Optimal Security Design under Asymmetric Information and Profit Manipulation*

Kostas Koufopoulos[†] Roman Kozhan[‡] Giulio Trigilia[§]

Abstract

We consider a model of external financing in which entrepreneurs are privately informed about the quality of their projects and seek funds from competitive financiers. The literature restricts attention to monotonic or 'manipulation proof' securities and finds that straight debt is the uniquely optimal contract. Monotonicity is commonly justified by the argument that it would arise endogenously if the entrepreneur can window dress the realized earnings before contract maturity. We explicitly characterize the optimal contracts when entrepreneurs can engage in window dressing, and derive necessary and sufficient conditions for straight debt to be optimal. Contrary to conventional wisdom, debt is often suboptimal and it is never uniquely optimal. Optimal contracts are non-monotonic and induce profit manipulation in equilibrium. They can be implemented as performance-sensitive debt. Our results remain qualitatively unchanged if we allow for output diversion.

Key words: security design, capital structure, asymmetric information, profit manipulation, window dressing.

JEL classification: D82; D86; G32; M40.

*We would like to thank Robert Akerlof, Dan Bernhardt, Peter Buisseret, Peter DeMarzo, Bhaskar Dutta, Marina Halac, Peter Hammond, Ilan Kremer, Dimitri Migrow, Motty Perry, Herakles Polemarchakis, Uday Rajan and David Ronayne. All mistakes are our own.

[†]University of Leictester, School of Business, email: kkoufopoulos@gmail.com.

[‡]University of Warwick, Warwick Business School, email: Roman.Kozhan@wbs.ac.uk.

[§]University of Rochester, Simon Business School, email: giulio.trigilia@simon.rochester.edu.

1 Introduction

Since Myers and Majluf (1984), asymmetric information has been used to explain the prevalence of debt as a means of external financing. The idea is that borrowers are better informed than lenders about the distribution of future cash flows. If they cannot credibly signal their quality, better borrowers pay an information cost through the underpricing of the securities they issue. Because better borrowers are less likely to generate low cash flows, this information cost is minimized if the securities issued offer the greatest downside protection and the lowest upside gain that are feasible and that (jointly) secure that the lenders obtain their desired return. Thus, if only debt and equity are available, debt is preferred to equity by high quality borrowers. Innes (1993) and Nachman and Noe (1994) consider the general security design problem and identify two sufficient conditions for debt to be optimal: (i) cash flows distributions are monotonic in the borrower's quality, or type; (ii) the retention of performance-based bonuses by borrowers is assumed away.[1]

In this paper we explore when and why ruling out such bonuses is optimal, and we derive conditions on fundamentals under which the optimal securities are monotonic. To begin with, one has to understand why performance-based bonuses pose a threat to the optimality of straight debt. It is so because, relative to debt, they reduce the upside gains of lenders and, therefore, they strengthen the downside protection (for the lenders to break even). In the security design language, their payoff crosses that of a debt contract from the right. This is precisely the desired goal of better quality borrowers.

So, what prevents bonuses from being used? The common explanation appeals to window dressing possibilities. A performance-based bonus is paid when the realized cash flows exceed some pre-specified benchmark. Now, suppose that a borrower can window dress the reported cash flows - perhaps by secretly borrowing from a friend. She clearly has an incentive to window dress when the realized cash flows do not hit the benchmark, in order to cash the bonus. As a result, window dressing effectively transforms debt with bonuses into straight debt, with 'real' (correctly anticipated) face value equal to the contractual face value minus the bonus. The above story is sound if: (i) window dressing possibilities are unbounded (the 'friend' has a deep pocket); and (ii) borrowers may never

[1] Hebert (2015) studies a model where effort moral hazard and risk shifting are simultaneously present. He shows that the optimal contract is always a mixture between non-monotonic securities and equity. Sometimes, the optimal security is equivalent to a debt contract. In different contexts, Antic (2014), Lee and Rajan (2016) and Malenko and Tsoy (2018) show that the optimal contract is monotonic when ex ante asymmetric information is complemented with ambiguity aversion on the financier's side.

be caught cheating. However, often some of these conditions fails empirically.

In this paper, we generalize the results of Nachman and Noe (1994) to encompass these cases. We characterize optimal securities and derive novel testable predictions relating the size of bonuses to cross-sectional differences in asymmetric information and to the efficacy of the legal system. In particular, we explicitly model window dressing by assuming that, in the interim period, the borrower has access to a credit line from a 'friend'. The credit line enables her to window dress earnings up to a fixed upper bound (with respect to which we perform comparative static exercises).[2]

Absent window dressing possibilities, we know from Innes (1993) that the optimal contract is not debt. In contrast, it is non-monotonic and it features a bonus that equals the face value of debt and is paid whenever the firm does not default. Namely, optimal securities are akin to *asset-or-nothing binary options*. Our analysis shows that, contrary to the conventional wisdom, such options remain optimal (and they dominate debt) even when window dressing is feasible, provided that it is not too severe.

The key observation driving the results is as follows. Consider three possible realizations of the cash flows: 10$, 20$ and 30$. Further, suppose that cash flows can be window dressed up to 15$. Evidently, an entrepreneur with a realized cash flow of 10$ can claim to have 20$, but not 30$. In contrast, an entrepreneur with 20$ can easily claim to have 30$. The situation described is known in the contracting literature as a failure of the *nested-range-condition* (NRC), as defined in Green and Laffont (1986). This condition is both necessary and sufficient for the revelation principle to hold – or, in our language, for concentrating attention without loss of generality on contracts that prevent any profit manipulation on the equilibrium path.

The immediate and, in our opinion, important consequence of a failure of NRC is that contracts involving profit manipulation in equilibrium implement allocations that cannot be achieved otherwise. In fact, we show that they strictly reduce the mispricing of the securities issued by better borrower types for some (most) parameter values. We characterize two parameter regions where the uniquely optimal security requires profit manipulation. In one region, the equilibrium is pooling and all borrowers issue non-monotonic securities, such as debt with a strictly positive performance-based bonus. In

[2]Another possible form of profit manipulation is output diversion. In the baseline model, we focus on window dressing because this is the main concern of the finance and accounting literature on profit manipulation. However, the model is extended to cover also output diversion opportunities in later sections, and the main results remain qualitatively unchanged.

the other region, the equilibrium is separating and better borrowers issue non-monotonic securities, whereas the worst borrowers can issue any other security, e.g. debt or equity.

Furthermore, for sufficiently large window dressing possibilities the aforementioned justification for monotonicity of securities holds, and every non-monotonic contract is ex post equivalent to straight debt, with 'real' face value equal to the nominal face value minus the bonus (as in Nachman and Noe (1994)). One contribution of our paper is to explicitly characterize what 'sufficiently large' means analytically, i.e., to derive necessary and sufficient conditions for debt to be optimal. We also show that such conditions are never satisfied if the distribution of future cash flows is unbounded above - this is especially relevant for practical applications, where often cash flows are assumed to be either log-normally or exponentially distributed. Moreover, debt is never uniquely optimal: there always exists a non-monotonic contract ex post equivalent to debt.

Importantly, we show analytically that our qualitative results remain unchanged when borrowers can engage in both window dressing, and output diversion.[3] We also discuss why they are not critically driven by the specific assumptions on the manipulation technology. Finally, we run simulations to confirm that the region where debt contracts belong to the set of optimal contracts is not only quite limited, but it also corresponds to the most extreme degrees of manipulation opportunities and adverse selection. In this region, there always exist optimal non-monotonic contracts that are ex post equivalent to debt. The paper unfolds as follows. Section 2 reviews the relevant literature. Section 3 presents the baseline model. Section 4 introduces the relevant securities, and discusses when and how they induce profit manipulation. Section 5 derives our main results. Section 6 extends the model allowing also for output diversion. Section 7 presents some parametric examples to streamline our findings. Section 8 discusses extensions such as considering more than two types and introducing moral hazard. Section 9 concludes.

2 Literature Review

Our paper is closely related to the literature on security design under asymmetric information. Myers and Majluf (1984) developed the 'pecking order' theory of debt optimality

[3] In this extension, we allow for asymmetric as well as symmetric boundaries for window dressing and output diversion, and our model takes a flavor similar to Lacker and Weinberg (1989). We could also add a costly manipulation technology but that would require optimal contracts to trade off two deadweight losses (manipulation and adverse selection), and the results may be complicated to derive.

under asymmetric information in a set up where only debt and (inside or outside) equity contracts were allowed. Noe (1988) first showed that their theory required somewhat restrictive assumptions on the distributions of earnings. Innes (1993) and Nachman and Noe (1994) revisited the theoretical argument allowing for a broader set of contracts than debt and equity. These papers found that to obtain debt as the optimal security some monotonicity constraint has to be imposed both on the type space, and on the set of feasible securities. The latter constraint restricts admissible contracts to those that are 'manipulation proof'. Since then, the monotonicity constraint has been widely used. Prominent examples include DeMarzo and Duffie (1999); DeMarzo et al. (2005); Inderst and Mueller (2006); Axelson (2007); Axelson et al. (2009); Gorbenko and Malenko (2011); Philippon and Skreta (2012); Scheuer (2013). Our contribution is to derive necessary and sufficient conditions for monotonicity to be without loss of generality.

Furthermore, our paper is related to the literature on optimal contracting under profit manipulation. The existing papers can be separated along two dimensions: (i) whether manipulations are assumed to be bounded (and a function of types) or not; and (ii) whether repayments can be extracted via additional tools such as verification, termination or liquidation of the firm. A literature originating from Townsend (1979) and Gale and Hellwig (1985) models unbounded manipulation with the possibility of verifying the earnings at a cost (the so-called 'costly state verification' - CSV - models). Bolton and Scharfstein (1990) and Hart and Moore (1998) study related models where verification is substituted with the threats of termination and liquidation. In contrast, Green and Laffont (1986) consider a set up with bounded manipulation possibilities but no verification. They provide a necessary and sufficient condition for the revelation principle to hold - the *nested range condition* (NRC) - which fails naturally in financial contracting models where the set of feasible manipulations is likely to be convex and depends on the type.

3 The Economy

There are two dates $\{0, 1\}$, an *entrepreneur* and a competitive *financier*. Both agents are risk-neutral and maximize date one consumption. The entrepreneur has a project that generates stochastic date one earnings $x \in X$ and requires a fixed input of $I > 0$ at date zero. The financier has wealth $W > I$, and can either lend it to the entrepreneur or store it without depreciation. The set of possible earnings realizations is $X \equiv [0, K]$. When we allow for unbounded future earnings, we let K approach infinity. There are two types of projects (entrepreneurs), $t \in T \equiv \{l, h\}$. Types differ according to their distribution of earnings. The cumulative distribution function (*cdf*) over X for a type t project is $F_t(x)$. The project's type is private information of the entrepreneur. Outside financiers only know that a fraction $\lambda_l \in (0,1)$ are type l projects, and a fraction $\lambda_h = (1-\lambda_l)$ are type h projects. All projects have positive net present value, and the firm's assets in place are assumed to be zero. Denote by $\mathbb{E}_t[x] = \int_0^K x\, dF_t(x)$ the full information expected value of a type t project. We assume that all projects have positive net present value:

Assumption 1: $\mathbb{E}_t[x] \geq I > 0$ for every $t \in T$.[4] **(A1)**

In addition, we make the following standard assumptions on the distributions of earnings:

Assumption 2: **(A2)**

1. *The cumulative distribution functions are mutually absolutely continuous;*
2. *Strict Monotone Likelihood Ratio Property (MLRP):* $\frac{\partial}{\partial x}\left(\frac{f_h(x)}{f_l(x)}\right) > 0$ *for every* x.

Continuity simplifies the analysis and it prevents contracts that penalize realizations with strictly positive probability only for one type. Strict MLRP implies that $\mathbb{E}_h[x] > \mathbb{E}_l[x]$. Both assumptions are standard in the literature (see for instance DeMarzo et al. (2005)). The timing of the game is as follows:

- **date 0:** The entrepreneur of type t issues publicly a *portfolio of securities* (financial contract) denoted by s. Each financier simultaneously quotes a price $P(s)$ at which he is willing to buy the securities. If a contract is signed (securities are sold), the entrepreneur collects $P(s)$. Subsequent investment is observable and verifiable;
- **date 1:** Realized earnings $x \in X$ are perfectly but privately observed by the entrepreneur. He can costlessly manipulate reported earnings by secretly borrowing from friends up to $\bar{\eta}(x) \geq 0$, reporting earnings to be $m \in M(x) \equiv [x, x + \bar{\eta}(x)]$. Hence, the profit manipulation technology $M(x)$ for a generic ex post realization x is fully characterized by $\bar{\eta}(x)$. The possibility of secret borrowing and its magnitude is common knowledge at date 0;[5]
- **date 2:** Claims are settled on the basis of the borrower's self reported earning and the game ends.

[4] **A1** guarantees that investment is risky, because $I > 0$ and strict positivity of $f_t(x)$ for every x imply that $F_t(I - \epsilon) > 0$ for all $t \in T$, for $\epsilon > 0$.

[5] In Section 6 we extend the manipulation technology and also consider the case of output diversion.

Timeline

• **Contracting stage**	• **Entrepreneur observes** x **and reports** $m(x|s)$	• **Claims are settled**
$t = 0$	$t = 1$	$t = 2$

The novel ingredient that differentiates our findings from existing results is the possibility of ex post profit manipulation (window dressing). We summarize the restrictions imposed on the manipulation technology in the following assumption:

Assumption 3: *The set of admissible profit manipulation technologies for any realized earnings* $x \in X$ *is given by:* **(A3)**

$$M(x) \equiv [x,\, x + \bar{\eta}(x)],$$

where $\bar{\eta} \in \mathbb{C}^1$ *such that for every* x *we have* $0 \leq \bar{\eta}(x) \leq K - x$ *and* $\bar{\eta}'(x) \geq -1$.[6]

Condition $\bar{\eta}'(x) \geq -1$ guarantees that the higher the realized output, the higher the output that can be reported. We allow for unbounded secret borrowing to nest Nachman and Noe (1994) results as a special case of our framework.

Since window dressing possibilities are the key innovation of our model relative to the existing work, a few comments are due. First, we could have called this an *upward profit manipulation technology.* Although this would have been a more direct way of expressing Assumption 3, we use window dressing for two reasons. First, to be in line with most of the previous security design papers that refer to window dressing possibilities as the main problem affecting non-monotonic security designs. Second, and most importantly, because our assumption is *equivalent* to a simple formal model of window dressing in the literal sense, that is, a model of *secret borrowing* from friends. Indeed, we could think about a third party who is directly related to the borrower (and so not subject to ex post informational asymmetries) and who provides a short-term bridge financing at the interim stage. The bounds on window dressing possibilities would naturally corresponds to the depth of this friend's pocket, hence such model would be equivalent to ours.

In addition, although we believe that there is a potential risk of strategic default as well as other issues in raising the necessary funds from a 'friend?, in this model we assume

[6]The bounds on $\bar{\eta}(\cdot)$ guarantee that the upper bound of the set $M(x)$ belong to X for every $x \in X$.

that all those impediments are absent for the following reason. The literature justifies the monotonicity constrain due to the possibility of window dressing and concludes that debt arises as the optimal contract under this assumption. We claim in the paper that this is not the case (as it is true only in some extreme cases). To make this point we make this manipulation as easy as possible for the entrepreneur (subject to our manipulation boundaries) and show that even in this hypothetical scenario manipulation does not eliminate non-monotonicity of the equilibrium contact. Surely, if one makes manipulation even more difficult the monotonicity constraint is even harder to justify.

The possibility of earnings misreporting means that a security $s(\cdot)$ cannot be a function of x as in the previous literature. Instead, it is a function of the reported earnings $m(x|s)$. Because $M(x)$ is a compact set for every $x \in X$, we know that for every security s and every x there exists a best message $m^*(x|s)$ defined so that:

$$m^*(x|s) \equiv \underset{m \in M(x)}{\arg\min} \{s(m)\}.$$

Further, because $M(x) \subseteq X$, we know that $m^*(x|s) \in X$.[7] Hence, the expected repayment of a security (or its *real payoff*) is a function $s(m^*(x|s)) : X \to \mathbb{R}$. It should be noticed that the ex post verification problem created by the possibility of profit manipulation prevents the application of the revelation principle, since $M(x)$ does not need to satisfy the nested-range-condition of Green and Laffont (1986).

The only restriction we impose on the contract space is that each security must satisfy limited liability, as appropriately redefined in terms of messages:

Assumption 4: *The set of admissible securities is given by:* **(A4)**

$$S \equiv \{s(m) \mid 0 \leq s(m) \leq m, \ \ \forall m \in X\}.$$

If the borrower declares m and cannot repay $s(m)$ to his financier, then the financier becomes the legitimate owner of borrower's assets.[8]

[7]Our formulation restricts attention to direct mechanisms, where the set of messages is a subset of the set of states X. It is easy to see that the restriction is without loss of generality.

[8]Since the limited liability constraint must be defined in terms of messages rather than realized output, we should consider the case in which the entrepreneur declares earnings that exceed true earnings, and does not have the resources to repay the contractual obligation. In this case, the fraud becomes observable and verifiable: it is revealed that he is either lying about x or refusing to make the payment he committed to make. We implicitly assume that when the fraud is revealed the agent receives a large punishment.

Denote by V_t the profits of an entrepreneur of type t whose offered security s has been priced at P by the financier, and by V_f the financier's profits. Then we can write:

$$V_t = P - I + \mathbb{E}_t\big[x - s(m^*(x|s))\big], \tag{1}$$

$$V_f = \mathbb{E}_{\boldsymbol{\lambda}(s)}\big[s(m^*(x|s))\big] - P(s). \tag{2}$$

The expectation in (2) is given by the sum across types (weighted by the posterior belief $\lambda(t|s)$ that type t is issuing the contract s) of the final payoff of the security after manipulation takes place:

$$\mathbb{E}_{\boldsymbol{\lambda}(s)}\big[s(m^*(x|s))\big] \equiv \sum_{t\in T}\lambda(t|s)\left[\int_{x\in X} s(m^*(x|s))dF_t(x)\right].$$

Notice that we can write $m^*_t(x|s) = m^*(x|s)$ because the cost and benefits of committing accounting fraud ex post are not type-dependent.

Here we adopt the concept of Perfect Bayesian Equilibrium (PBE):

PBE: *A strategy profile* $\big(s^*_t,\, m^*(x|s),\, P^*(s)\big)$ *and a common posterior belief* $\lambda^*(t|s)$ *for* $t \in T$ *form a PBE of the game if the following conditions are satisfied:*

1. *For every* $x \in X$ *and for* $s \in S$*:* $m^*(x|s) = \arg\min_{x\in M(x)} \big\{s(m)\big\}$;
2. *For every* $t \in T$, s^*_t *maximizes* $V_t(s_t, P^*(s_t), m^*)$ *subject to the limited liability constraint* $(s_t \in S)$*;*
3. *The posterior belief* $\lambda^*(t|s_t)$ *is obtained from Bayes' Rule whenever possible;*
4. *Competitive Rationality: for* $s_t \in S$, $P^*(s_t) = \mathbb{E}_{\boldsymbol{\lambda}^*(t|s_t)}[s_t]$

As standard, a PBE is said to be *separating* if $s^*_h \neq s^*_l$, and *pooling* otherwise. Notice that, because investment is observable and verifiable, in every equilibrium it must be the case that either $P^*(s_t) = 0$ (no investment), or $P^*(s_t) \geq I$ (investment takes place).

To rule out 'unreasonable' equilibria, we refine the off-equilibrium-path beliefs using the Intuitive Criterion by Cho and Kreps (1987). Denote by $V_t(s^*_t,\, e^*)$ the expected utility of type t entrepreneur issuing s^*_t at the equilibrium e^*, and by $\Pi^*(s|T)$ the set of *all* possible Bayesian Nash Equilibria of the game played by financiers given an observed $s \in S$[9].

[9]Each element of the set can be parameterized by a posterior belief $\boldsymbol{\lambda}(s) \in \Delta_T$, where we adopt the convention that bold symbols represent vectors.

The Intuitive Criterion:[10] *A PBE is* not reasonable *if there exist an out-of-equilibrium security $s' \in S$ such that only a subset of types $\tau \subset T$ may benefit from deviating to s'. That is, for every $t \in \tau$ and $t' \in T \setminus \tau$*

$$V_t(s_t^*, e^*) \leq \max_{P^* \in \Pi^*(s'|T)} V_t(s', e^*),$$

$$V_{t'}(s_t^*, e^*) > \max_{P^* \in \Pi^*(s'|T)} V_{t'}(s', e^*).$$

In words, suppose there are two types. Consider a pooling equilibrium, and a deviant security that could only benefit the high type (if accepted), for some off-equilibrium beliefs, and never the low type. The Intuitive Criterion prevents equilibria that are sustained by the off-equilibrium belief that such a security would be offered by a low type with positive probability. The next section introduces the key properties of the two contracts which are relevant in this framework: debt and bonus contracts.

4 Debt and Bonus Contracts

It is useful to stress again the distinction that arises in this model (unlike the existing literature) between the *promised expected payoff* and the *real expected payoff* of a security. The *promised* expected payoff is given by $\mathbb{E}_{\boldsymbol{\lambda}^*(s)}[s(m = x)]$, where each realized x is assumed to be reported truthfully. In contrast, the *real* expected payoff is given by $\mathbb{E}_{\boldsymbol{\lambda}^*(s)}[s(m^*(x|s))]$, where $m^*(x|s)$ solves condition (1) of a PBE, i.e. it maximizes the entrepreneur's ex post payoff. The characteristic features of a debt contract are: (i) the fixed repayment in non-bankruptcy states; and (ii) seniority in bankruptcy states. If we denote the face value of debt by d, then whenever $m \geq d$, the debt security specifies $s = d$. If, instead, $m < d$, a bankruptcy state, the debt holder is a senior claimant on the assets, obtaining repayment $s(m) = m$.

Definition 1. *A security $s \in S$ is a* debt contract *if $s(m) = min\{m, d\}$ for $d \in X$.*

Since debt contracts are monotonic, there cannot be any benefit from overstating earnings: the real and promised payoffs of debt coincide (see Figure 1).

Insert Figure 1 about here

[10] We provide a definition of the Intuitive Criterion for a generic set of types T because we extend our main result to the case of $|T| > 2$.

The contract that turns out to be generically optimal (we call it a bonus contract) takes the following form:

Definition 2. *A security $s \in S$ is a* bonus contract *if, for $(b, d) \in X^2$:*

$$s(m) = \begin{cases} m, & \text{if } m < b \\ d, & \text{otherwise.} \end{cases} \tag{3}$$

Since any admissible security satisfies limited liability ($s \in S$), we must have $d \in \left[0,\ b\right]$.[11] Moreover, if $d = 0$ the bonus contract becomes an asset-or-nothing binary option (or, equivalently, debt with a strictly positive performance-based bonus). Figure 2 depicts the *promised* payoff of contracts as defined in (3).

Insert Figure 2 about here

We next characterize the optimal amount of secret borrowing under bonus contracts. To do so we need to introduce a final piece of notation. Define a function $\bar{\bar{\eta}}(x) \equiv x + \bar{\eta}(x)$. This function is continuous and strictly increasing on $[0, K]$. Consider a bonus contract (b, d) such that $\bar{\eta}(0) < b$. Given that $\bar{\bar{\eta}}(b) \geq b$ by definition, the intermediate value theorem implies that there exists $\tilde{x} \in (0, b]$ such that $\bar{\bar{\eta}}(\tilde{x}) = \tilde{x} + \bar{\eta}(\tilde{x}) = b$. Monotonicity of the function $\bar{\bar{\eta}}(x)$ ensures the uniqueness of such a point. Define a function $\tilde{x} \colon [d, K] \to [0, K]$ as follows:

$$\tilde{x}(b) = \begin{cases} 0, & \text{if } \bar{\eta}(0) > b, \\ \tilde{x}, & \text{otherwise.} \end{cases}$$

The implicit function theorem and Assumption 3 imply that this function is strictly monotonic on the interval $(\bar{\eta}(0), K)$. For a given b, the value $\tilde{x}(b)$ specifies the smallest realized earnings so that the entrepreneur can reach the cut-off point b by means of secret borrowing.

Lemma 1. (The Real Payoff of a Bonus Contract) *Given any bonus contract s with fixed repayment d and threshold b, we have two cases:*

[11] Standard debt contracts are special cases of (3) where $d = b$. For this reason we shall always make explicit whether the contracts we discuss must feature a strictly positive bonus or not.

1. *If $\tilde{x}(b) > d$, then the entrepreneur optimally reports*

$$m^*(x|s) = \begin{cases} x, & \text{if } \ x < \tilde{x}(b), \\ \max\{x, b\}, & \text{if } \ x \geq \tilde{x}(b). \end{cases}$$

2. *If $\tilde{x}(b) \leq d$, then the entrepreneur optimally reports*

$$m^*(x|s) = \begin{cases} x, & \text{if } \ x < d, \\ \max\{x, b\}, & \text{if } \ x \geq d, \end{cases}$$

and the real payoff of a bonus contract is equivalent to that of a debt contract.

Proof. See the Appendix.

Insert Figure 3 about here

Figure 3 depicts the real payoff of a bonus contract for the cases of two different levels of profit manipulation. In Panel A of Figure 3, $\tilde{x}(b) > d$ and the real payoff is *not* ex post monotonic. In Panel B, $\tilde{x}(b) \leq d$ and the real payoff is ex post equivalent to that of a debt contract with face value d.

5 Optimal Security Design

In this section we solve for the optimal securities. We first consider the case in which the separation can be achieved and then turn attention to pooling equilibria.

5.1 Separating Equilibria

In this section we characterize the set of Separating Perfect Bayesian Equilibria (SPBE). Such equilibria never arise with the exogenous monotonicity constraint. The intuition behind the SPBE is the following: the most productive type tries to distinguish himself from the less productive one by offering securities with high downside protection and a low upside payoff for the financier (such as bonus contracts). By doing so, high types impose a relatively higher cost on low types should they try to mimic.

In a SPBE, $s_l \neq s_h$. Moreover, given the offered security s_t, the posterior belief that it is offered by type t is one, i.e. $\lambda(t|s_t) = 1$ for every $t \in T$. Incentive compatibility for type t requires:

$$\mathbb{E}_t[x - s_t(m^*(x|s))] \geq \mathbb{E}_t[x - s_{t'\neq t}(m^*(x|s))],$$

or, equivalently: $\mathbb{E}_t[s_t(m^*(x|s))] \leq \mathbb{E}_t[s_{t'\neq t}(m^*(x|s))]$. Further, at any SPBE it must be that $\mathbb{E}_t[s_t(m^*(x|s))] = I$, for every $t \in T$, so we can rewrite the incentive constraint as:

$$\mathbb{E}_{t'}[s_{t'}(m^*(x|s))] - \mathbb{E}_t[s_{t'}(m^*(x|s))] \leq 0.$$

Finally, it is trivial to show that the only incentive constraint that may be binding is that for the l type not to mimic the h type, i.e:

$$\mathbb{E}_h[s_h(m^*(x|s))] - \mathbb{E}_l[s_h(m^*(x|s))] \leq 0. \tag{4}$$

This formulation of the incentive constraint allows us to proceed and solve for the optimal contract as it will become clear below.

Suppose that s_h is a bonus contract (b_h, d_h). Then (4) would read:

$$\int_0^{\max\{d_h,\tilde{x}_h(b_h)\}} x\cdot\Big[dF_h(x)-dF_l(x)\Big]+\Big[F_l\big(\max\{d_h,\tilde{x}_h(b_h)\}\big)-F_h\big(\max\{d_h,\tilde{x}_h(b_h)\}\big)\Big]d_h \leq 0.$$

Notice that if $\max\{d_h, \tilde{x}_h(b_h)\} = d_h$, the incentive constraint can never be satisfied because the real payoff of the bonus contract is monotonic. To see this, integrate the above expression by parts and get:[12]

$$\underbrace{\int_0^{d_h} 1\cdot\Big[F_l(x) - F_h(x)\Big]dx}_{>0\text{ because MLRP implies FOSD, and by Assumption 3}} + \underbrace{\Big[F_l\big(d_h\big) - F_h\big(d_h\big)\Big]\Big[d_h - d_h\Big]}_{=0} > 0.$$

Hence, for the rest of this section suppose that $\max\{d_h, \tilde{x}_h(b_h)\} = \tilde{x}_h(b_h)$. Note also, that because $\tilde{x}_h(b_h) = b_h - \bar{\eta}(\tilde{x}_h) > d_n$, it implies that $b_h > d_h$. Rewrite the incentive compatibility constraint as:

$$\int_0^{\tilde{x}_h(b_h)} x\cdot\Big[dF_h(x) - dF_l(x)\Big] + \Big[F_l\big(\tilde{x}_h(b_h)\big) - F_h\big(\tilde{x}_h(b_h)\big)\Big]d_h \leq 0. \tag{5}$$

[12] As standard, MLRP is the acronym for *Monotone Likelihood Ratio Property*, and FOSD for *First Order Stochastic Dominance*.

In a SPBE, competitive financing yields $\mathbb{E}_h[s_h] = I$. Substituting this into (5) we get:

$$\int_0^{\tilde{x}_h(b_h)} x \cdot \Big[dF_h(x) - dF_l(x)\Big] + \Big[F_l\big(\tilde{x}_h(b_h)\big) - F_h\big(\tilde{x}_h(b_h)\big)\Big]\left(\frac{I - \int_0^{\tilde{x}_h(b_h)} x dF_h(x)}{1 - F_h(\tilde{x}_h(b_h))}\right) \leq 0.$$

Integrating by parts and rearranging, we obtain:

$$\begin{aligned} \text{IC} \equiv & \underbrace{\int_0^{\tilde{x}_h(b_h)} \Big[F_l(x)\big(1 - F_h(\tilde{x}_h(b_h))\big) - F_h(x)\big(1 - F_l(\tilde{x}_h(b_h))\big)\Big] dx}_{\geq 0 \text{ by FOSD and by Assumption 3}} \\ & + \underbrace{\Big[F_l\big(\tilde{x}_h(b_h)\big) - F_h\big(\tilde{x}_h(b_h)\big)\Big]}_{\geq 0 \text{ by FOSD}} \underbrace{\Big[I - \tilde{x}_h(b_h)\Big]}_{\text{sign?}} \leq 0. \end{aligned} \tag{6}$$

Recall that $\tilde{x}_h(b_h)$ is defined as the threshold such that for every $x \leq \tilde{x}_h(b_h)$ truthful reporting is weakly preferred to secret borrowing, and vice versa for $x > \tilde{x}_h(b_h)$. Hence, inequality (6) highlights the key mechanism that underlies separation: setting a threshold b_h that makes $\tilde{x}_h(b_h)$ high enough so that the last bracket becomes not just negative, but low enough that the second term in expression (6) counterbalances the first one.

The key properties of (6) that are useful in the analysis are given in Lemma 2:

Lemma 2. (Incentive Compatibility) *If $\exists\, b_h \in [0, K]$ that satisfies (6), then:*

1. *There is a unique b_h at which the inequality binds. We denote it by b_h^1;*
2. *For every $b_h < b_h^1$ the inequality is violated;*
3. *For every $b_h \geq b_h^1$ the inequality is satisfied.*

Proof. See the Appendix.

The argument to prove Lemma 2 is not immediate, as inequality (6) is a non-monotonic function of b_h. The proof relies on the fact that if the inequality is satisfied for some $b_h < K$, one can show that the set of b_h such that the inequality is binding is a singleton, and the inequality is always satisfied for values $b_h \geq b_h^1$, and never otherwise. If (6) is satisfied, then a contract is incentive compatible and leaves the financier at his participation constraint. However, it remains to guarantee that the underlying contract belongs to the set of admissible securities, i.e. that $d_h \geq 0$.

Denote by $\tilde{x}_h^{\max}$ the solution to the zero profit condition in a SPBE for type h when the face value of debt $d_h = 0$, and by b_h^{max} the corresponding contractual threshold such that $\tilde{x}_h^{\max} = \tilde{x}_h(b_h^{max})$ (which exists and is unique by the monotonicity of $\tilde{x}_h$). We have:

$$\int_0^{\tilde{x}_h^{\max}} x f_h(x)dx = I. \tag{7}$$

In the financing region equation (7) is guaranteed to have a solution, and it implies that all feasible thresholds are such that $b_h \leq b_h^{max}$. So, obtain the following theorem:

Theorem 1. (SPBE) *If b_h^1 exists and $b_h^1 \leq b_h^{max}$ then:*

1. *There exists a separating equilibrium e_s^* in which:*
   (a) *A type h entrepreneur issues a bonus contract (b_h^*, d_h^*) such that the financiers make zero profits, for any $b_h^* \in [b_h^1, b_h^{max}]$;*
   (b) *Type l entrepreneurs are indifferent between any contract such that $\mathbb{E}_l[s] = I$, as long as it is not a bonus contract with $d_l^* \leq d_h^*$;*
2. *No pooling equilibrium satisfies the Intuitive Criterion.*

Proof. See the Appendix.

Intuitively, when $b_h^1 \leq b_h^{max}$ separation may be achieved because MLRP implies that the low type $(t = l)$ expects to repay more relative to the high type. Thus, by choosing a sufficiently high threshold for the bonus contract (and a sufficiently low face value of debt) the high type can make the cost of mimicking for the low type excessively high, and credibly signal his type to the uninformed financiers.

It should be stressed that this equilibrium is not unique in terms of the securities used for achieving it. However, the equilibrium allocation is unique in the sense that in any of the separating equilibria the securities issued are fairly priced. Thus, in any separating equilibrium each type of entrepreneur get the net present value of his project. The multiplicity of separating equilibria is the feature of all models where the signal does not imply the deadweight loss (non-dissipative signals, see Bhattacharya (1980)).

Nevertheless, our choice of focusing on bonus contracts still needs to be justified. Does a separating equilibrium exist outside the region covered by Theorem 1? And if so, what contracts support it? The answers to these questions are negative: *if separation is not implementable through bonus contracts, then credible signaling cannot happen under any other security that satisfies limited liability.* This occurs because under a bonus contract the full reported earnings are transferred to the financier if they lie between zero and the threshold b. Because the probability that the low type reports earnings below b is higher, the bonus contract maximizes the cost of mimicking for the l type. Given limited liability, no other contract can achieve a higher expected repayment for the low type in this region. In other words, the conditions in Theorem 1 are *both necessary and sufficient* for separating equilibria to exist:

Corollary 1. (Necessity of bonus contracts for separation) *If b_h^1 does not exist or $b_h^1 > b_h^{max}$, then any PBE of the game must be pooling.*

Proof. See the Appendix.

The intuition for this result is as follows: a higher threshold for the bonus contract (and a lower face value of debt) increases the cost of mimicking for the low type. This cost is maximized when $d_h = 0$ and the threshold is b_h^{max}. If the distributions are such that the incentive constraint for the low type is violated at this contract, then separation is impossible and every equilibrium must be pooling. The argument is sometimes referred to in the literature as showing that 'no security in S crosses the repayment function of a bonus contract from the left'. We characterize the pooling equilibria next.

5.2 Pooling Equilibria

Since Nachman and Noe (1994) seminal paper, the literature has adopted a stronger refinement than the intuitive criterion to deal with pooling equilibria: the D1 criterion. As is well known, the intuitive criterion does not bind in the pooling region of such models. The reason is that both types may benefit from any deviation depending on the posterior belief of the financier. D1 refines the equilibrium set and obtain a unique equilibrium because it is a condition on the range of beliefs for which a deviation is profitable.

Denote by V_t' the utility of type t entrepreneur at the deviant contract, and by V_t^* the utility of type t entrepreneur at the equilibrium contract. Moreover, denote by $D(t|s')$ the set of responses of the financier that would deliver strictly higher utility to type t entrepreneurs than the utility he would obtain at the equilibrium contract. Formally:

$$D(t|s') \equiv \{P^*(s') \geq I : V_t' > V_t^*\}$$

where by competitive rationality, $P^*(s') = \mathbb{E}_{\boldsymbol{\lambda}^*(s')}[s']$ for *all* $\lambda^*(s') \in \Delta_T$, as beliefs off-the-equilibrium path are arbitrary. Finally, define the indifference set $D^0(t|s')$:

$$D^0(t|s') \equiv \{P^*(s') \geq I : V_t' = V_t^*\}.$$

The D1 restriction can be defined as follows[13]:

[13] The D1 restriction is stronger than the intuitive criterion, hence Theorem 1 goes through unchanged

D1: *Suppose $s' \in S$ is observed off-the-equilibrium path. Then for all $t \in T$:*

$$\lambda_t^*(s') = \begin{cases} 0, & if \quad \exists\, t' \in T\ s.t.\ t' \neq t, \text{ and } D(t|s') \cup D^0(t|s') \subset D(t'|s'), \\ 1, & if \quad D(t'|s') \cup D^0(t'|s') \subset D(t|s'),\ \forall t' \neq t \in T, \\ 1 - \lambda_{t' \neq t}, & \text{otherwise.} \end{cases}$$

The pooling zero profit condition at a bonus contract $(b, 0)$ is given by:

$$ZP^1(b) \equiv \lambda_h \left[\int_0^{\tilde{x}(b)} x f_h(x) dx \right] + (1 - \lambda_h) \left[\int_0^{\tilde{x}(b)} x f_l(x) dx \right] = I. \tag{8}$$

Applying D1 yields:

Theorem 2. (PPBE, part (a)) *If $b_h^1 > b_h^{max}$ (or b_h^1 does not exist) and there exists $b_{\boldsymbol{\lambda}} \in (0, K)$ solving (8), then there is a unique pooling equilibrium e_p^* which satisfies D1. At e_p^*, all types issue a contract $(b_{\boldsymbol{\lambda}}, 0)$.*

Proof. See the Appendix.

Theorem 2 characterizes the set of equilibria that satisfy D1 when separation is not feasible ($b_h^1 > b_h^{max}$), but the window dressing possibilities are relatively low ($b_{\boldsymbol{\lambda}} < K$). In this region, the uniquely optimal contract is a bonus contract with zero face value of debt. The intuition for the result is similar to that of Theorem 1: bonus contracts minimize the mispricing of securities issued, even though they do not reduce it to zero.

To conclude the characterization, we consider two final cases:

Theorem 3. (PPBE, part (b)) *If $b_h^1 > b_h^{max}$ (or b_h^1 does not exist) and for any $b \in (0, K)$ we have $ZP^1(b) < 0$, then there is a unique pooling equilibrium e_p^* that satisfies D1, at which all types issue a bonus contract with $d_p^* > 0$.*

Proof. See the Appendix.

Unlike the previous cases, where debt has been shown to be suboptimal, the result in Theorem 3 allows debt contracts to arise in equilibrium, since $d_p^* > 0$. To be precise:

Corollary 2. (Optimality of straight debt) *The optimal bonus contract (b_p^*, d_p^*) is ex post equivalent to straight debt if and only $\exists\, b < d_p^*$ such that:*

$$\lambda_h \left[\int_0^{\tilde{x}(b)} x f_h(x) dx + b(1 - F_h(\tilde{x}(b))) \right] + (1 - \lambda_h) \left[\int_0^{\tilde{x}(b)} x f_l(x) dx + b(1 - F_l(\tilde{x}(b))) \right] = I$$

---

if D1 is imposed.

The Corollary follows immediately from two observations. First, straight debt is the optimal monotonic contract (Nachman and Noe, 1994). Second, if $b_{\boldsymbol{\lambda}} < d_p^*$, there does not exist a feasible non-monotonic contract that is ex post distinguishable from debt. Importantly, debt is never uniquely optimal because its payoff can *always* be replicated by a bonus contract with a higher threshold and identical face value d_p^*. The Corollary gives us a necessary and sufficient condition for the optimality of debt.

Finally, observe that both Theorem 3 and its corollary rely on the distribution of earnings being bounded above. For this reason, whenever the earnings distribution is not bounded above they describe empty sets. Such result would hold, for instance, whenever earnings belong to the normal or exponential family.

Theorem 4. (Unbounded support) *If the distribution of earnings is unbounded above, i.e. $K \to \infty$, then straight debt is suboptimal regardless of parameter values.*

Proof. See the Appendix.

In this section we characterized the set of equilibria of the model. Bonus contracts are always optimal, and they provide necessary and sufficient conditions to characterize the unique equilibrium allocation of the game. Debt contracts only arise as a corner solution, when limited liability is binding and the earnings distribution is bounded above.

6 Secret Borrowing and Output Diversion

In general, manipulation could be performed both upwards (window dressing) and downwards (diverting output). In this section we extend our model allowing the entrepreneur both to divert the output and to engage in window dressing.

In particular, assume that the profit manipulation technology $M(x)$ is characterized by two functions: $\underline{\eta}(x)$ and $\overline{\eta}(x)$. The entrepreneur can divert resources up to $x - \underline{\eta}(x)$ and window dress (perhaps by means of secret borrowing) the earnings up to $x + \overline{\eta}(x)$. Hence, the convex set of feasible messages when the realized earnings are x is given by $M(x) \equiv [x - \underline{\eta}(x),\, x + \overline{\eta}(x)]$. As before we assume that the manipulation technology is common knowledge at $t = 0$.

In addition to Assumption 3, we impose the following restrictions on the lower bound of the manipulation technology:

Assumption 3a: *The set of admissible profit manipulation technologies for any re-*

alized earnings $x \in X$ is given by: **(A3a)**

$$M(x) \equiv [x - \underline{\eta}(x),\ x + \overline{\eta}(x)]$$

where $\underline{\eta}(x)$ and $\overline{\eta}(x)$ are $\mathbb{C}^1$ functions such that for every x: (i) $\underline{\eta}(x) \leq x$ and $\underline{\eta}'(x) \in [0, 1)$; (ii) $\overline{\eta}(x) \leq K - x$ and $\overline{\eta}'(x) \geq -1$.

The following comments are due:

1. We rule out unbounded diversion which would trivially lead to no financing;
2. We could have modeled diversion as *output destruction*, in which case the entrepreneur could not put the diverted amount in his pocket. However, in such a scenario the entrepreneur would be indifferent between diverting and not in equilibrium, making such possibilities useless;
3. Finally, we assume that $M(x)$ is a convex set. This is not without loss of generality, however we find it natural because if the entrepreneur can divert k dollars from the project to his own accounts, we believe he should be able to divert also $k - \epsilon$ for $\epsilon > 0$. Further, the assumption greatly simplifies the analysis.

Let us consider debt and bonus payoffs contracts under the output diversion possibilities. Like any other security with positive expected value, a debt contract provides incentives to divert output for some $x \in X$. In order to characterize the real payoff of a standard debt contract, we need to introduce some additional notation. In particular, consider a debt contract with a face value d and suppose that $K - \underline{\eta}(K) > d$. Define as δ the highest threshold such that output diversion is profitable for entrepreneurs, i.e.:

$$\delta \equiv \max_{x \in X} \{x | x - \underline{\eta}(x) < d\}.$$

Such point exists and is unique by the intermediate value theorem due to continuity and monotonicity of function $x - \underline{\eta}(x)$ (see Assumption 3a) and the fact that $K - \underline{\eta}(K) > d$ and $-\underline{\eta}(0) < d$. If $K - \underline{\eta}(K) \leq d$, instead, then simply set $\delta \equiv K$. Lemma 4 provides a characterization of the real payoff of a standard debt contract.

Lemma 5. (The Real Payoff of Debt under Output Diversion) *Given any debt security s with fixed repayment d, the entrepreneur optimally reports:*

$$m^*(x|s) = \begin{cases} x - \underline{\eta}(x), & \text{if } x \leq \delta, \\ x, & \text{otherwise.} \end{cases}$$

Proof. The Lemma follows trivially from the definition of δ, which is guaranteed to exist and be unique by Assumption 3a. *Q.E.D.*

The dashed curve in Figure 7 depicts the real payoff of a standard debt contract.

Insert Figure 7

We next characterize the optimal ex post accounting fraud under bonus contracts. Consider a bonus contract (b, d) such that $\bar{\eta}(0) < b$.

Lemma 6. (The Real Payoff of a Bonus Contract under Output Diversion) *Given any bonus contract s with fixed repayment d and threshold b, we have two cases:*

1. *If $b - d > \underline{\eta}(b)$, then the entrepreneur optimally reports,*

$$m^*(x|s) = \begin{cases} x - \underline{\eta}(x), & \text{if } \ x < \max\{\tilde{x}(b), \delta\}, \\ b, & \text{if } \ x \in [\max\{\tilde{x}(b), \delta\}, b), \\ x, & \text{otherwise.} \end{cases}$$

2. *If $b - d \leq \underline{\eta}(b)$, then the real payoff of a bonus contract is equivalent to that of a debt contract (see Lemma 5).*

Proof. See the Appendix

Figure 5 depicts the real payoff of a bonus contract for the cases of two different levels of profit manipulation. In Panel A of Figure 5, $\tilde{x}(b) > \delta$ and the real payoff is *not* ex post monotonic. In Panels B and C of Figure 5, $\tilde{x}(b) < \delta$ and the real payoff is ex post equivalent to that of a debt contract with face value d.

Insert Figure 5

6.1 Security Design under Output Diversion

In this section we characterize the set of Perfect Bayesian Equilibria under secret borrowing and output diversion. Condition (4) can be rearranged as

$$\int_0^{\max\{\delta_h, \tilde{x}_h(b_h)\}} \big(x \;-\; \underline{\eta}(x)\big)\Big[dF_h(x) - dF_l(x)\Big] \\ + \;\; \Big[F_l\big(\max\{\delta_h, \tilde{x}_h(b_h)\}\big) - F_h\big(\max\{\delta_h \tilde{x}_h(b_h)\}\big)\Big] d_h \leq 0.$$

Similarly to the previous section (no output diversion), the incentive constraint can only be satisfied when $\max\{\delta_h, \tilde{x}_h(b_h)\} = \tilde{x}_h(b_h)$. This condition imply that $\tilde{x}_h(b_h) - \underline{\eta}(b_h) \geq d_h$, which together with the fact that $\tilde{x}_h(b_h) = b_h - \bar{\eta}(\tilde{x}_h)$ lead to $b_h - d_h > \underline{\eta}(b_h)$. Rewrite the incentive compatibility constraint as:

$$\int_0^{\tilde{x}_h(b_h)} \big(x - \underline{\eta}(x)\big) \Big[dF_h(x) - dF_l(x)\Big] + \Big[F_l\big(\tilde{x}_h(b_h)\big) - F_h\big(\tilde{x}_h(b_h)\big)\Big] d_h \leq 0. \tag{9}$$

Substituting $\mathbb{E}_h(s_h) = I$ into (9) we get:

$$\int_0^{\tilde{x}_h(b_h)} \big(x - \underline{\eta}(x)\big) \Big[dF_h(x) - dF_l(x)\Big] + \Big[F_l\big(\tilde{x}_h(b_h)\big) - F_h\big(\tilde{x}_h(b_h)\big)\Big] \left(\frac{I - \int_0^{\tilde{x}_h(b_h)} \big(x - \underline{\eta}(x)\big) dF_h(x)}{1 - F_h(\tilde{x}_h(b_h))}\right) \leq 0.$$

Integrating by parts and rearranging yields:

$$\begin{aligned}\mathrm{IC} \equiv & \underbrace{\int_0^{\tilde{x}_h(b_h)} \Big[F_l(x)\big(1 - F_h(\tilde{x}_h(b_h))\big) - F_h(x)\big(1 - F_l(\tilde{x}_h(b_h))\big)\Big](1 - \underline{\eta}'(x))dx}_{\geq 0 \text{ by FOSD and by Assumption 3a}} \\ & + \underbrace{\Big[F_l\big(\tilde{x}_h(b_h)\big) - F_h\big(\tilde{x}_h(b_h)\big)\Big]}_{\geq 0 \text{ by FOSD}} \underbrace{\Big[I - \tilde{x}_h(b_h) + \underline{\eta}(\tilde{x}_h(b_h))\Big]}_{\text{sign?}} \leq 0. \end{aligned} \tag{10}$$

Recall that $\tilde{x}_h(b_h)$ is defined as the threshold such that for every $x \leq \tilde{x}_h(b_h)$ diversion is weakly preferred to window dressing, and vice versa for $x > \tilde{x}_h(b_h)$. Hence, inequality (10) highlights the key mechanism that underlies separation: setting a threshold b_h that makes $\tilde{x}_h(b_h)$ high enough so that the last bracket becomes not just negative, but low enough that the second line counterbalances the first.

We have a result that is equivalent to Lemma 2:

Lemma 7. (incentive compatibility) *If* $\exists\, b_h \in [0, K]$ *that satisfies* (10)*, then:*

1. *There is a unique* b_h *at which the inequality* (10) *binds. We denote it by* b_h^2*;*
2. *For every* $b_h < b_h^2$ *the inequality* (10) *is violated;*
3. *For every* $b_h \geq b_h^2$ *the inequality* (10) *is satisfied.*

Proof. See the Appendix.

As before, denote by $\tilde{x}_h^{\max}$ the solution to the zero profit condition in a SPBE for type h when the face value of debt $d_h = 0$, and by b_h^{max} the corresponding contractual threshold such that $\tilde{x}_h^{\max} = \tilde{x}_h(b_h^{max})$. We have:

$$\int_0^{\tilde{x}_h^{\max}} \big(x - \underline{\eta}(x)\big) f_h(x)dx = I. \tag{11}$$

The zero-profit condition for a bonus contract $(b, 0)$ in the case of output diversion is:

$$ZP^2(b) \equiv \lambda_h \Bigg[\int_0^{\tilde{x}(b)} \big(x - \underline{\eta}(x)\big) f_h(x)dx\Bigg] + (1-\lambda_h)\Bigg[\int_0^{\tilde{x}(b)} \big(x - \underline{\eta}(x)\big) f_l(x)dx\Bigg] = I. \tag{12}$$

We can formulate an analogous result to Theorems 2, 3 for the case with output diversion:

Theorem 5. (Optimal security design under output diversion)

1. *If b_h^2 exists and $b_h^2 \leq b_h^{max}$ then:*
   a. *there exists a separating equilibrium e_s^* in which a type h entrepreneur issues a bonus contract (b_h^*, d_h^*) such that the financiers make zero profits, and $b_h^* \in [b_h^2, b_h^{max}]$;*
   b. *type l entrepreneurs are indifferent between any contract such that $\mathbb{E}_l(s) = I$, as long as it is not a bonus contract with $d_l^* \leq d_h^*$;*
   c. *no pooling equilibrium satisfies the Intuitive Criterion.*

2. *If $b_h^2 > b_h^{max}$ (or b_h^2 does not exist) and $ZP^2(K) \geq 0$ then:*
   a. *if there exists a solution $b_{\boldsymbol{\lambda}}$ of (12) such that $b_{\boldsymbol{\lambda}} < K - \underline{\eta}(b_{\boldsymbol{\lambda}})$, then there is a unique pooling equilibrium e_p^* satisfying D1. At e_p^*, all types issue a bonus contract $(b_p^*, 0)$;*
   b. *if either the solution $b_{\boldsymbol{\lambda}}$ of (12) satisfies $b_{\boldsymbol{\lambda}} \geq K - \underline{\eta}(b_{\boldsymbol{\lambda}})$ or $ZP^2(b) < 0$ for all $b \in (0, K)$, then there is a unique pooling equilibrium e_p^* satisfying D1, at which all types issue a bonus contract with $d_p^* > 0$.*

3. If $ZP^2(K) < 0$ then there is no financing.

Proof. See the Appendix.

A necessary and sufficient condition for debt to be optional under output diversion is:

Corollary 3. (Optimality of straight debt) *The optimal contract is ex post equivalent to straight debt if and only if there exists a $b - d_p < \underline{\eta}(b)$ such that it solves:*

$$\lambda_h \Bigg[\int_0^{\tilde{x}(b)} \big(x - \underline{\eta}(x)\big) f_h(x)dx + b(1 - F_h(\tilde{x}(b)))\Bigg] + (1-\lambda_h)\Bigg[\int_0^{\tilde{x}(b)} \big(x - \underline{\eta}(x)\big) f_l(x)dx + b(1 - F_l(\tilde{x}(b)))\Bigg] = I$$

Finally, note that Theorem 2 holds also in the case of output diversion.

7 Examples

We now show how our results translate both for some families of widely used distributions that satisfy **A2**: the exponential, normal and log-normal distribution. We also present the case of the distribution with linear density function. The probability distribution functions for this family are linear and given by:

$$f_t(x) = \frac{1}{K}\left[\frac{K-2x}{(\mu_t+1)K}+1\right] \tag{13}$$

with $\mu_{t=l} > 0$ and $\mu_{t=l} < \mu_{t=h}$. This family satisfies strict MLRP, because for any $(t,t') \in T^2$ such that $t' < t$ we have:

$$\frac{\partial}{\partial x}\left(\frac{f_t(x)}{f_{t'}(x)}\right) = \underbrace{\frac{2K(1+\mu_{t'})}{(K(2+\mu_{t'})-2x)^2(1+\mu_t)}}_{>0}[\mu_t - \mu_{t'}]$$

and $\mu_{t'} < \mu_t$ whenever $t' < t$. We solve for optimal contracts for a range of parameter values and for two cases: when output diversion is forbidden and when it is allowed along with window dressing. In this exercise we demonstrate under which parameters we observe separating vs. pooling equilibria as well as equilibria with monotonic vs. non-monotonic contracts. Since debt can only arise only in cases of distributions with bounded support we consider truncated versions of the above mentioned distribution families. The examples are solved under the assumption that (i) $\underline{\eta}(x) = 0$ and $\overline{\eta}(x) = \eta$ for every x, for $\eta \in [0, 10]$ for the case of window dressing only and (ii) $\underline{\eta}(x) = \overline{\eta}(x) = \eta$ for every x, for $\eta \in [0, 4]$ for the case of window dressing and output diversion.[14] Table 1 summarizes the parameter values that we assume.[15]

Insert Table 1 about here

Figures 6 and 7 shows the characterization of equilibria for the examples. The black region is where a separating equilibrium exists (and it is unique, in terms of allocations).

[14]More precisely, at the boundaries of the set X we assume that: (i) whenever $x-\eta < 0$, then $\underline{\eta}(x) = x$; and (ii) whenever $x+\eta > K$, then $\overline{\eta}(x) = K - x$.

[15]The examples include *all* projects such that $\$1 \approx I \leq \mathbb{E}_l(x) < \mathbb{E}_h(x) \approx \$4$.

The gray regions are where a pooling equilibrium exists, and it is unique. Dark gray area corresponds to pooling equilibrium with non-monotonic contract, while light gray area represents cases with debt being the unique pooling equilibrium contract. Finally, the white region is where no financing occurs. Figure 6 corresponds to the case when entrepreneurs can engage only in window dressing and Figure 7 draws types of equilibria when both window dressing and output diversion are allowed.show that the regions described in Theorems 1 and 2 are non-empty.

Insert Figures 6 and 7 about here

8 Extensions

Our model is deliberately stylized in many respects. We now discuss some extensions and we show that the main insights of our analysis do not depend on (i) the type of asymmetric information assumed; (ii) the cardinality of the type space.

Moral hazard. Suppose that in addition to (or instead of) adverse selection, the capital market is subject to moral hazard: borrowers can increase the expected value of their projects by exerting costly (unobservable) effort. As Innes (1990) has shown, as long as the effort decision generates a family of distributions which satisfy the MLRP ordering, non-monotonic contracts dominate debt. Innes (1990) assumes that output is perfectly verifiable, hence his conclusions do not directly apply here. However, it is clear what the driving force of the result is: by choosing a non-monotonic contract borrowers have incentives to exert higher effort, because their payoff is zero unless they obtain high earnings. The optimal contract display a pay-for-performance payoff. In our setup, where output is only coarsely verifiable, optimal contracts are constrained by the profit manipulation possibilities, which reduce the effort exerted by borrowers relative to that in Innes (1990). However, the results are qualitatively similar.

Beyond the two-type case. Consider our assumption that there are just two types. One wonders how the results depend on it. To show that their qualitative properties of the optimal contracts we derive extend to richer type spaces, we characterize the set of separating equilibria for an example that admits a closed form solution.[16]

[16] As we mentioned previously, the existence of separating equilibria is what contrasts the most with the existing results. The solutions for the other, less interesting, cases are available upon request.

In particular, consider the family of linear density functions described by (13). This family has a convenient property: the densities all cross at the same point. Indeed, it is easy to verify that for any pair of types $(t, t') \in T$, $f_t(x) = f_{t'}(x)$ if and only if $x = K/2$, in which case $f_t(x) = 1/K$ for every t.

Consider now the incentive compatibility constraint (6) at the limit bonus contract given by the solution to (7). Differentiating the LHS with respect to the type t' yields[17]:

$$\frac{(\mu_{t'} - \mu_t)(2\eta + 3K - 4b_t^{\max})(b_t^{\max} - 2\eta)^2}{6K(1 + \mu_{t'})(1 + \mu_n)} \tag{14}$$

To achieve separation we must have that $b_t^{\max} > 2\eta$, so the expression is negative if and only if:

$$b_t^{\max} < \frac{\eta}{2} + \frac{3K}{4} \tag{15}$$

We know that the two inequalities describe a non-empty set of earnings realizations if $K > 2\eta$. Using (15), we can sign the derivative of the incentive constraint given by (14) with respect to type t', which is always *negative* in the relevant range, i.e. for every $x \leq K/2$.

The result has an immediate economic interpretation. It tells us that if a type t can separate from a type $t' < t$, then it can separate from any other $t'' \in (t', t)$. We can now restate our Theorem 1 for this case:

Theorem 6. (Optimal security design with N types and linear densities) *Suppose that the pdfs are described by (13) for every $t \in T$, and suppose that $T = \{t_1, t_2, ..., t_N\}$. If there exists a bonus contract with threshold $b_2 \leq b_2^{max}$ that satisfies the incentive constraint for the pair (t_2, t_1), then:*

1. *There exists a fully separating equilibrium e_s^* in which every $t \in T \setminus \{t_1\}$ issue a contract as in (3) such that the financiers make zero profits. The contracts are such that $d_n < ... < d_2$;*

2. *Type t_1 is indifferent between any contract such that $\mathbb{E}_1(s) = I$. If it is a bonus contract, though, it must be such that $d_1 > d_2$;*

3. *No pooling equilibrium satisfies the Intuitive Criterion.*

[17]Technically, we can take such a derivative only if we assume a continuum type space. We suppose so, and later we shall draw a finite set of types from such a continuum.

Proof. See the Appendix

Theorem 6 is given for a specific distribution, as the general case is difficult to analyze.[18] However, it clearly shows that our main result on the signaling property of capital structure does not depend on the two-type assumption.

9 Conclusion

We have shown that the optimal financial contract under ex ante asymmetric information, limited liability and ex post profit manipulation (window dressing) has the following features: (i) it is non-monotonic in earnings; (ii) it exhibits profit manipulation on-the-equilibrium path. That is, the standard justification for restricting attention to monotonic, manipulation-proof securities is not sound. The results suggest that ex ante asymmetric information is not sufficient to theoretically justify the optimality and the widespread use of debt contracts. We derive necessary and sufficient conditions for monotonic securities to arise in equilibrium. Monotonic securities are never uniquely optimal, and they may prevail only if both earnings are bounded, and feasibility is binding.

References

Antic, N., "Contracting with unknown technologies," *Working paper*, 2014.

Axelson, U., "Security design with investor private information," *The Journal of Finance*, 2007, *62* (6), 2587–2632.

_, P. Strömberg, and M.S. Weisbach, "Why are buyouts levered? The financial structure of private equity funds," *The Journal of Finance*, 2009, *64* (4), 1549–1582.

Bhattacharya, S., "Nondissipative Signaling Structures and Dividend Policy," *Quarterly Journal of Economics*, 1980, *95* (1), 1–24.

Bolton, P. and D. S. Scharfstein, "A theory of predation based on agency problems in financial contracting," *The American Economic Review*, 1990, pp. 93–106.

[18] To prove that the incentive constraints are ordered in the type space one deals with two countervailing forces: on the one hand, lower quality types have more to gain by mimicking higher ones. But, on the other hand, they are the ones for whom the costs of mimicking are the highest. Which of these two forces prevails is clear with the analytically tractable linear densities, but not for general distributions. One can prove graphically that the order of incentive constraints holds also for the exponential and truncated normal distributions. Such results are available from the authors upon request.

Cho, I. and D. M. Kreps, "Signaling games and stable equilibria," *The Quarterly Journal of Economics*, 1987, *102*, 179–221.

DeMarzo, P. and D. Duffie, "A liquidity-based model of security design," *Econometrica*, 1999, *67* (1), 65–99.

_ **, I. Kremer, and A. Skrzypacz**, "Bidding with Securities: Auctions and Security Design," *American Economic Review*, 2005, *95* (4), 936–959.

Gale, D. and M. F. Hellwig, "Incentive-compatible debt contracts: the one period problem," *Review of Economic Studies*, 1985, *52*, 647–664.

Gorbenko, A. S. and A. Malenko, "Competition among sellers in securities auctions," *The American Economic Review*, 2011, *101* (5), 1806–1841.

Green, J. R. and J. J. Laffont, "Partially verifiable information and mechanism design," *The Review of Economic Studies*, 1986, *53* (3), 447–456.

Hart, O. and J. Moore, "Default and Renegotiation: A Dynamic Model of Debt," *Quarterly Journal of Economics*, 1998, pp. 1–41.

Hebert, B., "Moral Hazard and the Optimality of Debt," *Available at SSRN 2185610*, 2015.

Inderst, R. and H. M. Mueller, "Informed lending and security design," *The Journal of Finance*, 2006, *61* (5), 2137–2162.

Innes, R., "Limited liability and incentive contracting with ex-ante action choices," *Journal of Economic Theory*, 1990, *52*, 45–67.

_ , "Financial contracting under risk neutrality, limited liability and ex ante asymmetric information," *Economica*, 1993, *60*, 27–40.

Lacker, J. M. and J. A. Weinberg, "Optimal contracts under costly state falsification," *The Journal of Political Economy*, 1989, pp. 1345–1363.

Lee, S. and U. Rajan, "Robust security design," 2016.

Malenko, A. and A. Tsoy, "Asymmetric Information and Security Design under Knightian Uncertainty," 2018.

Myers, S. C. and N. S. Majluf, "Corporate financing and investment decisions when firms have information that investors do not have," *Journal of financial economics*, 1984, *13* (2), 187–221.

Nachman, D. C. and T. H. Noe, "Optimal security design," *Review of Financial Studies*, 1994, *7*, 1–44.

Noe, T. H., "Capital structure and signaling game equilibria," *Review of Financial Studies*, 1988, *1* (4), 331–355.

Philippon, Th. and V. Skreta, "Optimal intervention in markets with adverse selection," *American Economic Review*, 2012, *102*, 1–28.

Scheuer, F., "Adverse selection in credit markets and regressive profit taxation," *Journal of Economic Theory*, 2013, *148* (4), 1333–1360.

Townsend, R. M., "Optimal contracts and competitive markets with costly state verification," *Journal of Economic Theory*, 1979, *21*, 265–293.

10 Figures and Tables

Figure 1: The promised payoff of a debt contract

This figure depicts the promised payoff of a debt contract as a function of realized output.

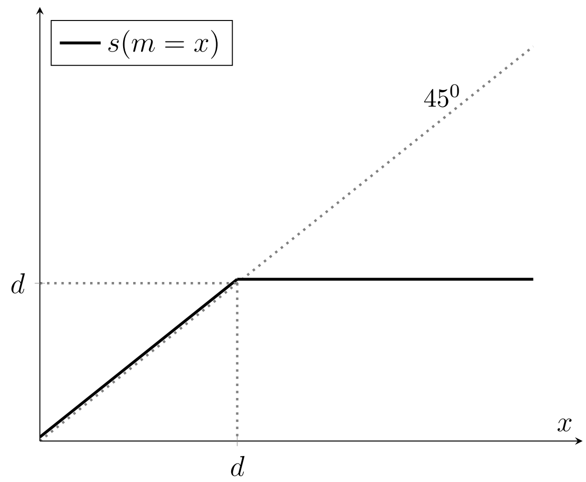

Figure 2: The promised payoff of a bonus contract

This figure depicts the promised payoff of a bonus contract as a function of realized output.

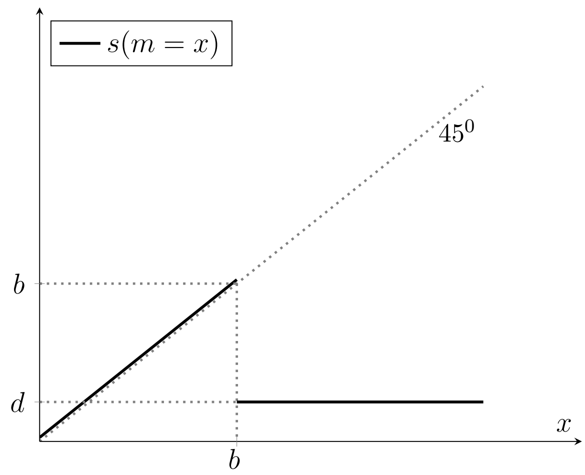

Figure 3: The real payoff of a bonus contract

This figure depicts real (dashed line) and promised (solid line) payoffs of a bonus contract for the cases of two different levels of profit manipulation. The promised payoff is based on the realized output while the promised payoff is based on the declared output. In Panel A, $\tilde{x}(b) > d$ and the real payoff is *not* ex post monotonic. In Panel B, $\tilde{x}(b) \leq d$ and the real payoff is ex post equivalent to that of a debt contract with face value d.

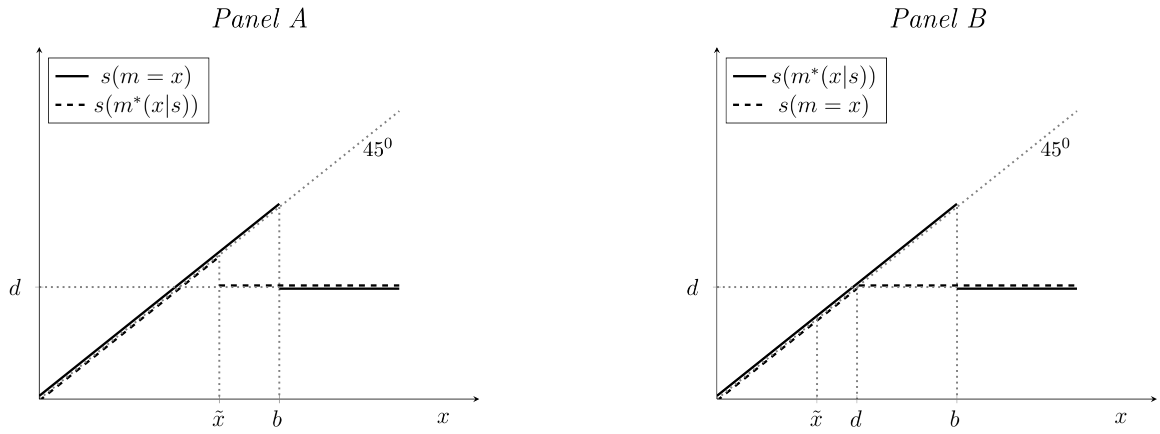

Figure 4: The real payoff of debt under output diversion.

This figure depicts real (dashed line) and promised (solid line) payoffs of a standard debt contract when entrepreneurs have ability to divert the output. The promised payoff is based on the realized output while the promised payoff is based on the declared output.

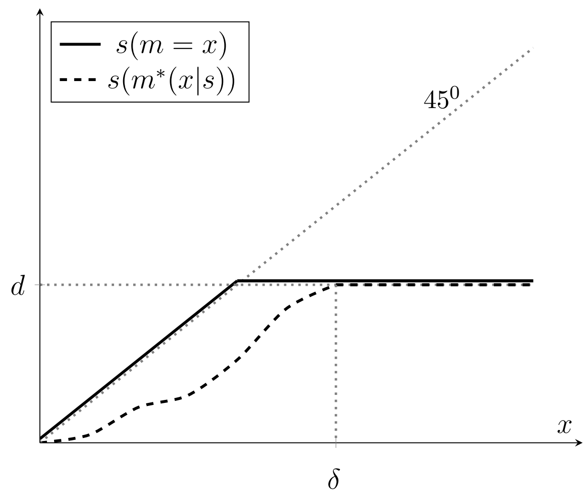

Figure 5: The real payoff of a bonus contract

This figure depicts real (dashed line) and promised (solid line) payoffs of a bonus contract for the cases of two different levels of profit manipulation. The promised payoff is based on the realized output while the promised payoff is based on the declared output. In Panel A, $\tilde{x}(b) > \delta$ and the real payoff is *not* ex post monotonic. I n Panels B and C, $\tilde{x}(b) < \delta$ and the real payoff is ex post equivalent to that of a debt contract with face value d.

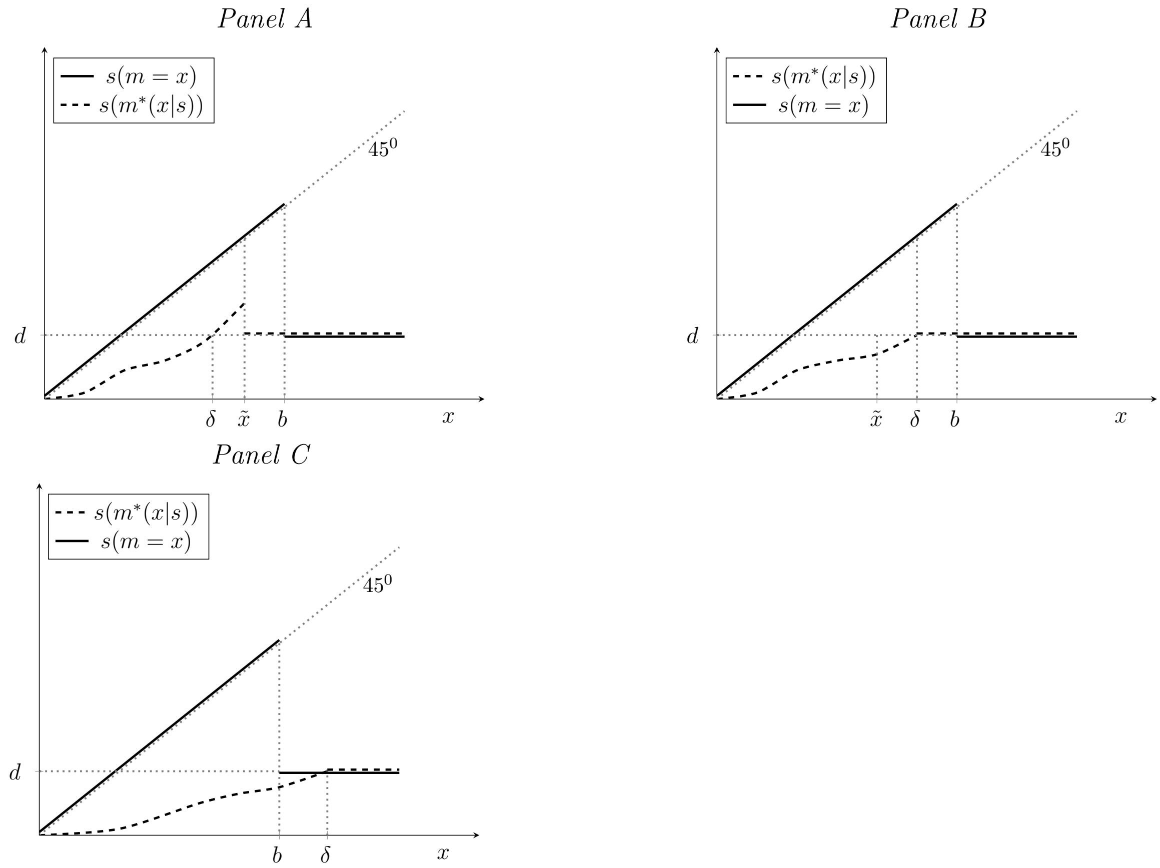

Figure 6: Numerical example: Equilibria types under window dressing only

This figure shows the characterization of equilibria for different distributions of the output. The black region corresponds to parameters values where a unique separating equilibrium exists. The dark gray region is where a unique pooling equilibrium in bonus contracts exists, and it is unique. The light gray area corresponds to monotonic (debt contract) pooling equilibrium. Finally, the white region is where no financing occurs. Four panels corresponds to four different distributions: Exponential (truncated), Normal (truncated), Lognormal (truncated) and Linear.

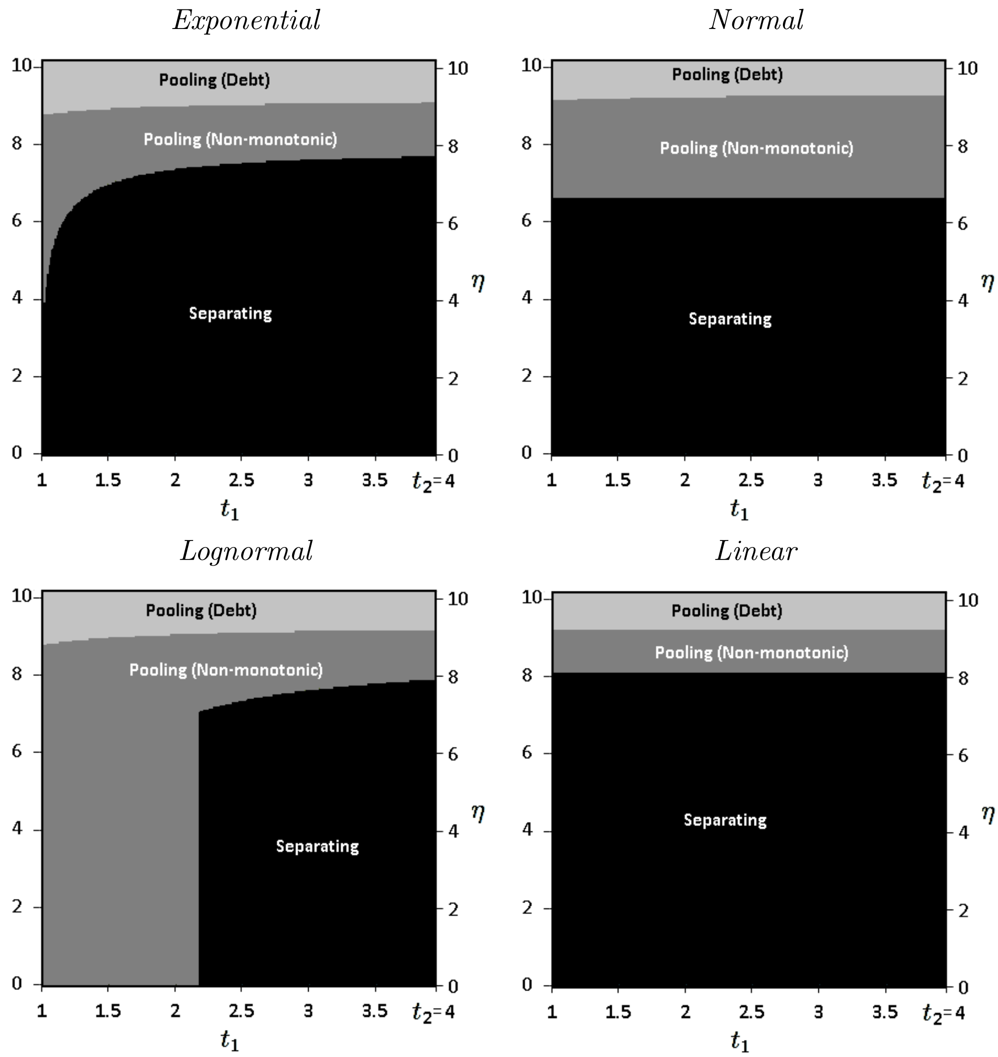

Figure 7: Numerical example: Equilibria types under window dressing and output diversion

This figure shows the characterization of equilibria for different distributions of the output. The black region corresponds to parameters values where a unique separating equilibrium exists. The dark gray region is where a unique pooling equilibrium in bonus contracts exists, and it is unique. The light gray area corresponds to monotonic (debt contract) pooling equilibrium. Finally, the white region is where no financing occurs. Four panels corresponds to four different distributions: Exponential (truncated), Normal (truncated), Lognormal (truncated) and Linear.

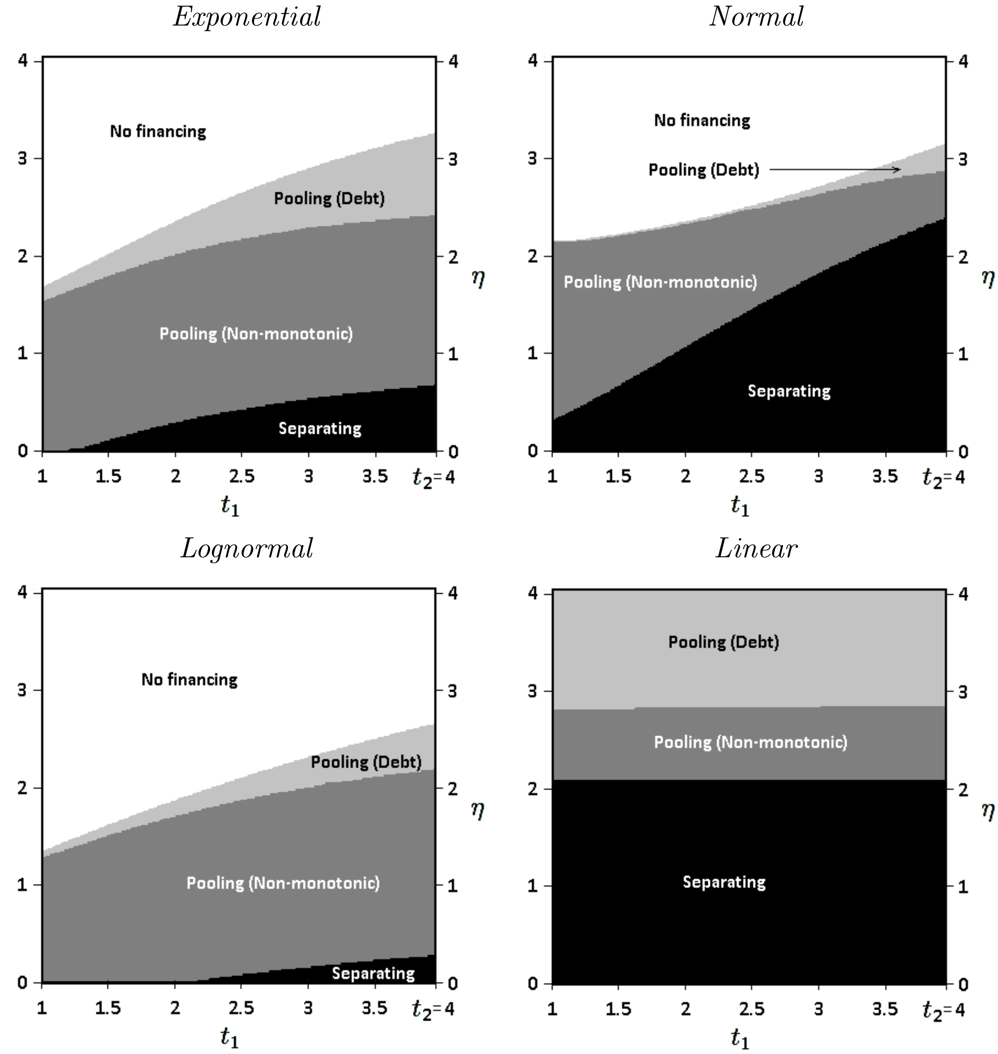

Table 1: Parameter Assumptions

This table summarizes the parameter values for a set of statistical distributions that are used for the numerical calculation of equilibria. The examples are solved under the assumption that $\underline{\eta}(x) = \overline{\eta}(x) = \eta$ for every x, for $\eta \in [0, 4]$. Examples include exponential (truncated), normal (truncated), lognormal (truncated) distributions and a distribution with linear density function.

Family	CDF	I	Parameter
Truncated Exponential	$F_t(x) = \begin{cases} 0, & x < 0, \\ \frac{1-e^{-x/t}}{1-e^{-K/t}}, & x \in [0, K], \\ 1, & x > K \end{cases}$	$K = 10\$$ $I = 1\$$	$t_l \in [1, 4)$ $t_h = 4$
Truncated Normal	$F_t(x) = \begin{cases} 0, & x < 0, \\ \frac{\mathrm{erf}((x-t)/\sqrt{2})-\mathrm{erf}(t/\sqrt{2})}{\mathrm{erfc}((K-t)/\sqrt{2})-\mathrm{erfc}(t/\sqrt{2})}, & x \in [0, K], \\ 1, & x > K \end{cases}$	$K = 10\$$ $I = 1\$$	$t_l \in [1, 4)$ $t_h = 4$
Truncated Lognormal	$F_t(x) = \begin{cases} 0, & x < 0, \\ \frac{\mathrm{erfc}((\ln(x/t)+0.5)/\sqrt{2})}{\mathrm{erfc}((\ln(K/t)+0.5)/\sqrt{2})}, & x \in [0, K], \\ 1, & x > K \end{cases}$	$K = 10\$$ $I = 1\$$	$t_l \in [1, 4)$ $t_h = 4$
Linear	$F_t(x) = \begin{cases} 0, & x < 0, \\ \frac{K(2+t)x-x^2}{K^2(1+t)}, & x \in [0, K], \\ 1, & x > K \end{cases}$	$K = 10\$$ $I = 1\$$	$t_l \in [1, 4)$ $t_h = 4$

Appendix

Proof of Lemma 1

1). Suppose that $\tilde{x}(b) > d$ and $x < \tilde{x}(b)$. In this case secret borrowing it is not optimal as the entrepreneur cannot reach the non-monotonic region of the contract and is better off with truthful reporting. For any $x \in [\tilde{x}(b), b]$ we have that $\bar{\eta}(x) \geq \bar{\eta}(\tilde{x}(b)) = b$ and therefore the entrepreneur can report b. Since $x \geq \tilde{x}(b) > d$ this makes him better off than truthful reporting. Finally, for any $x > b$ secret borrowing does not benefit the entrepreneur and he truthfully reports x.

2). Assume now that $\tilde{x}(b) \leq d$. For any $x < d$ it is not optimal to window dress as the entrepreneur is better off with truthful reporting. For any $x > d \geq \tilde{x}(b)$ the entrepreneur borrows secretly and reports b. *Q.E.D.*

Proof of Lemma 2

Proof. From the definition of b_h^1 we know that, if b_h^1 exists, it must generate a threshold $\tilde{x}_h^1 = \tilde{x}_h\left(b_h^1\right)$ such that:

$$\int_0^{\tilde{x}_h^1}\Big(F_l(x)\big(1-F_h(\tilde{x}_h^1)\big) - F_h(x)\big(1-F_l(\tilde{x}_h^1)\big)\Big)dx + \Big[F_l\big(\tilde{x}_h^1\big) - F_h\big(\tilde{x}_h^1\big)\Big]\Big[I - \tilde{x}_h^1\Big] = 0.$$

The proof consists on showing that the derivative of the incentive constraint with respect to $\tilde{x}_h$ evaluated at $\tilde{x}_h^1$ is strictly negative.

Differentiating the incentive constraint (6) with respect to $\tilde{x}_h$ yields:

$$\begin{aligned}
\frac{\partial IC}{\partial \tilde{x}_h} &= \big(f_l(\tilde{x}_h) - f_h(\tilde{x}_h)\big)[I - \tilde{x}_h] - \big(F_l(\tilde{x}_h) - F_h(\tilde{x}_h)\big) \\
&+ \big(F_l(\tilde{x}_h)[1 - F_h(\tilde{x}_h)] - F_h(\tilde{x}_h)[1 - F_l(\tilde{x}_h)]\big) \\
&- f_h(\tilde{x}_h)\left[\int_0^{\tilde{x}_h} F_l(x)dx\right] + f_l(\tilde{x}_h)\left[\int_0^{\tilde{x}_h} F_h(x)dx\right] \\
&+ \left\{f_l(\tilde{x}_h)\left[\int_0^{\tilde{x}_h} F_l(x)dx\right] - f_l(\tilde{x}_h)\left[\int_0^{\tilde{x}_h} F_l(x)dx\right]\right\}
\end{aligned}$$

where the last row is obtained adding and subtracting the same expression to the derivative, and it is introduced so that the derivative simplifies to:

$$\frac{\partial IC}{\partial \tilde{x}_h} = \big(f_l(\tilde{x}_h) - f_h(\tilde{x}_h)\big)\left[I - \tilde{x}_h + \int_0^{\tilde{x}_h} F_l(x)dx\right] - f_l(\tilde{x}_h)\left[\int_0^{\tilde{x}_h}\big(F_l(x) - F_h(x)\big)dx\right]$$

Evaluating the derivative at $\tilde{x}_h^1$ yields:

$$\frac{f_l(\tilde{x}_h^1) - f_h(\tilde{x}_h^1)}{F_l(\tilde{x}_h^1) - F_h(\tilde{x}_h^1)}\left[\int_0^{\tilde{x}_h^1}\Big(F_h(x)\big(1 - F_l(\tilde{x}_h^1)\big) - F_l(x)\big(1 - F_h(\tilde{x}_h^1)\big)\Big)dx\right] +$$
$$\big(f_l(\tilde{x}_h^1) - f_h(\tilde{x}_h^1)\big)\left[\int_0^{\tilde{x}_h^1} F_l(x)dx\right] - f_l(\tilde{x}_h^1)\left[\int_0^{\tilde{x}_h^1}\big(F_l(x) - F_h(x)\big)dx\right] =$$
$$= \frac{f_l(\tilde{x}_h^1)(1 - F_h(\tilde{x}_h^1))}{F_l(\tilde{x}_h^1) - F_h(\tilde{x}_h^1)}\left[\int_0^{\tilde{x}_h^1}(F_h(x) - F_h(x))dx\right] - \frac{f_h(\tilde{x}_h^1)(1 - F_l(\tilde{x}_h^1))}{F_l(\tilde{x}_h^1) - F_h(\tilde{x}_h^1)}\left[\int_0^{\tilde{x}_h^1}(F_h(x) - F_h(x))dx\right]$$

which can be rewritten as:

$$\underbrace{\frac{\int_0^{\tilde{x}_h^1}(F_l(x) - F_h(x))dx}{F_l(\tilde{x}_h^1) - F_h(\tilde{x}_h^1)}}_{>0 \text{ by FOSD}}\underbrace{\Big[f_l(\tilde{x}_h^1)[1 - F_h(\tilde{x}_h^1)] - f_h(\tilde{x}_h^1)[1 - F_l(\tilde{x}_h^1)]\Big]}_{<0 \text{ because MLRP implies HRO}} < 0$$

The fraction is strictly positive whenever $\tilde{x}_h^1 > 0$, which is clearly satisfied at every contract that implements investment. The second bracket is negative because we assumed strict MLRP, and it is well known that strict MLRP implies the strict HRO - hazard rate ordering - which in turns guarantees that the bracket is strictly negative.

An immediate consequence of the strict inequality is that if the incentive constraint crosses zero, it must do so only once. Lemma 2 follows. *Q.E.D.*

Proof of Theorem 1

Proof. **Claims 1 and 2:** Follow from the previous discussion, and the fact that if t_l issues a bonus contract with $d_l^* \leq d_h^*$ that breaks even on his type, the good type would mimic him and he would end up with a rate of repayment higher than one.

Claim 3: Suppose that all agents are in the pooling equilibrium $\hat{e}$ of the game. Type $t = h$ (the better type) is certainly paying a strictly positive net rate of return to the investors. No type other than $t = h$ is in the set Θ for a security s' that satisfies (6). Hence, the Intuitive Criterion implies that the investor must believe that the deviation comes from type h with probability one. If this is so, the deviation is profitable and the pooling equilibrium does not satisfy the Intuitive Criterion. *Q.E.D.*

Proof of Corollary 1

Proof. To establish this result, some preliminary steps are required.

Denote the bonus contract with $b_h = b_h^{max}$ and $d_h = 0$ as s^*, and compare it with another generic security s such that $\mathbb{E}_h[s^*] = \mathbb{E}_h[s] = I$. Define the following sets:

$$\Pi_+(s) \equiv \{m | s^*(m = x) > s(m = x)\}$$

$$\Pi_-(s) \equiv \{m | s^*(m = x) < s(m = x)\}$$

Lemma 3. *For every pair $(m_l = x_l, m_h = x_h)$ in X^2 such that $m_l \in \Pi_+$ and $m_h \in \Pi_-$ we have $m_h > m_l$. Moreover, $m^*(x_l|s^*) \geq m^*(x_l|s)$ and $m^*(x_h|s^*) \leq m^*(x_h|s)$.*

Proof. First notice that $\Pi_+(s) = \emptyset$ if and only if $\Pi_-(s) = \emptyset$, because $f_t(x) > 0$ for every $x \in [0, K]$, for every $t \in T$. In this case the lemma is not very useful, but it is still satisfied. Suppose $\Pi_+(s)$ is non-empty. Because of limited liability, it must be the case that $m_h > b_h^{max}$ for every $m_h \in \Pi_-$, and $m_l < b_h^{max}$ for every $m_l \in \Pi_+(s)$. As for the claim about the real payoff, it follows directly from the shape of s^*. *Q.E.D.*

Lemma 4. *Denote the bonus contract with $b_h = b_h^{max}$ and $d_h = 0$ as s^*. For any generic security s such that $\mathbb{E}_h[s^*] = \mathbb{E}_h[s] = I$, we have that $\mathbb{E}_l[s^*] > \mathbb{E}_l[s]$.*

Proof. The only interesting case is, again, when $\Pi_+(s)$ is non-empty (else the lemma holds trivially). Suppose so. Furthermore, suppose we move from s^* toward s through a series of steps such that in each step we create a security s' such that $\mathbb{E}_t[s'] = I$, but there exists a small interval $dx_a \in \Pi_+(s)$ such that $s'(m^*(dx_a)) < s^*(m^*(dx_a))$ and this change is compensated by inducing a change in the real payoff for another small interval $dx_b \in \Pi_-(s)$ so that $s'(m^*(dx_b)) > s^*(m^*(dx_b))$[19]. Then,

$$\begin{aligned}
\mathbb{E}_l[s^*] - \mathbb{E}_l[s'] &= f_l(x_a)\big[s^*(m^*(dx_a)) - s'(m^*(dx_a))\big] + \ f_l(x_b)\big[s^*(m^*(dx_b)) - s'(m^*(dx_b))\big] \\
&= \underbrace{\big[s^*(m^*(dx_b)) - s'(m^*(dx_b))\big]}_{<\,0 \text{ by construction}} \underbrace{\left(\frac{f_l(x_b)}{f_h(x_b)} - \frac{f_l(x_a)}{f_h(x_a)}\right)}_{<\,0 \text{ by MLRP}} f_t(x_b) > 0,
\end{aligned}$$

where the second equality comes from $\mathbb{E}_h[s^*] = \mathbb{E}_h[s']$. The iteration of this procedure one step at a time concludes the proof. *Q.E.D.*

Because of Lemma 4 we know that $\mathbb{E}_h[s^*] - \mathbb{E}_l[s^*] < \mathbb{E}_h[s] - \mathbb{E}_l[s]$, for every $\mathbb{E}_h[s^*] = \mathbb{E}_h[s] = I$. The Corollary follows. *Q.E.D.*

Proof of Theorem 2

[19]In both cases, construct the interval such that it is of equal length as the pdf centered at the two points: $f(x_a)$, $f(x_b)$

Proof. **Existence:** Suppose there exists an $b_{\boldsymbol{\lambda}}$ that satisfies the pooling zero profit condition. Define the security s_p so that: $d_p = 0$ and $b_{\boldsymbol{\lambda}}$ solves the pooling zero profit condition. Moreover, suppose that the market posterior is equal to the prior at s_p, and it is $\lambda_h = 0$ at any other $s' \neq s_p$ such that $s' \in S$. Then, all types issuing s_p is an equilibrium. It remains to show that it satisfies D1. In particular, we need to prove that $D(1|s') \cup D^0(1|s') \not\subset D(2|s')$ for every $s' \neq s_p$ such that $s' \in S$. There are two cases:

1. If $\mathbb{E}_l[s'] < \mathbb{E}_l[s_p]$, then $D(1|s') = [I, \infty)$. Hence $D(2|s') \subseteq D(1|s') \cup D^0(1|s')$;

2. If $\mathbb{E}_l[s'] \geq \mathbb{E}_l[s_p]$, Lemma 4 implies $\mathbb{E}_h[s'] \geq \mathbb{E}_h[s_p]$ as well. But we can say more:

   Suppose we move from s_p to s' through a series of consecutive steps (i.e. interim contracts s'') such that in each step we induce an increase in the real payoff of s_p by raising $s''(m_k = x_k)$ for some $x_k \in X$. Clearly, it must be that $x_k \geq b_{\boldsymbol{\lambda}}$. Notice that because s_p is a pooling equilibrium, it must be that it does not satisfy (6). Hence, because of MLRP, at x_k we must have $f_l(x_k) < f_h(x_k)$ - i.e. x_k must exceed the (unique) crossing point of the two densities. Therefore:

   $$\begin{aligned}\mathbb{E}_l[s''] - \mathbb{E}_l[s_p] &= f_l(x_k)\big[s''(m^*(x_k|s'')) - s_p(m^*(x_k|s_p))\big] \\ &= f_l(x_k)(s''(m^*(x_k|s'')) < f_h(x_k)(s''(m^*(x_k|s'')) = \mathbb{E}_h[s''] - \mathbb{E}_h[s_p].\end{aligned}$$

   Iterating the same logic we conclude that $\mathbb{E}_h[s'] - \mathbb{E}_h[s_p] > \mathbb{E}_l[s'] - \mathbb{E}_l[s_p]$. It follows that at e_p^* it must be the case that, for all $P^* \geq I$:

   $$\big(V_h' - V_h^*\big) - \big(V_l' - V_l^*\big) = \big(\mathbb{E}_l[s'] - \mathbb{E}_l[s_p]\big) - \big(\mathbb{E}_h[s'] - \mathbb{E}_h[s_p]\big) < 0,$$

   which implies that $D(2|s') \subseteq D(1|s') \cup D^0(1|s')$ again.

Uniqueness: From Corollary 1 we know that there can only exist other pooling equilibria if the conditions required for Theorem 1 to apply do not hold. We now show that if there exists an $b_{\boldsymbol{\lambda}} \in (b_h^{max}, K)$ such that (8) is satisfied, then every pooling equilibrium e' of the game such that $e' \neq e_p^*$ does not satisfy D1.

Consider a generic $e' \neq e_p^*$. The above analysis and Lemma 4 imply that there exists s' such that $\mathbb{E}_l[s'] \geq \mathbb{E}_l[s_p]$ but $\mathbb{E}_h[s'] < \mathbb{E}_h[s_p]$. Then the logic of the previous proof (point 2 above) is reversed. We conclude that such a equilibrium does not satisfy D1. *Q.E.D.*

Proof of Theorem 3

Proof. Part (1) can be proved in the same fashion as Theorem 2, with a twist: now it must be the case that a bonus contract with $d = 0$ cannot satisfy the pooling zero profit condition. Hence, we start by finding the minimum $d > 0$ such that the condition can be satisfied. Then, the result follows from the logic of the previous proof.

Part (2) follows from the fact that with a contract as in (1) we are hitting the upper bound of the distribution of earnings. If such a contract does not exist, then any other security could not break even for the financier. *Q.E.D.*

Proof of Theorem 4

Proof. When $K \to \infty$ there always exists $b_h^{\max}$ such that the pooling zero profit condition is satisfied for $d_p = 0$, because $f_t(x) > 0$ for every $x \in X$ and $t \in T$. Moreover, regardless of the extent profit manipulation, as long as it is bounded, the pooling contract with $d_p = 0$ has a real payoff which is non-monotonic. As a result, any contract with a monotonic real payoff cannot be part of an equilibrium that satisfies D1. *Q.E.D.*

Proof of Lemma 6

1. Suppose that $b - d > \underline{\eta}(b)$ and $\tilde{x}(b) \leq \delta$. In this case, for any $x < \delta$ it is not optimal to window dress as the entrepreneur is better off with output diversion. For any $x \in [\delta, b)$ we have that $\bar{\bar{\eta}}(x) \geq \bar{\bar{\eta}}(\tilde{x}(b)) \geq b$ and therefore the entrepreneur can report b which makes him better off than diverting the output. Finally, for any $x > b$ neither output diversion nor windows dressing benefits the entrepreneur and he truthfully reports x.

Suppose now that $\tilde{x}(b) > \delta$. In this case the entrepreneur diverts output for any $x < \tilde{x}(b)$ since it is impossible to reach the bonus region $\{x \colon x \geq b\}$ by means of window dressing. For any $x \in [\tilde{x}(b), b)$ window dressing is beneficial since $d = \delta - \underline{\eta}(\delta) < x - \underline{\eta}(x)$. Finally, for any $x > b$, as above, the entrepreneur truthfully reports x.

2. Assume now that $b - d \leq \underline{\eta}(b)$. Note that in this case $\delta > b$. For any $x < \delta$ it is not optimal to window dress as the entrepreneur is better off with output diversion. For any $x > \delta > b$ the entrepreneur truthfully reports x. *Q.E.D.*

Proof of Lemma 7

Similarly to the proof of Lemma 2. *Q.E.D.*

Proof of Theorem 5

The proof is analogous to Theorems 2 and 3. *Q.E.D.*

Proof of Theorem 6

From the analysis above we know that if the incentive constraint for the pair (t, t_1) is satisfied, then the one for any pair (t, t') with $t' \in T \setminus \{t_1\}$ also holds. Because of our distributional assumption, if the condition holds for t_h, then it holds for all $t \in T \setminus \{t_1, t_2\}$ and a fully separating equilibrium in which financiers make zero profits exists. That no pooling equilibrium is reasonable can be proved as in the two-type case. *Q.E.D.*