptions, monotonicity and sorting (Assumption C and D) and the LCIC payoff profile is SPO. This sheds more light on what conditions are responsible for the striking non-existence result of RS.

□ *Related Literature.* Apart from RSW, several authors have examined the canonical insurance paradigm. ???To begin with, note that neither RS nor Wilson specify an explicit game among insurance companies but rather provide properties that a competitive equilibrium needs to satisfy. Moreover, the analysis of RS lacks formal and rigorous arguments, that are so prevalent in modern economic theory, and it is mostly based on graphical arguments. The definition of equilibrium however is in accordance with Nash equilibrium. Each company takes the actions of the rival companies as given when selecting what contract to offer. In Wilson on the contrary, insurance companies are able to predict what would happen if other companies were going bankrupt. Based on that (non-myopic) definition, an equilibrium exists. The equilibrium definition proposed by Wilson is known as reactive equilibrium (or E2 equilibrium). ????

Miyazaki (1977) extended the reactive equilibrium (E2 Equilibrium) of Wilson by letting companies offer menus of insurance policies and showed that a reactive equilibrium exists and is constrained Pareto optimal.[2] Riley (1979) also analysed a similar market with asymmetric information and showed that an equilibrium exists if insurance companies are able to add contracts. A serious drawback in Wilson, Riley and Miyazaki is that they did not examine games but rather defined properties of competitive equilibria

[2]The menu of contracts (or allocation) that results as a unique equilibrium in Miyazaki is also known as Miyazaki-Wilson (MW) or sometimes Spence-Miyazaki-Wilson, after Spence (1978)[13].

and examined insurance policies that might satisfy these properties in the canonical insurance market. Standard textbook treatments, such as Mas Colell et al. (1995) and Jehle and Reny (2011) , have examined competition in the RSW economy and confirmed the results of RS. Rosenthal and Weiss (1984), and Dasgupta and Maskin (1984a,b) . Hellwig (1987) and Engers and Fernadez (1987) specified games for the models of Wilson [14] that considerably diverted from Bertrand-type competition.

To the best of our knowledge the only closely related study to our paper is that by Rustichini and Siconolfi (2008). They analyse a walrasian market where agents are price takers in a standard perfectly competitive market and they trade lotteries over net trades.They first examine pure exchange competitive equilibria in which there are no intermediary (insurance) firms. They prove the existence of equilibria but . Then they introduce a constant- returns- to- scale intermediary firm that acts as a pool of risk. In this economy they reach similar sufficient conditions as in our paper and they show that in RSW economies, equilibria may not exist. In our paper, we study Bertrand insurance markets. We also examine in detail broader classes of environments for which the results of RSW apply and we identify those attributes that are responsible for the non- existence of equilibrium.

The rest of our paper is organised as follows: In Section II, we describe the general insurance market, we analytically characterise competition in menus of insurance contracts and we state primitive assumptions. In Section III, we characterise the set of LCIC allocations and we show that when the LCIC payoff profile is SPO, then an equilibrium exits. Furthermore, we show that in economies that satisfy two commonly- imposed assumptions, monotonicity and sorting, the set and companies must earn zero profits in equilibrium, an equilibrium exists if and only if the LCIC payoff profile is SPO.

II. The General Insurance Market

■ *Consumers and Insurance Companies.* There is a set of consumers. Each consumer belongs to a certain class (type). We assume that the set of types is finite $\theta = 1, ..., N$. The restriction to finite sets is not essential but to avoid measurability problems that will unnecessarily complicate the analysis. All our results extend, under technical difficulties, to uncountably infinite sets. The type of a consumer may include any non-observable characteristic such as riskiness, attitude towards risk, income etc, or any observable characteristic that cannot be used from insurance companies for discrimination such as gender, income, race etc. Hence, the environment we study is one of multi-dimensional heterogeneity, as opposed to most studies in insurance markets where heterogeneity concerns only one characteristic. The measure of type θ consumers in the population is $\lambda^\theta < \infty$.

We assume that there are N active insurance companies in the market $i = 1, ..., I$. Insurance companies are pools of risk, (expected) profit maximisers.

□ *Insurance Contracts.* Consumers and insurance companies sign insurance contracts. A contract is denoted by x and the space of available contracts by X. Type θ has utility function that is type specific denoted as $u^\theta : X \to \mathbb{R}$. The status quo utility of type θ is $\underline{u}^\theta$ and to allow each type to have access to his status quo, we denote the "no-insurance" contract as x_0. The net cost function of company i is type dependent and denoted as

$c_i^\theta : X \times \mathbb{R}_+ \to \mathbb{R}$. Note that we assume that the contract space is common among insurance companies which is rather realistic as an assumption. For instance, any company is able to design any insurance contract if the relevant insurance contingencies are commonly known. We also assume that, the same contract leads the type θ regardless of which company has been provided. In other words, insurance companies are undifferentiated Bertrand competitors in the product market.[3] Nonetheless, a contract may have a different cost depending on which company has provided it. The cost function may have any returns to scale. An insurance environment is $< \Theta, (\lambda^\theta)_{\theta\in\Theta}, X, (u^\theta)_\theta, (c_i^\theta)_{\theta,i} >$.

An allocation $\mathbf{x} = (x^\theta, i)_{\theta\in\Theta} \in \times_{\theta=1}^N \times_{i=1}^I X$ is a set of contracts indexed by the set of types. An allocation $\mathbf{x} = (x^\theta)_{\theta\in\Theta}$ is incentive compatible if and only if for all $\theta, \theta' \in \Theta$, $u^\theta(x^\theta) \geq u^\theta(x^{\theta'})$. The set of incentive compatible allocations is denoted as $\boldsymbol{X}^{IC}$. The (net) cost of allocation $\mathbf{x}$ is given by $\xi(\mathbf{x}) = \sum_{\theta\in\Theta} \lambda^\theta c^\theta(x^\theta)$. An allocation $\mathbf{x}$ is a strong Pareto optimum (SPO) if: (i) it is incentive compatible, (ii) individually rational, and, (iii) there exists no other incentive compatible and non-negative profit allocation $\tilde{\mathbf{x}}$ such that for each $\theta \in \Theta$, $u^\theta(\tilde{x}^\theta) \geq u^\theta(x^\theta)$ with the inequality being strict for at least one $\theta \in \Theta$. An allocation is a weak Pareto optimum (WPO) if we take all inequalities to be strict. We denote the set of SPO and WPO allocations of economy E as $\boldsymbol{X}^{SPO}$ and $\boldsymbol{X}^{WPO}$ respectively. A utility profile is $\mathbf{u} = (u^\theta)_{\theta\in\Theta}$, where $u^\theta = u^\theta(x^\theta)$. The set of utility profiles from all SPO and WPO allocations is denoted as $\mathcal{U}^{SPO}$ and $\mathcal{U}^{WPO}$ respectively.

□ *Competition.* Let σ_i denote the menu of insurance contracts offered by company $i = -1, 1$. A menu of insurance contracts is simply an (unordered) finite set of insurance contracts. We denote the set of all possible menus of insurance contracts as Σ. We assume that $x_0 \in \sigma_i$ for every $\sigma_i \in \Sigma$. Company $i = -1, 1$ has the right not to enter the market if it wishes. We denote this decision as $\underline{\sigma}$. A strategy profile is denoted as $\boldsymbol{\sigma} = (\sigma_{-1}, \sigma_1)$.

The set of all payoff maximising contracts for type θ when the action profile is $\boldsymbol{\sigma}$ is denoted as $\Xi^\theta(\boldsymbol{\sigma})$. We also denote as $\boldsymbol{u}(\boldsymbol{\sigma}) = \left(u^\theta(\boldsymbol{\sigma})\right)_{\theta\in\Theta}$ the payoff - maximising payoff profile when the action profile is $\boldsymbol{\sigma}$.[4] Consumers select those policies that are payoff-maximising among all the available contracts in the market. Note that given the generality of the environment, consumers that belong to the same class may purchase a different policy offered from the same or a different insurance company. Evidently, all consumers of the same class will have the same payoff both on and off the equilibrium path. This is why we specify an equilibrium payoff profile as a function of the menus of contracts that are available in the market. We consider only exclusive contracts, therefore a consumer selects a unique contract from a unique insurance company, after having observed all contracts that are available in the market.

For each company, a set of demand functions consists of functions that map each of its contracts to a positive real number. For company i a set of demand functions is $d_i : \sigma_i \to \mathbb{R}_+$, for every $\sigma_i \in \Sigma$. A demand profile is $\boldsymbol{d} = (d_{-1}, d_1)$. Demands are determined

[3] This may be a little misleading because insurance companies may provide different insurance contracts and hence different products. What we mean is that when a contract has been supplied by all companies, consumers are indifferent what company to choose. In other words, the "brand" is irrelevant for every consumer as long as the contract terms are the same.

[4] Formally, $\Xi^\theta(\boldsymbol{\sigma}) = \underset{x\in\sigma_{-1}\cup\sigma_1}{\arg\max}\; u^\theta(x)$ and $U^\theta(\boldsymbol{\sigma}) = \underset{x\in\sigma_{-1}\cup\sigma_1}{\max}\; u^\theta(x)$.

endogenously in equilibrium along with the actions of insurance companies. Given a set of demand functions, the expected profit of company i by offering σ_i when company $-i$ offers σ_{-i} is

$$\Pi_i[\sigma_i, \sigma_{-i}; d_i] = \sum_{\theta \in \Theta} \sum_{x \in \sigma_i} \mathbb{1}_{\{\sigma_i \cap \Xi^\theta(\boldsymbol{\sigma})\}}(x) d_i(x) c^\theta(x)$$

where $\mathbb{1}_{\{\sigma_i \cap \Xi^\theta(\boldsymbol{\sigma})\}}(x)$ is the indicator function. We assume that $\Pi_i(\underline{\sigma}, \sigma_{-i}; d_i) = 0$ for every σ_{-i} so that if company i decides not to enter the market its payoff is zero.

EQUILIBRIUM: A Nash equilibrium is a profile of actions and demands $(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{d}})$ such that:

1. For every $i = -1, 1$, $\hat{\sigma}_i \in \arg\max_{\sigma_i \in \Sigma} \quad \Pi_i[\sigma_i, \hat{\sigma}_{-i}; \hat{d}_i]$
2. For every $\boldsymbol{\sigma}$ and $\theta \in \Theta$, $\sum_{i=-1,1} \sum_{x \in \sigma_i \cap \Xi^\theta(\boldsymbol{\sigma})} \hat{d}_i(x) = \lambda^\theta$

According to this definition, an equilibrium consists of a profile of actions and demands such that: (1) The action of company i is optimal given the action of company $-i$ and the demand function of company i, (2) For every possible action profile, the sum of the demands for the payoff maximising contracts for type θ equals the ex ante measure of type λ^θ. Condition 2 is a simple market-clearing condition. The set of Nash equilibria is denoted as $\mathcal{Q}^B$.

□ *Assumptions.* Our aim is to study the existence and properties of Nash equilibria in general insurance markets. We state four assumptions that play a key role in this study.

ASSUMPTION A: X is a compact subset of a topological, linear space and $u^\theta : X \to \mathbb{R}$, $c^\theta : X \to \mathbb{R}$ are continuous, linear functionals for every $\theta \in \Theta$.

It is perhaps surprising at first sight to assume that utility functions are linear functionals. Usually, it is assumed that utility functions are strictly concave and the cost function is linear. Nonetheless, incentive constraints create non-convexities that limit the effectiveness of competition. In particular, if utility functions are strictly concave, the requirement of incentive compatibility restricts the usual price-cutting analysis, in accordance with elementary Bertrand markets for a homogeneous product. This is why in studies of general insurance markets, e.g. Rustichini and Siconolfi (2008), Citanna and Siconolfi (2013), agents trade lotteries over net trades. The main purpose of this is to make the utility functions linear functionals on the space of traded products, i.e. contracts in our environment. We simplify, without much loss of generality, by assuming that utility and cost functions are linear functionals on a subset of a topological, linear space.[5]

The second assumption is supposed to make the environment more "competitive":[6]

[5]In our environment X is supposed to be a set of lotteries over some space of deterministic contracts.

[6]This statement is perhaps misleading. What we mean is that Assumption A alone is not sufficient to show that companies compete by undercutting prices in order to conquer the market as in a price competition. Assumption B along with Assumption A create such a competitive environment.

ASSUMPTION B: There exist $\bar{x} \in \cap_{\theta\in\Theta} \arg\max_{x\in X} u^{\theta}(x) \neq \emptyset$ such that $c^{\theta}(\bar{x}) < 0$ for every $\theta \in \Theta$ and $\underline{x} \in \cap_{\theta\in\Theta} \arg\min_{x\in X} u^{\theta}(x) \neq \emptyset$ such that $\sum_{\theta\in\Theta} \lambda^{\theta} c^{\theta}(\bar{x}) > 0$.

In words, Assumption B states that there exist a "maximum" and a "minimum" contract in the space of available contracts such that the former is the (weakly) most preferred contract by all types but makes strictly negative (expected) profits if taken by all types and the latter is the (weakly) least preferred contract by all types but makes strictly positive (expected) profits for all types. Assumption B is always satisfied in most insurance environments in which consumers are VNM expected utility maximisers and differ with respect to some characteristics. Assumption B is sufficient to guarantee that insurance companies earn zero profits in equilibrium in accordance to the marginal/ average cost equilibrium pricing rule in Bertrand markets for a homogeneous product.

Two commonly-imposed assumption follows that place a substantial structure in the space of types and utility functions are monotonicity and single crossing. Monotonicity can be formally stated as follows:

ASSUMPTION C: $\forall\, x \in X$, $c^1(x) \leq c^2(x) \leq ... \leq c^n(x)$.

ASSUMPTION D: $\forall\, x \in X$ and $\theta \in \Theta$, there exists $x' \in X$ such that $u^{\eta}(x') > u^{\eta}(x)\ \forall\ \eta \geq \theta$ and $u^{\eta}(x') < u^{\eta}(x)\ \forall\ \eta < \theta$.

In other words, Assumption C states that for every possible contract x, its expected profit (cost) can always be ordered in the same order of the set of types. The idea is that higher types are less costly to trade with than lower types.[7] According to Assumption D, for every possible policy x and type θ, there exists some policy x' such that all types that are higher in the rank than type θ strictly prefer policy x' over x, and those types that are lower in the rank than type θ strictly prefer x over x'. In that sense, types can always be "sorted" in a certain way.

III. Existence of Equilibrium

■ *LCIC Allocation and Existence.* As we have already explained, our main goal is to study the existence (or non existence) of pure strategy Nash equilibrium in general insurance markets. RS have shown under very restrictive conditions that an equilibrium may exist. We can now ask whether an equilibrium exists in our more general environment under similar circumstances.

To begin with, let us characterise the set of incentive compatible allocations with the following properties: Each insurance contract for each of the types has a non-negative cost and is payoff maximising. We call this set, the set of *Least Costly Incentive Compatible* (LCIC) allocations.[8] The significance of LCIC has also been identified and studied in other environments.[9] In order to recover this set consider the following recursive program:

[7] This assumption is so common that it is not explicitly stated most of the times.

[8] The set of LCIC allocations is sometimes called the set of Rothschild-Stiglitz-Wilson allocations because it is prevalent in the analysis of RSW.

[9] See for instance Maskin and Tirole (1992).

$$\Psi^0 = \left\{ (x^\theta)_{\theta\in\Theta} : u^\eta(x^\eta) \geq u^\eta(x^\iota) \;\; \forall \;\; \eta, \iota \in \Theta, c^\eta(x^\eta) \geq 0, u^\eta(x) \geq \underline{u}^\eta \;\; \forall \;\; \eta \in \Theta \right\}$$

For every $\theta = 1, ..., n$:

$$\Psi^\theta = \underset{(x^\theta)_{\theta\in\Theta}}{\arg\sup} \left\{ u^\theta(x) : (x^\theta)_{\theta\in\Theta} \in \Psi^{\theta-1} \right\}$$

The set of LCIC allocations is Ψ^n. The payoff for type θ from a LCIC allocation is denoted as $\mathring{u}^\theta$ and is unique even if the set of LCIC allocations is not a singleton. Therefore, we denote as $\mathring{\mathbf{u}} = (\mathring{u}^\theta)_{\theta\in\Theta}$ the payoff profile of any LCIC allocation.

The set of LCIC allocations can shed more light in the characterisation of the equilibrium set. In particular, it seems to be crucial whether the LCIC payoff profile belongs to the set of strictly Pareto optimal payoff profiles. The following general result formally establishes this relation:

PROPOSITION 1: If $\mathring{\mathbf{u}} \in \mathcal{U}^{SPO}$, then $\mathcal{Q}^B \neq \emptyset$.

In words, if the set of LCIC allocations is strongly Pareto optimal, then an equilibrium exists. Uniqueness is not guaranteed unless one imposes stricter conditions. Proposition 1 is a broad generalisation of the the result of RS with regards to the existence of equilibrium. Note that no assumptions are required to prove this result and therefore it is applicable to all insurance environments.

□ *Non-Existence.* Our next goal is to provide sufficient conditions for non existence of equilibrium, i.e. the "ground-breaking" result of RS. Proposition 1 is not very informative with regards to the non-emptiness of the equilibrium set when the LCIC payoff profiles is not strongly Pareto optimal.

We are now ready to state the following (non) existence result:

PROPOSITION 2: If Assumptions A, B, C and D are satisfied and $\mathring{\mathbf{u}} \notin \mathcal{U}^{SPO}$, then $\mathcal{Q}^B = \emptyset$.

Roughly speaking, this result reconfirms the findings of RS that in environments in which, when firms must earn zero profits because of competition, an equilibrium may fail to exist when the LCIC payoff profile is not SPO. Nonetheless, this can be only shown in environments that satisfy monotonicity and sorting. If these two conditions are not satisfied, we are agnostic with respect to the existence of equilibrium. The insurance environment of RSW satisfy these two conditions and hence it falls into Proposition 2.

V. Discussion

Appendix A: Formal Proofs

□ *Proof of Proposition 1.* Suppose that $\mathring{\mathbf{u}} \in \mathcal{U}^{SPO}$. Consider $(\mathring{x}^\theta)_{\theta\in\Theta} \in \cap_{\theta\in\Theta}\Psi^\theta$. If $\mathring{\mathbf{u}} \in \mathcal{U}^{SPO}$, then for every $(\vec{x}^\theta)_{\theta\in\Theta}$ that strictly Pareto dominates $(\mathring{x}^\theta)_{\theta\in\Theta}$, $\sum_{\theta\in\Theta} \lambda^\theta c^\theta(\vec{x}^\theta) < 0$.

Consider the action and demand profile $(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{d}})$, where $\hat{\sigma}_i = (\mathring{x}^\theta)_{\theta\in\Theta}$ and $\hat{d}_i(\mathring{x}^\theta) = \lambda^\theta/2$ for every $i = -1, 1$. We prove by contradiction that $(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{d}}) \in \mathcal{Q}^B$. Assume not. There exists i and $\tilde{\sigma}_i \neq (\mathring{x}^\theta)_{\theta\in\Theta}\Sigma$ such that

$$(1) \quad \Pi_i(\tilde{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i) > \Pi_i(\hat{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i) = \frac{1}{2}\sum_{\theta\in\Theta} \lambda^\theta c^\theta(\mathring{x}^\theta) \geq 0$$

with the the last inequality following directly from the definition of $\mathring{\mathbf{u}} \in \mathcal{U}^{SPO}$. Denote as $\tilde{x}^\theta \in \arg\max_{x\in\tilde{\sigma}_i}\ u^\theta(x)$. $\tilde{\sigma}_i \neq (\mathring{x}^\theta)_{\theta\in\Theta}$ means that there exists $\Phi \subseteq \Theta$ such that:

(i) $u^\theta(\tilde{x}^\theta) > \mathring{u}^\theta\ \forall\ \theta \in \Phi$

(ii) $\max_{x\in\tilde{\sigma}_i}\ u^\theta(x) \leq \mathring{u}^\theta\ \forall\ \theta \in \Theta - \Phi$

The allocation that consists of the union of the set of contracts $(\mathring{x}^\theta)_{\theta\in\Theta-\Phi}$ and $(\tilde{x}^\theta)_{\theta\in\Phi}$ is incentive compatible and has total cost $\sum_{\theta\in\Phi} \lambda^\theta c^\theta(\tilde{x}^\theta) + \sum_{\theta\in\Theta-\Phi} \lambda^\theta c^\theta(\mathring{x}^\theta)$. Note though that the first part is $\Pi_i(\tilde{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i)$ which is positive from (1) and the second part is also positive since $c^\theta(\mathring{x}^\theta) \geq 0$ by the definition of the LCIC allocations. Therefore $\sum_{\theta\in\Phi} \lambda^\theta c^\theta(\tilde{x}^\theta) + \sum_{\theta\in\Theta-\Phi} \lambda^\theta c^\theta(\mathring{x}^\theta) \geq 0$. Because of (i) and (ii), the new allocation weakly Pareto dominates allocation the LCIC allocation and has non-negative cost, contradicting that for every $(\vec{x}^\theta)_{\theta\in\Theta}$ that strictly Pareto dominates $(\mathring{x}^\theta)_{\theta\in\Theta}$, $\sum_{\theta\in\Theta} \lambda^\theta c^\theta(\vec{x}^\theta) < 0$. Q.E.D.

□ *Proof of Proposition 2.* We will assume throughout that Assumptions A, B, C and D are satisfied and we will prove the result through a series of lemmas. To begin with, note the following lemma.

LEMMA A.1: For every $\mathbf{x} \in \mathbf{X}^{WPO}$, $\sum_{\theta\in\Theta} \lambda^\theta c^\theta(x^\theta) = 0$.

PROOF: We prove the result by contraposition. Clearly, if $\mathbf{x} \in \mathbf{X}^{IC}$ is such that $\sum_{\theta\in\Theta} c^\theta(x^\theta) < 0$, then $\mathbf{x}$ does not satisfy the definition of WPO. We now show that $\mathbf{x} \in \mathbf{X}^{IC}$ where $\sum_{\theta\in\Theta} \lambda^\theta c^\theta(x^\theta) > 0$ does not satisfy the definition of WPO either. From Assumption B, there exists $\bar{x} \in \mathbf{X}^{IC}$ such that $u^\theta(\bar{x}) > u^\theta(x^\theta)$ and $\sum_{\theta\in\Theta} c^\theta(x^\theta) < 0$ for every $\theta \in \Theta$. From Assumption A, for every $0 < \alpha < 1$, there exists $\hat{x}^\theta = \alpha x^\theta + (1-\alpha)\bar{x} \in X$ such that $u^\theta(\hat{x}^\theta) > u^\theta(x^\theta)$ for every $\theta \in \Theta$. Furthermore, $u^\theta(\hat{x}^\theta) \geq u^\theta(\hat{x}^{\theta'})$ for every $\theta, \theta' \in \Theta$. The expected profit of $\hat{\mathbf{x}}$ is $\xi(\hat{\mathbf{x}}) = \sum_{\theta\in\Theta} \lambda^\theta c^\theta(\hat{x}^\theta)$ and because of Assumption A, $\sum_{\theta\in\Theta} \lambda^\theta c^\theta(\hat{x}^\theta) = \alpha \sum_{\theta\in\Theta} \lambda^\theta c^\theta(x^\theta) + (1-\alpha)\sum_{\theta\in\Theta} \lambda^\theta c^\theta(\bar{x})$. For $1 > \alpha > -\frac{\sum_{\theta\in\Theta} \lambda^\theta c^\theta(\bar{x})}{\sum_{\theta\in\Theta} \lambda^\theta [c^\theta(x^\theta) - c^\theta(\bar{x})]}$, $\sum_{\theta\in\Theta} \lambda^\theta c^\theta(\hat{x}^\theta) > 0$ and hence $\mathbf{x}$ does not satisfy the definition of WPO. Q.E.D.

A further necessary condition for any Nash equilibrium follows.

LEMMA A.2: If $(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{d}}) \in \mathcal{Q}^B$, then $\boldsymbol{u}(\hat{\boldsymbol{\sigma}}) \in \mathcal{U}^{WPO}$.

PROOF: We will prove the result by contradiction. Suppose $(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{d}}) \in \mathcal{Q}^B$ but $\boldsymbol{u}(\hat{\boldsymbol{\sigma}})\neg \in \mathcal{U}^{WPO}$. There are two possible cases.

(a) $\Pi_i[\hat{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i] = 0$ for every $i = -1, 1$

(b) $\Pi_i[\hat{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i] > 0$ for at least one $i = -1, 1$

From the definition of the equilibrium, for every i and $\sigma_i \in \Sigma$,

(2) $\quad \Pi_i[\sigma_i, \hat{\sigma}_{-i}; \hat{d}_i] \leq \Pi_i[\hat{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i]$

For Case (a), consider $\tilde{\sigma}_1 = (\alpha x^\theta + (1-\alpha)\underline{x})_{\theta\in\Theta}$, where $u^\theta(x^\theta) > u^\theta(\hat{\boldsymbol{\sigma}})$ for every $\theta \in \Theta$ and $(x^\theta)_{\theta\in\Theta} \in \mathbf{X}^{WPO}$. From Lemma A.1, it is true that $\sum_{\theta\in\Theta} \lambda^\theta c^\theta(x^\theta) = 0$. For $1 > \alpha > \max_{\theta\in\Theta} \frac{u^\theta(\hat{\boldsymbol{\sigma}}) - u^\theta(\underline{x})}{u^\theta(x^\theta) - u^\theta(\underline{x})}$, $u^\theta(\tilde{\sigma}_i) > u^\theta(\tilde{\boldsymbol{\sigma}})$ for every $\theta \in \Theta$ and hence for $x \in \tilde{\sigma}_i \cap \Xi^\theta(\tilde{\sigma}_i, \hat{\sigma}_{-i})$ it is true from the definition of equilibrium that $\hat{d}_i(x) = \lambda^\theta$. The profit of company i from action $\tilde{\sigma}_i$ is: $\Pi_i[\tilde{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i] = \alpha \sum_{\theta\in\Theta} \lambda^\theta c^\theta(x^\theta) + (1-\alpha)\sum_{\theta\in\Theta} \lambda^\theta c^\theta(\underline{x})$. The first part is zero from Lemma A.1 and the second part strictly positive from Assumption B. Therefore, $\Pi_i[\tilde{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i] > 0$ which contradicts (2) because $\Pi_i[\hat{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i] = 0$.

In Case (b), there necessarily exists i such that $\Pi_i[\hat{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i] < \sum_{i=-1,1} \Pi_i[\hat{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i]$. Consider $\tilde{\sigma}_i = (\alpha x^\theta + (1-\alpha)\ddot{x}^\theta)_{\theta\in\Theta}$, where $u^\theta(x^\theta) > u^\theta(\hat{\boldsymbol{\sigma}})$ for every $\theta \in \Theta$, $(x^\theta)_{\theta\in\Theta} \in \mathbf{X}^{WPO}$ and $\ddot{x}^\theta \in \arg\max_{x\in\sigma_i\cup\sigma_{-i}} u^\theta(x)$. For every $1 < \alpha < 0$ $u^\theta(\tilde{\sigma}_i) = \alpha u^\theta(x^\theta) + (1-\alpha)u^\theta(\ddot{x}^\theta) > u^\theta(\ddot{x}^\theta)$ and therefore from the definition of the equilibrium for every $x \in \tilde{\sigma}_i \cap \Xi^\theta(\tilde{\sigma}_i, \hat{\sigma}_{-i})$ $\hat{d}_i(x) = \lambda^\theta$. Hence, $\Pi_i[\tilde{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i] = \sum_{\theta\in\Theta} \lambda^\theta c^\theta(\alpha x^\theta + (1-\alpha)\ddot{x}^\theta) = (1-\alpha)\sum_{i=-1,1} \Pi_i[\hat{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i]$. For $\frac{\sum_{i=-1,1} \Pi_i[\hat{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i] - \Pi_i[\hat{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i]}{\Pi_i[\hat{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i]} > \alpha > 0$, we have that $\Pi_i[\hat{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i] < \Pi_i[\tilde{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i] < \sum_{i=-1,1} \Pi_i[\hat{\sigma}_i, \hat{\sigma}_{-i}; \hat{d}_i]$ which contradicts (2). Q.E.D.

LEMMA A.3: $c^\theta(\mathring{x}^\theta) = 0 \; \forall \; \theta \in \Theta$.

PROOF: We will prove the result by contraposition. Take $(x^\theta)_{\theta\in\Theta}$ such that for some θ $c^\theta(x^\theta) > 0$. Consider contract $\tilde{x}$ such that $u^\eta(\tilde{x}) > u^\eta(x^\theta) \; \forall \; \eta \geq \theta$ and $u^\eta(\tilde{x}) < u^\eta(x^\theta)$ $\forall \; \eta < \theta$. From Assumption D such a contract exists. If $c^\theta(\tilde{x}) \geq 0$, then $x^\theta\neg \in \Psi^n$ because $\tilde{x}$ satisfies all constraints and provides higher utility to type θ. If $c^\theta(\tilde{x}) < 0$, consider $\hat{x} = \alpha x^\theta + (1-\alpha)\tilde{x}$, where $\alpha = \frac{-c^\theta(\tilde{x})}{c^\theta(x^\theta) - c^\theta(\tilde{x})}$. $u^\eta(\hat{x}) = \alpha u^\eta(x^\theta) + (1-\alpha)u^\eta(\tilde{x}) > u^\eta(x^\theta) \; \forall$ $\eta \geq \theta$, $u^\eta(\hat{x}) = \alpha u^\eta(x^\theta) + (1-\alpha)u^\eta(\tilde{x}) < u^\eta(x^\theta) \; \forall \; \eta < \theta$ and $c^\eta(\hat{x}) \geq 0 \; \forall \; \eta > \theta$. Therefore, $x^\theta\neg \in \Psi^n$. Q.E.D

Suppose now that $\mathring{\mathbf{u}}\neg \in \mathcal{U}_E^{SPO}$. The result is an immediate consequence of the following auxiliary lemma:

LEMMA A.4: For every $\mathbf{u} \in \mathcal{U}^{WPO}$ such that $u^\theta(x^\theta) \geq \mathring{u}^\theta$ with the inequality being strict for at least one $\theta \in \Theta$, there exists $\bar{x} \in X$ and $\eta \in \Theta$ such that $c^\eta(\bar{x}) > 0$, $u^\iota(\bar{x}) > u^\theta(x^\eta) \; \forall$ $\iota \geq \eta$ and $u^\iota(\bar{x}) < u^\iota(x^\iota) \; \forall \; \iota < \eta$.

PROOF: Consider $\mathbf{u} \in \mathcal{U}^{WPO}$ such that $u^\theta(x^\theta) \geq \mathring{u}^\theta$ with the inequality being strict for at least one $\theta \in \Theta$. We can first prove with a straightforward argument that there exists $\eta \in \Theta$ such that $c^\eta(x^\eta) > 0$. If not, then either $\forall \; \theta \in \Theta$ $c^\theta(x^\theta) < 0$ or $\forall \; \theta \in \Theta$ $c^\theta(x^\theta) = 0$. In the former case there is an immediate contradiction with the definition of strong Pareto optimum. In the latter case, there is a contradiction with $\mathring{x}^\eta$ being a solution of Program R^η. Hence, there exists $\eta \in \Theta$ such that $c^\eta(x^\eta) > 0$. Given Assumption (III), there exists $\tilde{x}$

such that $u^\iota(\tilde{x}) > u^\iota(x^\iota)\ \forall\ \iota \geq \eta$ and $u^\iota(\tilde{x}) < u^\iota(x^\iota)\ \forall\ \iota < \eta$. With an argument similar to that in Proposition (2.ii) we can show that there exists $\bar{x}$ such that $c^\iota(\tilde{x}) > 0\ \forall\ \iota \geq \eta$ and $u^\iota(\bar{x}) > u^\iota(x^\iota)\ \forall\ \iota \geq \eta$, $u^\iota(\bar{x}) < u^\iota(x^\iota)\ \forall\ \iota < \eta$. Q.E.D.

References

[1] Citanna, A. and Siconolfi, P. Incentive efficient price systems in large insurance economies with adverse selection. *Columbia Business School Research Paper*, (13-45), 2013.

[2] Dasgupta, P. and Maskin, E. The existence of equilibrium in discontinuous economic games, I: Theory. *The Review of Economic Studies*, 53(1):1–26, 1986.

[3] Dasgupta, P. and Maskin, E. The existence of equilibrium in discontinuous economic games, II: Applications. *The Review of Economic Studies*, 53(1):27–41, 1986.

[4] Engers, M. and Fernandez, L. Market equilibrium with hidden knowledge and self-selection. *Econometrica*, 55(2):425–439, 1987.

[5] Hellwig, M. Some recent developments in the theory of competition in markets with adverse selection. *European Economic Review*, 31(1):319–325, 1987.

[6] Jehle, G. A. and Reny, P. J. Advanced microeconomic theory, 2001.

[7] Mas-Colell, A. and Whinston, M. D. and Green, J. R. *Microeconomic theory*, volume 1. Oxford university press New York, 1995.

[8] Miyazaki, H. The rat race and internal labor markets. *The Bell Journal of Economics*, 8(2):394–418, 1977.

[9] Riley, J. G. Informational equilibrium. *Econometrica*, 47(2):331–359, 1979.

[10] Rosenthal, R. W. and Weiss, A. Mixed-strategy equilibrium in a market with asymmetric information. *The Review of Economic Studies*, 51(2):333–342, 1984.

[11] Rothschild, M. and Stiglitz, J. Equilibrium in competitive insurance markets: An essay on the economics of imperfect information. *The Quarterly Journal of Economics*, 90(4):629–649, 1976.

[12] Rustichini, A. and Siconolfi, P. General equilibrium in economies with adverse selection. *Economic Theory*, 37(1):1–29, 2008.

[13] Spence, M. Product differentiation and performance in insurance markets. *Journal of Public Economics*, 10(3):427–447, 1978.

[14] Wilson, C. A model of insurance markets with incomplete information. *Journal of Economic theory*, 16(2):167–207, 1977.