

# Robust Opinion Aggregation and its Dynamics\*

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## Abstract

We consider a general model of non-Bayesian social learning. We start by the standard foundation of the DeGroot style models, but we relax the demanding parametric assumptions that are usually imposed on the data generating process and the payoff functions of the agents. With this, we obtain opinion aggregators that are axiomatically characterized by the functional properties of normalization, monotonicity, and translation invariance. Beside the more attractive microfoundation, the obtained opinion aggregators allow for several economically relevant patterns ruled out by the linear model. For instance, agents can feature dislike (or attraction) for extreme opinions, confirmatory bias, as well as discard information obtained from sources perceived as redundant. We also show that under this weaker assumptions is still possible to explore the standard questions addressed by the linear model, such as the convergence of limit opinions, and the properties of consensus and wisdom for this limit.

## 1 Introduction

Often, economists describe opinion dynamics in a social structure through non-Bayesian learning models. There are at least three good reasons to do so. First, Bayesian inference is not an easy task to implement under the complex information structures that arise in social networks. Indeed, the informational content encoded in others' actions and opinions is hardly assessed by real-life agents. Therefore, it is reasonable

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to expect them to use some form of simpler heuristic. Second, when modeling the evolution of Bayesian updates in a network, tractability is easily lost, especially outside the standard quadratic-Gaussian setting. More profoundly, it is not even clear that a search for the truth only motivates the evolution of opinions in a social network. Often, agents are just trying to either adapt to each other or to adjust on a belief shared by the entire society. In those cases, it is not clear why even a cognitively unbounded agent should adhere to Bayesian updating in adjusting her opinion and behavior.

**The benchmark: DeGroot’s linear model** The tractability issue is satisfactorily addressed by one of the simplest and most analyzed models of opinion aggregation in a group of agents: the *DeGroot’s linear model* (see DeGroot [16], DeMarzo, Vayanos, and Zwiebel [17], Golub and Jackson [27]). It is a discrete time dynamic model where a group of agents starts with initial estimates or opinions and then periodically update them by taking weighted averages of the estimates of each other. Therefore, the opinions of the agents in the society at period  $t$  are given by

$$x^t = Wx^{t-1}$$

where  $W$  is a stochastic matrix and each entry  $w_{ij}$  is interpreted as the *influence* of  $j$  over  $i$ . This model allows describing the dynamics induced by an initial opinion profile  $x^0$  and an influence matrix  $W$ , which is often derived from an underlying network structure and acts as the *opinion aggregator* in the society. The critical feature that makes this model so tractable is the ease to link the properties of the underlying network structure with the long-run evolution of opinions. In particular, conditions on  $W$  under which the sequence of updates converges (to a consensus or not) can be given. The usual interpretation associated with the dynamics just described is one of *naive learning*. Indeed, each agent naively updates her opinion by pooling the opinions she observes from the others and weighting them according to an exogenously given matrix. The resulting process does not follow from the acquisition of new exogenous information but rather from the reciprocal adaptation of the opinions in the group.

**Micro-foundation** DeMarzo, Vayanos, and Zwiebel [17] provide a microfoundation of the DeGroot linear updating based on a *Persuasion bias* model. They consider a directed graph of agents trying to estimate a fundamental parameter  $\mu$ . Agents observe the parameter plus i.i.d. normal error terms with different precisions. The first-period estimate of each agent is the *Maximum Likelihood Estimator* of  $\mu$  using the signal realizations in her neighborhood; however, they may hold wrong beliefs about the precisions of others. Under their assumptions, each estimate is a weighted average

of the signals.<sup>1</sup> In the following periods, agents update their estimates by naively using the same weights. Even if this is not consistent with a proper statistical procedure, DeMarzo, Vayanos, and Zwiebel argue that this can be seen as a boundedly rational heuristic capturing persuasion bias. With this, the resulting dynamics are the same as in DeGroot’s model.<sup>2</sup> Again, note that linearity of aggregation crucially relies on their assumptions on the error terms. If we weaken one among independence and normality of the signals or certainty about the precision of others, then linearity is lost. More generally, we consider agents that perform *robust estimation* (as defined in Huber [32]) given that they are uncertain about the data generating process.<sup>3</sup> This procedure leads to the class of robust opinion aggregators.

**Robust opinion aggregators** Surprisingly enough, the resulting aggregation of past opinions preserves some of the nice and tractable features of DeGroot’s model. These natural properties\axioms will define the class of *robust opinion aggregators*:<sup>4</sup>

1. **Normalization:** Every time the agents have reached a consensus none of them further updates their opinion;
2. **Monotonicity:** If we consider two profiles of opinions such that the first one dominates (according to the coordinatewise order) the second, then this relation is preserved after aggregation;
3. **Translation invariance:** The way every agent aggregates information does not change if the opinion of *every* agent is suddenly shifted by the *same constant*.

On the one hand, these simple properties are appealing because they naturally arise when we generalize the foundations of DeGroot’s model and allow for effects that are prevented by linear aggregation (e.g., distrust for extreme opinions, assortativeness, confirmatory bias). On the other hand, the resulting opinion dynamics would undoubtedly look different from the ones described by the standard linear updating

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<sup>1</sup>As argued by Golub and Sadler [30] and Golub and Jackson [28], under a very diffuse prior about  $\mu$ , also the Bayesian estimator of  $\mu$  takes the form of a weighted average with suitable weights.

<sup>2</sup>They also allow agents to vary over time the weight they give to their own past beliefs relative to the others. For the generalization of their procedure in our model see Section 6.6.

<sup>3</sup>In his seminal paper (Huber [32]), which is now almost sixty years old, Huber writes: “*It is interesting to look back to the very origin of the theory of estimation, namely to Gauss and his theory of least squares. Gauss was aware that his main reason for assuming an underlying normal distribution and a quadratic loss function was mathematical, i.e. computational, convenience. In later times, this was often forgotten, partially because of the central limit theorem. However, if one wants to be honest, the central limit theorem can at most explain why many distributions occurring in practice are approximately normal.*”

<sup>4</sup>See Section 2 for the formal definitions of these properties.

rule. Are these new dynamics completely undisciplined? Is it still possible to obtain convergence? Moreover, if the answer is yes, can we say something on the rate of convergence and the formation of consensus? Does the crowd become wise in the limit a la Golub and Jackson [27]? The second objective of the present work is to answer these and other questions that may arise given our general framework.

**The dynamics of robust aggregation** Once we have identified the class of naive updating rules that we want to study, we switch to the analysis of the dynamics that they induce. First, we show that the *time averages* of the sequences of opinions induced by robust aggregators *uniformly* converge. Rephrased in other words, the updates of the agents either converge or eventually oscillate. The proof of this first result exploits mathematical tools that are, to the best of our knowledge, novel for this literature. Despite being an essential result about what an external observer can learn by observing the evolution of opinions, the convergence of time averages is usually not satisfactory for analysis of learning and aggregation of agents' opinions. Therefore, we next look for conditions that characterize, or at least ensure, proper convergence of the iterates of our maps. It turns out that, for robust aggregators, *asymptotic regularity* characterizes both local and global convergence. Nevertheless, we aim to provide conditions that imply convergence and have transparent economic interpretations that can be obtained from our foundations. Hence, we analyze three properties of the network structure that guarantee convergence of the limit opinions:

1. **Self-influence:** For every given profile of opinions, the update of each agent is influenced by her own past opinion;
2. **Uniform-common influencer:** there is at least one source of information that is trusted by the entire society;
3. **Strong connectedness:** For every pair of agents in society, there is a sequence of agents connecting them.

Each of the previous conditions alone is sufficient to obtain convergence and, for the last two, we provide bounds for the corresponding rates. Furthermore, in the cases of 2 and 3, convergence to consensus is implied. Importantly, all these conditions are related to a network structure among agents that we derive from the aggregator. Our model suggests caution in concluding that consensus can be reached after having studied the *local* network properties. Indeed, our robust aggregators highlight that the choice of trusting someone cannot generally be disentangled from the opinion she is expressing.

Next, we zoom our focus on a subclass of robust aggregators that satisfy *comonotonic additivity*: the aggregation is linear whenever restricted to comonotonic vectors of opin-

ions. We call the elements of this class *Choquet aggregators* because they have a representation in the form of a Choquet Integral. The median together with all the quantile functions as well as the order statistics are examples of such aggregators. Aside from the nice representation, these aggregators have some useful properties and interpretation. From a tractability point of view, their aggregation procedure corresponds to a linear aggregation using a matrix that is selected from a finite set of possible alternatives. Which matrix is used depends on the particular ranking of opinions. On the interpretation, they can capture the fact that agents overweight either the extreme or the intermediate opinions in the networks.

**Vox populi, vox dei** Inspired by Golub and Jackson [27], we test the *wisdom of the crowd* under robust aggregation obtaining mixed findings. First, sufficient conditions for wisdom can be given. On the one hand, similarly to the linear case, wisdom occurs when the influence of each agent vanishes in the limit. On the other hand, this is obtained by further assuming symmetry of the distribution of agents' signals, which is irrelevant in the linear case but plays a key role under general robust aggregation. Furthermore, we link these conditions to our statistical foundation. Finally, we provide a negative result about the robustness of wisdom: every wise aggregator in the sense of Golub and Jackson is arbitrarily close to an unwise robust aggregator.

**Related Literature** Our work falls in the literature on non-Bayesian learning in social networks, pioneered by the seminal papers of DeGroot [16], DeMarzo, Vayanos, and Zwiebel [17], and Golub and Jackson [27]. Among the more recent papers, the one closest to us is Molavi, Tahbaz-Salehi, and Jadbabaie [47]. The first difference concerns the stochastic component of the model. They follow [48] in considering social learning when agents both repeatedly receive signals about an underlying state of the world and naively combine the belief of their neighbors. Instead, we follow the wisdom of the crowd approach of [27] and we study the long-run opinions as the size of the society grows to infinity. The second difference regards the direction of the relaxation of the linearity in the naive-updating rules of the agents. Both papers take an axiomatic approach, specifying some behavioral properties of the opinion aggregators, the main differences being between the assumptions of translation invariance and label neutrality. In the Online Appendix, we show that for the purpose of the the questions we explore, i.e., convergence of limit opinions and the wisdom of the crowd, loglinear aggregators a la [47] can be studied in an equivalent linear system. Since our class of robust aggregators encompasses the linear model, our results cover their aggregators, too. However, notice that the equivalence with linear system may be lost for a problem of learning with repeated signals as the one they analyze in their paper.

Our paper lies at the intersection of a few kinds of mathematical literature: namely, nonexpansive self-maps, fixed point theory, and discrete dynamical systems. We provide here a brief overview. The literature on discrete dynamical systems/repeated averaging shares a common theme. Agents aggregate opinions at each point in time following DeGroot’s rationale, with a time-varying aggregation matrix. This literature is typically concerned in providing the more general conditions possible on the sequence of matrices  $W_t$  which guarantee the convergence of the sequence of updates  $x^t$  to consensus. One of the first papers dealing with such a problem is Chatterjee and Seneta [13]. Krause [39] provides an excellent textbook exposition of the topic and a full characterization of convergence to consensus. In a nutshell, our results differ from the ones above in two dimensions. First, we do not necessarily always obtain convergence of the updates and, even when it happens, consensus does not always realize. Second, even when it is possible to reduce our framework to one of this literature, there are significant differences. Since our opinion aggregators are micro-founded, under mild conditions, they inherit the primitive network structure of the foundation. In turn, this imposes a strong discipline on the sequence of matrices  $W_t$ . This approach comes with two significant benefits. Conceptually, it makes explicit the fact that the matrix  $W_t$  not only depends on time but also on the value of  $x^{t-1}$ . This fact was never fully exploited, and mathematically, this turns out to simplify significantly our proofs which, when dealing with convergence to consensus, rely on a combination of operatorial techniques mixed with simple Markov chains arguments. Krause’s results are related to an underlying network structure by Muller-Frank [49]. Again, the results obtained in that paper rely on a weaker form of internality to the opinions in the neighborhood of each agent. Our opinion aggregators, in general, do not satisfy the strict internality properties. However, note that a particular but important, case of our Theorem 8 can be obtained as a corollary of Theorem 4 in Muller-Frank.

The other literature relevant to our work is the one about nonexpansive maps. Also here, we provide a brief overview.<sup>5</sup> This literature only shares our mathematical framework. It was developed for a completely different reason. In particular, the goal there is finding solutions to functional equations, or more in general fixed points of operators  $T$ , which are not contractions. We exploit some of the techniques coming from this literature (e.g., Baillon, Bruck, and Reich [3]). In particular, our micro-foundations yield that  $T$  might satisfy properties of monotonicity that are not present in this literature. These other properties allow us to obtain sharper results: namely, the rate of convergence is independent of the initial opinion and convergence attains as long as self-influentiality is satisfied.

In general, we consider three dimensions which are not present in the previous set

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<sup>5</sup>See Remarks 3 to 6.

of papers. First, we explore conditions to obtain wisdom of the crowd under robust aggregation. Second, we derive our class of aggregators by generalizing in natural ways the existing statistical and game-theoretical foundations. Finally, we also follow an axiomatic approach building on results from Decision Theory to obtain a useful representation of our operators.

**Outline** The paper is structured as follows. Section 2 presents the definitions of the mathematical objects used in our analysis. Section 3.1 shows that relaxing the functional form in the most standard micro-foundation of DeGroot’s model leads to our class of robust aggregators. Section 6 describes the long-run evolutions of opinions and the properties of the limit. Finally, Section 7 explores the conditions for obtaining the wisdom of the crowd a la Golub and Jackson [27]. All the proofs are collected in the Appendix.

## 2 Preliminaries

We consider a finite set  $N = \{1, \dots, n\}$  of agents, sometimes called players. We denote by  $I$  a closed interval of  $\mathbb{R}$  with nonempty interior. For example, if  $I = [0, 1]$ , then we interpret a number in this interval as either a measurement of agreement on a particular instance or a subjective probability about a specific event. In what follows, we study maps  $T : B \rightarrow B$  where  $B = \prod_{i=1}^n I = I^n$ . We call these selfmaps  $T$  *opinion aggregators*. With a small abuse of notation, we denote two objects by the letter  $I$ : a closed interval with nonempty interior, and the identity map  $I : B \rightarrow B$ . The context will always clarify unambiguously to which object we are referring. For all Cartesian products  $C = \prod_{i \in N} C_i \subseteq \mathbb{R}^n$  and  $i \in N$ , we define  $C_{-i} = \prod_{j \neq i} C_j$ .

We endow  $\mathbb{R}^n$  with the usual componentwise order. Given two vectors  $x, y \in \mathbb{R}^n$ , recall that they are comonotonic if and only if  $[x_i - x_j][y_i - y_j] \geq 0$  for all  $i, j \in N$ .<sup>6</sup> By  $e \in \mathbb{R}^n$ , we denote the vector whose components are all 1s. We denote by  $D$  the diagonal of  $B$ , that is, the subset of elements of  $B$  whose components are all equal. We denote by  $\Delta$  the collection of probability vectors in  $\mathbb{R}^n$ , that is,  $p \in \Delta$  if and only if  $p_i \geq 0$  for all  $i \in N$  and  $\sum_{i=1}^n p_i = 1$ . We endow  $B$  with the topology induced by the supnorm  $\|x\|_\infty = \sup_{i \in \{1, \dots, n\}} |x_i|$ . Given an opinion aggregator  $T : B \rightarrow B$  and  $x \in B$ , the sequence  $\{T^t(x)\}_{t \in \mathbb{N}}$  will be called the *sequence of updates of  $x$* . Often, in denoting a sequence of updates and particularly if interested in a local behavior, we will call  $x^0$  the initial value, rather than  $x$ .

Among other things, we are concerned about the convergence and the rate of convergence of these sequences. We will be dealing with two kinds of convergence: the

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<sup>6</sup>In this paper, vectors will always be interpreted as column vectors, unless otherwise specified.

standard one induced by the supnorm as well as Cesaro convergence, that is,

$$C - \lim_t T^t(x) \stackrel{def}{=} \lim_\tau \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x)$$

where the limit in the right-hand side is the standard limit. It is well known that if a sequence converges in the usual sense, then it converges a la Cesaro, while the opposite might not hold instead.

We denote by  $\mathcal{W}$  the collection of stochastic matrices, that is, all  $n \times n$  square matrices whose rows entries are positive and sum up to 1. Given  $\varepsilon \in (0, 1]$ , we denote by  $\mathcal{W}_{\varepsilon,k}$  the subset of  $W \in \mathcal{W}$  such that the  $k$ -th column has all the entries greater than  $\varepsilon$ , that is,  $w_{ik} \geq \varepsilon$  for all  $i \in N$ . Let  $\mathcal{W}_\varepsilon = \cup_{k \in N} \mathcal{W}_{\varepsilon,k}$ .<sup>7</sup> Given  $k \in N$ , we denote by  $J_k$  the matrix whose columns are all zero except for the  $k$ -th one which is 1 in each entry. Note that  $J_k \in \mathcal{W}_\varepsilon$  for all  $\varepsilon \in (0, 1]$  and  $k \in N$ .

We say that an opinion aggregator  $T$  is:

1. *normalized* if and only if  $T(ce) = ce$  for all  $c \in I$ ;
2. *monotone* if and only if for each  $x, y \in B$

$$x \geq y \implies T(x) \geq T(y);$$

3. *translation invariant* if and only if

$$T(x + ce) = T(x) + ce \quad \forall x \in B, \forall c \in \mathbb{R} \text{ s.t. } x + ce \in B;$$

4. *constant affine* if and only if

$$T(\lambda x + (1 - \lambda) ce) = \lambda T(x) + (1 - \lambda) ce \quad \forall x \in B, \forall c \in I, \forall \lambda \in [0, 1];$$

5. *comonotonic additive* if and only if

$$x \text{ and } y \text{ comonotonic} \implies T(x + y) = T(x) + T(y);$$

6. *linear* if and only if there exists a matrix  $W \in \mathcal{W}$  such that

$$T(x) = Wx \quad \forall x \in B;$$

7. *odd* if and only if

$$T(-x) = -T(x) \quad \forall x \in B \text{ s.t. } -x \in B;$$

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<sup>7</sup>The collection of matrices  $\bigcup_{\varepsilon \in (0,1]} \mathcal{W}_\varepsilon$  are also said to be Markov's matrices or that they satisfy "Doebelin's condition" (see, respectively, Seneta [57, Definition 4.7] and Stroock [58, p. 32]).

8. *nonexpansive* (i.e., Lipschitz continuous of order 1) if and only if

$$\|T(x) - T(y)\|_\infty \leq \|x - y\|_\infty \quad \forall x, y \in B.$$

Our foundations naturally yield opinion aggregators that have the following properties: normalization, monotonicity, and translation invariance. The following definition formalizes this.

**Definition 1** *Let  $T$  be an opinion aggregator. We say that  $T$  is robust if and only if  $T$  is normalized, monotone, and translation invariant.*

We call these aggregators robust for two reasons: 1) our statistical foundation builds on the theory of robust statistics (cf. Section 3.1), 2) more in general, our foundations generalize the ones of the linear model naturally, but they do not take a parametric approach specifying a specific functional form. It is easy to see that any convex linear combination of robust opinion aggregators is itself a robust opinion aggregator.

## 3 The model

### 3.1 Robust aggregators

Consider a finite set of agents  $N = \{1, \dots, n\}$  who wish to estimate a fundamental parameter  $\mu \in \mathbb{R}$ . Agents initially observe signals

$$X_i = \mu + \varepsilon_i,$$

where  $\varepsilon = (\varepsilon_i)_{i \in N}$  is a random vector with joint distribution  $F_\varepsilon$ . The latter is such that the random vector  $X = (X_i)_{i \in N}$  has support contained in  $B \subseteq \mathbb{R}^n$ . The period-0 estimate of every agent  $i$  coincides with the realization  $x_i^0$  of her signal.

In period 1, the agents communicate with each other to acquire new information on the parameter. We model the communication through a directed network  $(N, A)$ , where  $A$  is an adjacency matrix in  $\{0, 1\}^{N \times N}$  with the understanding that  $a_{ij} = 1$  if and only if  $i$  listens to (or is influenced by)  $j$ . In particular, every agent  $i$  collects the sample of realizations of signals (opinion stated) in her neighborhood  $N_i = \{j : a_{ij} = 1\}$  and then solves an estimation problem about  $\mu$  using these data collected. Here, we consider a generalization of the class of M-estimators for location parameters defined in Huber [32].<sup>8</sup> Formally, we endow every agent  $i$  with a lower semicontinuous loss function  $\phi_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$  and let her solve

$$\min_{c \in \mathbb{R}} \phi_i(x^0 - ce), \tag{1}$$

where  $x^0 = (x_j^0)_{j \in N}$  are the realizations of signals.<sup>9</sup> Given the profile of loss functions

<sup>8</sup>Our generalization falls within the class of Extremum estimators.

<sup>9</sup>We maintain the semicontinuity assumption throughout the rest of the paper.

$\phi = (\phi_i)_{i=1}^n$ , the updates  $x^1$  at period 1 belong to the set

$$\mathbf{T}(x^0) = \prod_{i \in N} \left\{ \arg \min_{c \in I} \phi_i(x^0 - ce) \right\} \subseteq B. \quad (2)$$

Formally, (2) defines an updating correspondence  $\mathbf{T} : B \rightrightarrows B$  that satisfies

$$\mathbf{T}(x + ke) = \mathbf{T}(x) + ke \quad \forall x \in B, k \in \mathbb{R},$$

where  $\mathbf{T}(x) + ke$  is the set of vectors in  $\mathbf{T}(x)$  shifted by  $k$ .<sup>10</sup> Let  $T : B \rightarrow B$  denote an arbitrary selection from  $\mathbf{T}$  and note that, if  $\mathbf{T}$  is single-valued (e.g., whenever each  $\phi_i$  is strictly quasi-convex), then it coincides with its unique selection  $T$ .

The minimization problem in (1) has the following interpretation: agent  $i$  optimally picks the estimate for  $\mu$  as to minimize a loss function of the induced vector of deviations  $\epsilon = x^0 - ce$ . In particular, the function  $\phi_i$  represents a belief-free reduced form of all the ex-ante information of agent  $i$  about both the network structure and the objective distribution of errors  $F_\epsilon$ .<sup>11</sup> For example, if  $i$  thinks that the signal of  $j$  is highly informative, then her loss function  $\phi_i$  will penalize relatively more the deviations from  $x_j$ . In general, such estimation procedure is justified here insofar the agents are non-Bayesian. Indeed, we implicitly assume that the high uncertainty that they face does not allow them to attach probabilistic beliefs to unknown variables (i.e., both the fundamental parameter and the data generating process).

In the successive periods, the agents do not receive any additional external information on  $\mu$  but rather keep iterating the same estimation procedure with respect to a new set of data points given by the last-period estimates of their neighbors. Formally, we have that  $x^t \in \mathbf{T}(x^{t-1})$  for every period  $t \in \mathbb{N}$ . In particular, whenever  $\mathbf{T} = T$  is single valued, the deterministic dynamics of the estimates in the population given the initial realization  $x^0$ , are described by the iteration of the operator  $T : B \rightarrow B$  at  $x^0$ , that is,

$$\{x^t\}_{t \in \mathbb{N}} = \{T^t(x^0)\}_{t \in \mathbb{N}}.$$

Conversely, a sequence of updates  $\{x^t\}_{t \in \mathbb{N}}$  is consistent with the profile  $\phi$  of loss functions and with the initial realizations  $x^0 \in B$  if

$$x^t \in \mathbf{T}(x^{t-1}) \quad \forall t \in \mathbb{N}.$$

<sup>10</sup>This fact is already highlighted in [32].

<sup>11</sup>The clear understanding is that the loss functions are adapted to the neighborhood of the agents: for every  $i \in N$ , and  $z, z' \in \mathbb{R}^n$

$$(z_j = z'_j \quad \forall j \in N_i) \implies \phi_i(z) = \phi_i(z').$$

One may be tempted to let  $\phi_i : \mathbb{R}^{N_i} \rightarrow \mathbb{R}_+$ . However, since our results do not rely on the assumption of adapted loss functions, and for notational convenience, we maintain  $\mathbb{R}^N$  as the domain.

There are at least two justifications for the iteration procedure just described. The first one has been extensively analyzed in DVZ and is related to a form of persuasion bias. Under this interpretation, the agents ignore the information redundancies in their neighbors' estimates and consider what they observe as brand new information. Despite the convincing arguments presented in DVZ in favor of this kind of behavior, this interpretation requires a certain degree of bounded rationality of the agents. Whether or not this is a closer description of the reality than Bayesian updating, we would like to exploit our belief-free framework to try to overcome the assumption of bounded rationality. To see this, assume for simplicity that the profile of loss functions  $\phi$  is such that  $\mathbf{T} = T$  is single valued and  $T$  is normalized.<sup>12</sup> If we define  $\varepsilon^1 = T(\varepsilon)$ , then the random vector describing the profile of opinions stated in period 1 is given by

$$X^1 = T(X) = T(\mu e + \varepsilon) = \mu e + \varepsilon^1,$$

where the last equality follows from translation invariance and normalization of  $T$ . Therefore, at period 1, every agent  $i$  observes the realizations of the estimates in her neighborhood, that is,  $(X_j^1)_{j \in N_i}$ . Note that each  $X_i^1$  is again a location experiment centered in  $\mu$  with error term equal to  $\varepsilon_i^1$ . With this, at period 2, every agent  $i$  faces a similar estimation problem to the one she faced at period 1 with respect to different set of data. Indeed, every agent is uncertain about the distribution of  $\varepsilon^1$  similarly as it was for  $\varepsilon$  in the previous period. It seems natural then that every  $i$  would repeat the same estimation procedure (i.e., robust estimation with respect to  $\phi_i$ ) of period 1. Inductively, the same reasoning illustrated before shows that, for every  $t \in \mathbb{N}$ , the random vector describing the  $t$ -period estimates of the agents

$$X^t = T^t(X) = \mu e + T^t(\varepsilon)$$

is a location experiment with errors  $\varepsilon^t = T^t(\varepsilon)$ . Therefore, for all initial realizations  $x^0$  of signals  $X$ , the dynamics followed by the agents' estimates are described by  $\{T^t(x)\}_{t \in \mathbb{N}}$ .

Finally, we note that the updating procedure proposed in DVZ is easily nested in our framework by considering quadratic loss functions. Formally, every agent  $i$  minimizes

$$\phi_i(x - ce) = \sum_{j \in N_i} w_{ij} (x_j - c)^2. \quad (3)$$

for a vector of weights  $w_i \in \Delta$  representing the belief of  $i$  about the precisions of the signals in her neighborhood.

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<sup>12</sup>As showed in Theorem 1, the class of loss functions we analyze always induce normalized selections from the solution correspondences.

**Characterization of robust aggregators** The robust aggregators analyzed in this paper emerge as a unifying class for robust estimation problems. Here, we study the general properties of  $\phi$  that *characterize* robust aggregators.

The following definition captures the most elementary form of trust in the signals observed.

**Definition 2** *The profile of loss functions  $\phi$  is sensitive if, for all  $i \in N$  and  $h \in \mathbb{R} \setminus \{0\}$ ,*

$$\phi_i(he) > \phi_i(0).$$

In words, if agent  $i$  observes an unanimous opinion (including herself) her loss is minimized by declaring this same opinion. Following definition is a form of complementarity in disagreeing with two or more agents from the same side.

**Definition 3** *The profile of loss functions  $\phi$  has increasing shifts if and only if, for all  $i \in N$ ,  $z, v \in \mathbb{R}^n$  such that  $z \geq v$ , and  $h \in \mathbb{R}_{++}$ ,*

$$\phi_i(z + he) - \phi_i(z) \geq \phi_i(v + he) - \phi_i(v). \quad (4)$$

*It has strictly increasing shifts if the above inequality is strict whenever  $z \gg v$ .*

We consider the property of increasing shifts because it is very permissive and naturally emerges from the characterization of Theorem 1. It is implied by stronger properties usually required on games played on networks, such as supermodularity and degree complementarity (see, e.g., Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv [24]). Moreover, we will see momentarily how this corresponds to the generalization of the convexity assumption imposed in robust statistics.

We call *robust* the profile of lower semicontinuous loss functions  $\phi = (\phi_i)_{i \in N}$  that are sensitive and satisfy increasing shifts. The set of robust loss functions is denoted by  $\Phi_R$ . Next, we formalize the relation between robust opinion aggregators and robust loss functions.<sup>13</sup>

**Theorem 1** *The following facts are true:*

1. *If  $T : B \rightarrow B$  is a robust opinion aggregator, then there exists  $\phi \in \Phi_R$  satisfying strictly increasing shifts such that, for every  $i \in N$ ,*

$$T_i(x) = \arg \min_{c \in I} \phi_i(x - ce) \quad x \in B.$$

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<sup>13</sup>Given  $i \in N$  and  $T : B \rightarrow B$ , we define  $T_i : B \rightarrow I$  as  $T_i(x) = (T(x))_i$  for all  $x \in B$ .

2. Conversely, if  $\phi \in \Phi_R$ , then the updating correspondence  $\mathbf{T}$  admits a robust selection  $T$ . Moreover, if each  $\phi$  satisfies strictly increasing shifts, then  $\mathbf{T} = T$  is a robust opinion aggregator.

Even though sensitivity and increasing shifts are the properties characterizing robust aggregation in our model, it might not be immediate to verify that a given profile of loss functions satisfies them. The following corollary of Theorem 1 is a useful tool in recognizing loss functions that induce robust opinion aggregators. We first need an additional definition. Recall that for all  $z, v \in \mathbb{R}^n$ ,

$$z \vee v = (\max \{z_i, v_i\})_{i=1}^n \quad \text{and} \quad z \wedge v = (\min \{z_i, v_i\})_{i=1}^n.$$

**Definition 4** *The profile of loss functions  $\phi$  is supermodular if and only if, for all  $i \in N$  and  $z, v \in \mathbb{R}^n$ ,*

$$\phi_i(z \vee v) + \phi_i(z \wedge v) \geq \phi_i(z) + \phi_i(v).$$

Let  $\Phi_R^*$  denote the set of profiles of continuous, convex, sensitive, and supermodular loss functions. One can show that  $\Phi_R^* \subseteq \Phi_R$ .<sup>14</sup>

**Corollary 1** *If  $\phi \in \Phi_R^*$ , then the updating correspondence  $\mathbf{T}$  admits a robust selection  $T$ . Moreover, if each  $\phi_i$  is strictly convex, then  $\mathbf{T} = T$  is a robust opinion aggregator.*

Within the class of robust aggregators, constant affine and odd aggregators play key roles in our convergence results in Sections 6 and 7. Natural properties on  $\phi$  permit to derive these aggregators.

**Definition 5** *The profile of loss functions  $\phi$  is positive homogeneous if and only if, for all  $i \in N$ , there exists a positive function  $\eta_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that*

$$\phi_i(\lambda z) = \eta_i(\lambda) \phi_i(z) \quad \forall z \in \mathbb{R}^n, \lambda \in \mathbb{R}_+.$$

The aggregators induced by  $L_p$ -seminorms considered in the next section, as well as the Choquet aggregators studied in Section 6.5, are prominent examples of constant affine operators.

**Definition 6** *The profile of loss functions  $\phi$  is symmetric if and only if, for all  $i \in N$ ,*

$$\phi_i(z) = \phi_i(-z) \quad \forall z \in \mathbb{R}^n. \quad (5)$$

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<sup>14</sup>See the proof of Corollary 1. For an example of a  $\phi_i$  that has increasing shifts but it is not supermodular, consider

$$\phi_i(z_1, z_2) = z_1^2 + z_2^2 + |z_1||z_2|.$$

Symmetry is always satisfied when the loss function only depends on the absolute value of the distance between opinions. However, notice that for loss functions that are not separately additive, the converse does not hold.<sup>15</sup>

**Proposition 1** *The following facts hold true:*

1.  $T : B \rightarrow B$  is a constant affine robust opinion aggregator if and only if there exists a positive homogeneous  $\phi \in \Phi_R$  with strictly increasing shifts such that, for every  $i \in N$ ,

$$T_i(x) = \arg \min_{c \in I} \phi_i(x - ce) \quad x \in B.$$

2.  $T : B \rightarrow B$  is an odd robust opinion aggregator if there exists a symmetric  $\phi \in \Phi_R$  with strictly increasing shifts such that, for every  $i \in N$ ,

$$T_i(x) = \arg \min_{c \in I} \phi_i(x - ce) \quad x \in B.$$

By inspection of the proof of Proposition 1, one can see that more is true. Indeed, even without strictly increasing shifts, whenever  $\phi \in \Phi_R$  is positive homogeneous, the updating correspondence admits a constant affine robust selection (and similarly for odd robust aggregators and symmetric and robust loss functions).

**i.i.d. signals and additive separable loss functions (Huber, 1964)** Here, we assume that agents commonly know that the errors  $\varepsilon = (\varepsilon_i)_{i \in N}$  are independently and symmetrically distributed according to an objective univariate distribution  $F_\varepsilon$ , which is still unknown from their perspective. In this case, at period 1, every agent  $i$  solves an estimation problem with an independent sample  $(X_j)_{j \in N_i}$ . Following Huber [32], each agent  $i$  uses an estimator  $T_i$  that, for every signals' realization  $x = (x_j)_{j \in N}$ , solves

$$\min_{c \in \mathbb{R}} \phi_i(x - ce) = \min_{c \in \mathbb{R}} \sum_{j \in N} a_{ij} \rho_i(x_j - c) \quad (6)$$

where  $\rho_i : \mathbb{R} \rightarrow \mathbb{R}_+$  is continuous, convex, strictly decreasing on  $\mathbb{R}_-$  and strictly increasing on  $\mathbb{R}_+$ . This shows that problem (1) is a proper generalization non i.i.d. sample of the estimation method proposed by Huber. Indeed, most of the loss functions used in the Robust Statistic literature initiated by Huber satisfies the properties we

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<sup>15</sup>To see this, consider

$$\phi_i(z_1, z_2) = z_1^2 + z_2^2 + (z_1 \vee 0)(z_2 \vee 0) + (z_1 \wedge 0)(z_2 \wedge 0).$$

exploit: the quadratic loss  $\rho_i(z) = z^2$ , the absolute loss  $\rho_i(z) = |z|$ , and the Huber loss defined as

$$\rho_i(z) = \begin{cases} z^2 & \text{if } |z| \leq k \\ 2k|z| - k^2 & \text{else} \end{cases} \quad (7)$$

for some  $k > 0$ , are examples from this family of loss functions.

Although not always justifiable in a network setting, player-wise separable losses are sometimes more tractable and, for this class, it is often easier to recognize when our properties hold.

**Definition 7** *The profile of loss functions  $\phi$  is additive separable if and only if there exist a stochastic matrix  $W \in \mathcal{W}$  and a profile of lower semicontinuous functions  $\rho = (\rho_i : \mathbb{R} \rightarrow \mathbb{R}_+)_{i \in N}$  such that, for all  $i \in N$ ,*

$$\phi_i(z) = \sum_{j=1}^n w_{ij} \rho_i(z_j) \quad \forall z \in \mathbb{R}^n.$$

Given a profile of additive separable loss functions  $\phi$ , we often identify it with the corresponding pair  $(W, \rho)$ .<sup>16</sup> We denote the set of robust additive separable loss functions with  $\Phi_A \subseteq \Phi_R$ . The following proposition characterizes the elements of  $\Phi_A$ .

**Proposition 2** *Let  $W \in \mathcal{W}$  and  $\rho = (\rho_i : \mathbb{R} \rightarrow \mathbb{R}_+)_{i \in N}$ . The following statements are equivalent:*

- (i)  $(W, \rho) \in \Phi_A$ ;
- (ii)  $(W, \rho) \in \Phi_A \cap \Phi_R^*$ ;
- (iii) *for every  $i \in N$ ,  $\rho_i$  is convex, strictly decreasing on  $\mathbb{R}_-$  and strictly increasing on  $\mathbb{R}_+$ .*

The previous proposition together with Theorem 1 immediately yield the following result.

**Corollary 2** *Let  $(W, \rho)$  be an additive separable profile of loss functions. If for all  $i \in N$ ,  $\rho_i$  is convex, strictly decreasing on  $\mathbb{R}_-$  and strictly increasing on  $\mathbb{R}_+$ , then the updating correspondence  $\mathbf{T}$  admits a selection  $T$  which is robust. Moreover, if each  $\rho_i$  is strictly convex, then the updating correspondence  $\mathbf{T} = T$  is a well defined robust opinion aggregator.*

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<sup>16</sup>The use of a stochastic matrix instead of arbitrary weights is without loss of generality given the additive structure since it can always be obtained applying a strictly monotone transformation to the original loss function.

The previous corollary is a useful tool in easily recognizing problems that admit a unique path of updates derived from a robust opinion aggregator. For example, it implies that the, for every  $W \in \mathcal{W}$ , the profiles of loss functions  $\phi = (W, \rho)$ , with each  $\rho_i$  defined as in (3) or (15) admit a unique robust updating function.

We close this section by analyzing an important class of loss functions which do not formally fall within any of the cases considered so far, but that, nevertheless, admit robust aggregators as updating functions. First, consider all the weighted  $L_p$ -seminorms, defined as

$$\phi_i(z) = \|z\|_{p_i, w_i} = \left( \sum_{j \in N} w_{ij} |z_j|^{p_i} \right)^{\frac{1}{p_i}} \quad \forall z \in \mathbb{R}^n \quad (8)$$

for some  $w_i \in \Delta$  and  $p_i \in [1, \infty)$ . It is straightforward to see that the minimization problem with loss function as in (8) admits an equivalent additive separable formulation. Indeed, Proposition 2 immediately yields that, for all  $W \in \mathcal{W}$  and  $p = (p_i)_{i \in N} \in [1, \infty)^N$ ,

$$\left( \|\cdot\|_{p_i, w_i}^{p_i} \right)_{i \in N} \in \Phi_R^* \cap \Phi_A.$$

Differently, the weighted  $L_\infty$ -seminorm, defined as

$$\phi_i(z) = \|z\|_{\infty, w_i} = \max_{j: w_{ij} > 0} |z_j| \quad \forall z \in \mathbb{R}^n,$$

is robust but not additive separable. In particular, one can show that the unique solution is

$$T_i(x) = \frac{1}{2} \left( \min_{j: w_{ij} > 0} x_j + \max_{j: w_{ij} > 0} x_j \right) \quad \forall x \in B, \quad (9)$$

which, by Theorem 1 satisfies all the properties defining robust aggregators.

All the weighted  $L_p$ -seminorms, including  $L_\infty$ , are positive homogeneous and symmetric. Therefore, inspection of the proof of Proposition 1 yields the following corollary. To save notation, whenever  $p_i = \infty$ , we let  $\|\cdot\|_{p_i, w_i}^{p_i} = \|\cdot\|_{\infty, w_i}$ .

**Corollary 3** *If  $\phi = \left( \|\cdot\|_{p_i, w_i}^{p_i} \right)_{i \in N}$  for some  $W \in \mathcal{W}$  and  $p \in [1, \infty]^N$ , then the updating correspondence  $\mathbf{T}$  admits a constant affine and odd selection  $T$ . If in addition  $p \in (1, \infty]^N$ , then the updating correspondence  $\mathbf{T} = T$  is a well defined constant affine, odd, and robust opinion aggregator.*

The previous corollary is relevant because it asserts that the robust aggregators induced by  $L_p$ -seminorms satisfy constant affinity, a property that we will exploit in the next sections to obtain a rate of convergence which is independent of the initial condition.

## 3.2 Additional statistical procedures

**Maximum Likelihood Estimation** The approach followed by Huber is completely nonparametric, in the sense that he did not postulate any functional form for the probability distribution of the error terms. However, an important intermediate case between Huber’s approach and assuming normality of the error terms is the one that performs a Maximum Likelihood Estimation of the errors with more general, or merely different, probability distribution functions. The following result shows how even in this case, under standard assumptions on the pdf, the aggregator obtained is robust.

As in DVZ, the agents may entertain incorrect point beliefs about the distribution of the errors, and in particular, we let  $f_i$  be the pdf of the errors as believed by agent  $i$ , and with  $f_{ij}$  its marginal with respect to the error term  $\varepsilon_j$ . Then, we have the following.

**Proposition 3** *If for all  $i \in N$ ,*

$$f_i(\varepsilon) = \prod_{j \in N} f_{ij}(\varepsilon_j),$$

*and for all  $j \in N$ ,  $f_{ij}$  is log-concave and with a unique local maximum in 0, then the MLE has a selection that is a robust opinion aggregator.*

These properties are quite mild and satisfied by several parametric families, such as the Normal, logistic, Gumbel, Laplace, and skewed Laplace ones. Also, many of the aggregators studied above can be directly mapped into these families. Indeed, the Normal case corresponds to the linear aggregator, the Gumbel to the variational one (see equation (15)), the Laplace to the median, and the skewed Laplace to the general quantiles (see [38, Remark 2.6.3]). Even if the pdf of the uniform distribution does not admit a *unique* local maximum in 0, a similar proof shows that its MLE admits a robust selection and that such selection is the Choquet aggregator given by equation (9).

This procedure reduces to the model in DeMarzo, Vayanos, and Zwiebel as a particular case by letting  $\varepsilon_i \sim N(0, \Sigma_i)$ , where  $\Sigma_i$  is a diagonal matrix with diagonal entries  $(1/\tau_{ij})_{i \in N}$ . Here,  $\tau_{ij}$  denotes the degenerate *subjective* belief of  $i$  about the precision of  $j$ . According to their model, in period 1 every agent  $i$  performs a Maximum Likelihood estimation of  $\mu$  given the initial realizations of the signals in her neighborhood  $(x_j^0)_{j \in N_i}$  and her beliefs  $(\tau_{ij})_{j \in N_i}$ . At period 1, agent  $i$  estimate is the weighted average

$$x_i^1 = T_i(x) = \sum_{j \in N_i} w_{ij} x_j^0, \quad (10)$$

where the weights  $w_{ij} = \frac{\tau_{ij}}{\sum_{\ell \in N_i} \tau_{i\ell}}$  are proportional to the signal precisions. In the successive periods  $t \in \mathbb{N}$ , agents naively keep updating their opinions by combining their neighbors’ estimates with the weights just defined, that is,  $x_i^t = T_i(x^{t-1})$ .

The rationale offered to justify this kind of behavior is persuasion bias: agents ignore redundancies and treat the new estimates they observe as brand new information.<sup>17</sup> With this, they provide a foundation for the linear DeGroot model based on optimal information acquisition with a behavioral component. Despite this line of reasoning is convincing, we note that the exact functional form of (10) essentially relies on several restrictive hypothesis: the original signals  $(X_i)_{i \in N}$  are independently and normally distributed, and every agents correctly conjectures the distributions of the signals in her neighborhood up to the exact value of the precisions.

**Bayesian estimation** The updating rule proposed by DVZ has been also rationalized as a quasi-Bayesian procedure with a diffuse prior (see [17, Footnote 17] and [30]). Formally, keep assuming that agents observe the realization of an independent location experiment  $X_i = \mu + \varepsilon_i$  and they have an improper prior (i.e., the uniform measure  $\Lambda$  over the entire real line). Once she observes the realization of the signals in her neighborhood, she updates her belief about  $\mu$ :

$$\Lambda\left(\mu \mid (x_j)_{j \in N_i}\right) = \frac{\prod_{j \in N_i} f_{ij}(x_j - \mu)}{\int_{-\infty}^{\infty} \prod_{j \in N_i} f_{ij}(x_j - \mu') d\mu'} \quad \forall x \in \mathbb{R}^n, \mu \in \mathbb{R}.$$

With this, the posterior expectation of each  $i$  given  $x$  is defined as

$$T_i(x) = \mathbb{E}_{\Lambda}\left[\mu \mid (x_j)_{j \in N_i}\right] = \frac{\int_{-\infty}^{\infty} \mu \prod_{j \in N_i} f_{ij}(x_j - \mu) d\mu}{\int_{-\infty}^{\infty} \prod_{j \in N_i} f_{ij}(x_j - \mu') d\mu'}.$$

Note that, whenever each  $f_{ij}$  is Gaussian,  $T_i$  is a linear function of  $x$ . However, if we relax normality, then  $T$  is only a robust opinion aggregator.

**Proposition 4** *If for all  $i \in N$ :*

$$f_i(\varepsilon) = \prod_{j \in N} f_{ij}(\varepsilon_j),$$

*and for all  $j \in N$ ,  $f_{ij}$  is log-concave and symmetric around 0, then the posterior expectation is a robust opinion aggregator.*

**L-estimators** A last procedure that induces robust opinion aggregators is the use of L-estimators. When using an L-estimator, an agent takes a linear combination of the order statistics in the observed sample. They seem particularly appealing for agents trying to make inference in a network for two reasons. First, they are very robust

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<sup>17</sup>More precisely, DeMarzo, Vayanos, and Zwiebel also consider a certain degree of stickiness to the previous estimate at each round of updating. For a detailed presentation of the dynamics of their model, see Section 6.6.

to misspecification of the data generating process for the observed opinions. Second, they are very easy to compute, since they simply consists into a weighted average procedure. In addition to their normative appeal in a network structure, they allow for descriptively relevant biases in information aggregation, since they can be naturally used to capture the overweighting (as well as the neglecting) of extreme realizations, see Section 6.5. Finally, we also notice that L-estimators are comonotonic additive, henceforth constant additive and robust. Such mathematical property significantly simplifies the analysis of their limit behavior.

**Proposition 5** *If every agent  $i \in N$  uses an L-estimator, the induced opinion aggregator is comonotonic additive.*

## 4 Examples

**Model uncertainty** Notice that nonlinear robust opinion aggregators may also arise when the agents think that the signals are normally distributed, but they are uncertain about the precision of their neighbors. For example, suppose that agent  $i$  has a subjective belief that the variance of the signal of agent  $j$  is distributed as

$$\sigma_{ij} = \frac{\sqrt{2\nu_i}}{\tau_{ij}}$$

and  $\nu_i \sim Exp(1)$ . Then, it is well known (see, e.g., [38, Proposition 2.2.1]) that

$$X_{ij} = \mu + \varepsilon_{ij} = \mu + \sigma_{ij}Z$$

where  $Z$  is a standard normal, has a Laplace  $\left(\mu, \frac{1}{\tau_{ij}}\right)$  distribution. Since the MLE estimators for Laplace observations is the weighted median, the previous argument provides additional motivation for the use of nonlinear aggregators even when signals are normally distributed. Formally,

$$T_i(x) = \min \left\{ c \in I : \sum_{j:x_j \leq c} w_{ij} \geq 0.5 \right\} \in \mathbf{T}_i(x) \quad \forall x \in B,$$

where  $w_{ij} = \frac{\tau_{ij}}{\sum_{\ell \in N_i} \tau_{i\ell}}$ . In the previous robust estimation framework, such an estimator is alternatively obtained when agents use the absolute distance as the loss function, i.e., agent  $i$  minimizes

$$\phi_i(x - ce) = \sum_{j=1}^n w_{ij} |x_j - c| \quad \forall x \in \mathbb{R}^n, c \in \mathbb{R} \quad (11)$$

where, as before,  $w_i \in \Delta$ . Surprisingly, such a small modification induces dramatically different dynamics. We analyze more in detail this case in Section 6.5. For now, let us consider the following example.

**Example 1** A group of agents  $N = \{1, 2, 3, 4\}$  share their opinions  $x = (x_1, x_2, x_3, x_4) \in [0, 1]^4$ . The subjective beliefs about the precisions of the signals is represented by the matrix

$$\begin{pmatrix} 0.4 & 0.3 & 0.3 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0.1 & 0.1 & 0.2 & 0.6 \\ 0 & 0 & 0.6 & 0.4 \end{pmatrix}.$$

where the entry in row  $i$ , column  $j$  represents  $\tau_{ij}$ . It is immediate (see e.g., Proposition 1 in Golub and Jackson [27]) that aggregation through weighted *averages* would imply consensus in the limit. However, the dynamics induced using the median are qualitatively different.

- If  $x^0 = (0, 1, 1, 1)$ , then the block of agents agreeing on the higher opinion is sufficiently large to attract agent 1 to the same opinion, and the limit (consensus) opinion of  $(1, 1, 1, 1)$  is reached in one round of updating;
- However, the prediction of consensus is lost if the initial opinion of player 2 is slightly lowered. Let  $x^0 = (0, 1/2, 1, 1)$ , then their first round of updating gives  $x^1 = (1/2, 1/2, 1, 1)$  and this polarization will be the limit outcome: A strongly connected society fails to reach consensus without a sufficiently large block of initial agreement;
- Finally, consider  $x^0 = (0, 1/2, 0, 1)$ . Then agents' first update is  $x^1 = (0, 0, 1, 0)$  and agents 1 and 2 will never change opinion again, whereas agents 3 and 4 will keep reciprocally switching their opinions. This example shows that also convergence may not be guaranteed. ▲

This is just an example of a more general fact. By [59], we know that every distribution in the exponential power family can be obtained as a mixture of normals. Since the exponential power densities satisfy the assumptions of Proposition 3, it immediately follows that robust opinion aggregators rationalize the behavior in face of various forms of uncertainty about the precision of the signals of the other agents.

More generally, agents may hold a belief about the profile of precision of their neighbors. In principle, they may feature different combinations of perceived uncertainty and attitudes towards that uncertainty (see Hansen and Marinacci [31]). In the previous example, uncertainty about the precision is captured by a Laplace prior, and the agents have neutral attitudes towards uncertainty in that they treat their *subjective* uncertainty about the precision and the *objective* randomness of the signals symmetrically. However, it may well be the case that the agents are more averse to the subjective component of uncertainty, see [31] and the references therein. For example,

given a belief  $\xi \in \Delta(\Delta(N))$  where an element  $w_i \in \Delta(N)$  is a profile of (normalized) precision of the agents, he may solve a problem of the form

$$\min_{c \in \mathbb{R}} \int f\left(\sum_{j:x_j \geq c} w_{ij}(x_j - c)^2\right) + f\left(\sum_{j:x_j < c} w_{ij}(c - x_j)^2\right) d\xi(w_i). \quad (12)$$

In words, the agent is minimizing the expected disutility of right and left deviations from the observed signals. Here, the term  $f$  captures the extent of the aversion to subjective uncertainty. Whenever the function  $f$  is convex, the agents are averse to subjective uncertainty and, by Proposition 1 the resulting opinion aggregator is robust.

The second weakness of (3) is that upward and downward discrepancies from the observed opinions are felt as equally harming by every agent. However, it might well be the case that (some) agents dislike more one or the other. One easy example of this kind of behavior is the asymmetric version of (11). Formally, we define

$$\phi_i(x - ce) = \alpha_i \sum_{j:x_j \geq c} w_{ij}(x_j - c) + (1 - \alpha_i) \sum_{j:x_j < c} w_{ij}(c - x_j) \quad \forall x \in \mathbb{R}^n, c \in \mathbb{R} \quad (13)$$

where  $\alpha_i \in [0, 1]$ . Note that, whenever  $\alpha_i = 1/2$ , the loss function in (13) reduces to the one in (11). Moreover, (one of) the solution functions of the minimization of (13) are given by<sup>18</sup>

$$T_i^{\alpha_i}(x) = \begin{cases} \min_{j:w_{ij}>0} x_j & \text{if } \alpha_i = 0 \\ \min \left\{ c \in I : \sum_{j:x_j \leq c} w_{ij} \geq \alpha_i \right\} & \text{if } \alpha_i \in (0, 1) \\ \max_{j:w_{ij}>0} x_j & \text{if } \alpha_i = 1 \end{cases} \quad \forall x \in B, \quad (14)$$

which is exactly the (weighted) quantile function for the distribution of observed opinions. Quantile functions capture the behavior of agents who have a bias in favor of relatively extreme stances ( $\alpha_i$  close to 0 or 1) or relatively moderate ones ( $\alpha_i$  close to 1/2).

The quantile functions are nondifferentiable and, even though we do not rely on differentiability properties for our main results, smooth aggregators can be more easily analyzed in most applications. A smooth and tractable robust opinion aggregator is obtained by letting  $\lambda \in \mathbb{R} \setminus \{0\}$  and considering the following loss function:

$$\phi_i^\lambda(x - ce) = \sum_{j \in N} w_{ij} (\exp(\lambda(x_j - c)) - \lambda(x_j - c)) \quad \forall x \in \mathbb{R}^n, c \in \mathbb{R} \quad (15)$$

In particular, whenever  $\lambda > 0$ , upward deviations from  $i$ 's current opinion  $c$  are more penalized than downward deviations and vice-versa whenever  $\lambda < 0$ . Figure 1 compares the quadratic loss with the asymmetric one with  $\lambda = 1$ . Some interesting comparative statics hold for the loss function in (15).

<sup>18</sup>See, for example, Section 1.3 in [36].

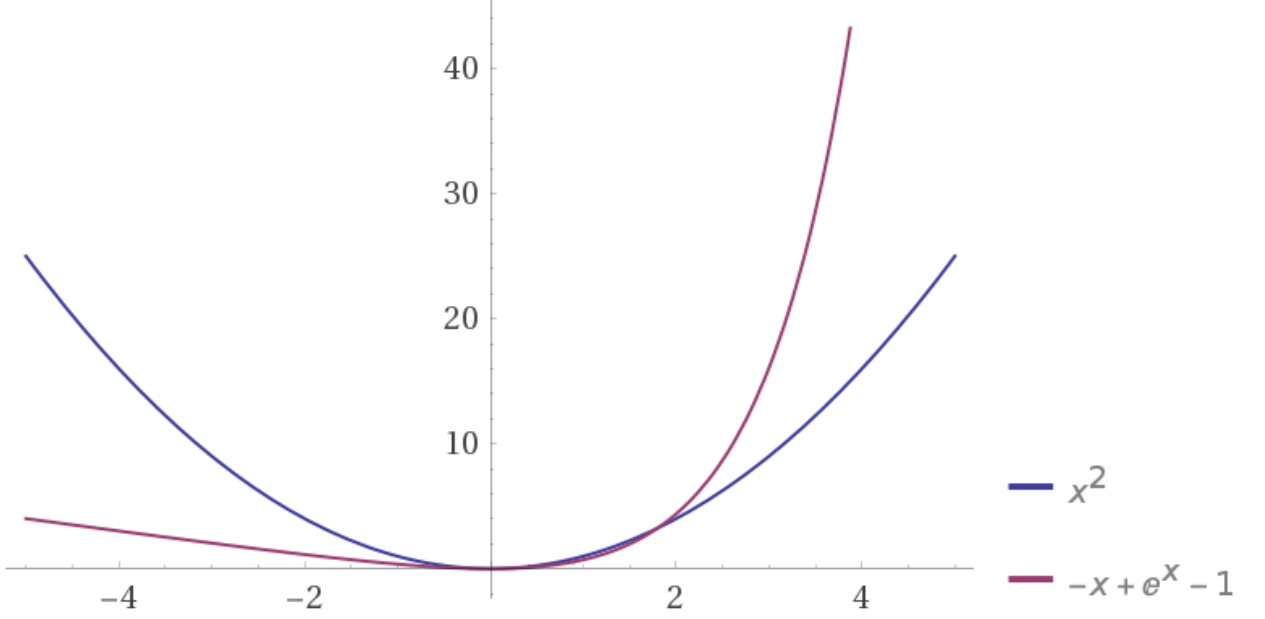


Figure 1: Smooth asymmetric loss

**Proposition 6** *If the loss function is as in (15) with  $W \in \mathcal{W}$  and  $\lambda \in \mathbb{R} \setminus \{0\}$ , the following facts hold true:*

1. *The solution function is given by*

$$T_i^\lambda(x) = \frac{1}{\lambda} \ln \left( \sum_{j=1}^n w_{ij} \exp(\lambda x_j) \right) \quad \forall x \in B.$$

2. *It captures the DeGroot model as a limit case:*

$$\lim_{\lambda \rightarrow \hat{\lambda}} T_i^\lambda(x) = \begin{cases} \max_{j:w_{ij}>0} x_j & \text{if } \hat{\lambda} = \infty \\ \sum_{j=1}^n w_{ij} x_j & \text{if } \hat{\lambda} = 0 \\ \min_{j:w_{ij}>0} x_j & \text{if } \hat{\lambda} = -\infty \end{cases} \quad \forall x \in B.$$

3. *If the underlying network  $W$  structure is such that the linear aggregator would have obtained consensus in the limit, i.e., there exists an influence vector  $s \in \Delta$  such that*

$$\lim_{t \rightarrow \infty} W^t x = \left( \sum_{i \in N} s_i x_i \right) e$$

then

- *the limit opinion is*

$$\lim_t (T^\lambda(x))^t = \left( \frac{1}{\lambda} \log \sum_{i \in N} s_i \exp(\lambda x_i) \right) e$$

and the limit influence of each agent is increasing (resp., decreasing) in his initial opinion when  $\lambda > 0$  ( $\lambda < 0$ )

$$\frac{\partial}{\partial x_j} \lim_t (T^\lambda(x))^t = \frac{s_j \exp(\lambda x_j)}{\sum_i s_i \exp(\lambda x_i)}.$$

- Given two agents  $i, j$  sharing the same influence under the linear model i.e., such that  $s_i = s_j$ , if their initial opinions are more dispersed (resp. more concentrated) in the second order dominance sense, then the limit consensus is higher (lower) when  $\lambda > 0$  ( $\lambda < 0$ ).

Again, we see another prediction of the linear model reversed. It is not just the network structure  $W$  that determines the limit influence of each agent, but the initial opinion also plays a key role. Indeed, when  $\lambda > 0$ , the higher the initial signal realization of an individual, the higher her weight in the limit.

## 5 Conditions for convergence

Next, we explore how the network in our statistical foundation can be connected with the conditions guaranteeing convergence to consensus.

**Proposition 7** *Let  $\phi \in \Phi_R$  be differentiable with gradient locally Lipschitz and such that*

$$\phi'_i(z + he_j; e) - \phi'_i(z; e) \geq \varepsilon_{ij}h \quad \forall z \in \mathbb{R}^n, h \in \mathbb{R}_+. \quad (16)$$

*for some  $\varepsilon_{ij} > 0$ . Then, for every robust selection  $T$  of  $\mathbf{T}$ ,  $j$  strongly influences  $i$  when  $I$  is compact.*

The condition of equation (16) has already been used to obtain a monotone best response in economics, see the concept of state monotonicity in Frankel, Morris, and Pauzner [21]. In the case of a separable  $\phi_i$ , an easy to check sufficient condition for (16) can be given.

**Corollary 4** *If  $(W, \rho)$  is such that for all  $i \in N$ ,  $\rho_i$  is twice continuously differentiable and strongly convex, then  $\mathbf{T}$  is single valued and*

$$w_{ij} \geq 0 \iff j \text{ strongly influences } i$$

*when  $I$  is compact.*

Examples of loss functions for which equation (16) can be used to check monotonicity include the quadratic loss, the exponential loss, the pseudo-Huber loss, the loss

inducing the variational weighted average, as well as the loss for uncertainty averse agents (12) when  $f$  is twice differentiable with second derivative strictly larger than 0.

Finally, the results in Section 7 highlight the importance of not having a single agent being too influential for the others. The next result bounds the weight of agent  $j$  opinion on agent  $i$ , as the number of source of information of  $i$  (i.e.,  $N_i$ ) increase.

**Proposition 8** *If  $(W, \rho)$  is such that for all  $i \in N$   $\rho_i$  is twice continuously differentiable with  $\rho_i'' > 0$*

$$w_{ij} \in \left\{ 0, \frac{1}{N_i} \right\}$$

*and  $I$  is compact, then  $T_i$  is differentiable and there exists  $\bar{c}$  such that*

$$\nabla T_i(x) \leq \frac{\bar{c}}{\min_k |N_k|} e \quad \forall i \in N.$$

## 6 Convergence

In this section, we study the long-run evolution of opinions in a society characterized by a robust opinion aggregator. We already know from Example 1 that the induced dynamics may be qualitatively different from the linear case: the extent of polarization or consensus attained in the limit, as well as the identity of the groups agreeing on a particular issue depends on the initial distribution of opinions in the society.

Here, we proceed by showing that robust opinion aggregators always induce a weaker form of convergence of opinions, i.e., the convergence of the time averages. Given this result, we give general conditions under which standard convergence is obtained, and we study the stability and consensus properties of the limit.

### 6.1 Convergence of time averages

Section 3 shows that a more robust foundation of opinion aggregation leads to aggregators that are not linear. In light of this, given an initial opinion  $x^0$  the study of its evolution via the sequence of updates  $\{T^t(x^0)\}_{t \in \mathbb{N}}$  cannot rely on the results developed for the classical DeGroot's model (e.g., Berger [7], Jackson [35, Chapter 8], and Golub and Jackson [27]). In the nonlinear case, it is not even obvious what kind of dynamics might arise. For example, a priori one cannot rule out that the behavior of the sequence of updates might depend heavily on the initial condition  $x^0$  (e.g., convergence vs non-convergence, rate of convergence, etc.). The following section is devoted to studying the properties of robust opinion aggregators and the behavior of their sequences of updates  $\{T^t(x^0)\}_{t \in \mathbb{N}}$ .

As mentioned, the use of the adjective robust is based on the foundation of these opinion aggregators. Despite the name though, one is left to wonder whether or not

these aggregators might generate chaotic dynamics. Indeed, chaotic behavior for discrete deterministic dynamical systems is often defined by allowing for dynamics which are *not* robust to the initial conditions.

The following property, dubbed *sensitive dependence on initial conditions*, is often interpreted as the mathematical translation of the “butterfly effect” and, to the best of our knowledge, is a requirement in most of the available formal definitions of chaotic maps (see Devaney [18, p. 49] and Robinson [54, Section 3.5]):<sup>19</sup>

$$\exists r > 0, \forall x \in B, \forall \varepsilon > 0, \exists y \in B, \exists t \in \mathbb{N} \text{ s.t. } \|x - y\|_\infty < \varepsilon \text{ and } \|T^t(x) - T^t(y)\|_\infty \geq r. \quad (17)$$

In other words, a small change  $\varepsilon$  in the initial condition, say from  $x$  to  $y$ , might generate very different dynamics. The  $t$ -th iterates are separated, no matter how close  $x$  and  $y$  might be and despite  $T$  being potentially continuous. This is a troublesome feature if computer simulations are involved since different approximations of either the initial condition or the iterates might generate very different sequences of updates.<sup>20</sup> Chaotic behavior would be exactly the opposite of robustness when it comes to the dynamics of the updates. The following simple result shows that a robust opinion aggregator violates (17) and therefore its dynamics will not be chaotic.

**Lemma 1** *If  $T$  is a robust opinion aggregator, then  $T^t$  is nonexpansive for all  $t \in \mathbb{N}$ . In particular,  $T$  is nonexpansive and violates condition (17).*

The previous lemma rules out the possibility of chaotic behavior, but it is also mute in terms of the limit behavior of the updates. In particular, one might be interested in the convergence of the sequence  $\{T^t(x^0)\}_{t \in \mathbb{N}}$  given a specific  $x^0$  or, more in general, in the convergence of  $\{T^t(x)\}_{t \in \mathbb{N}}$ , irrespective of the  $x$  chosen. The first one is a local property, and the second one is global. We crystallize these observations in a definition.

**Definition 8** *Let  $T$  be an opinion aggregator. We say that  $T$  is convergent at  $x^0$  if and only if  $\lim_t T^t(x^0)$  exists. Moreover, we say that  $T$  is convergent if and only if  $T$  is convergent at each  $x$  in  $B$ .*

We cannot expect to obtain that robust aggregators are convergent in general since this statement is already false for the linear case. The next example using a simple linear aggregator illustrates our first convergence result.

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<sup>19</sup>An example of a chaotic map, thus satisfying (17), is the tent map which is defined by assuming  $n = 1$ ,  $I = [0, 1]$ , and

$$T(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \frac{1}{2} < x \leq 1 \end{cases} \quad \forall x \in [0, 1].$$

Note that  $T$  is continuous.

<sup>20</sup>See Devaney [18, p. 49] for a discussion.

**Example 2** Let  $B = [0, 1]^2$  and consider the case in which aggregation is linear with matrix

$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consider  $x^0 \in B$ . Clearly,  $\{T^t(x^0)\}_{t \in \mathbb{N}}$  converges if the components of  $x^0$  coincide, that is  $x^0 \in D$ , while  $\{T^t(x^0)\}_{t \in \mathbb{N}}$  oscillates with period 2 whenever the first component of  $x^0$  is different from the second one. At the same time, this feature allows us to say that the *time averages* of the updates converge, no matter what is the initial condition, that is,

$$C - \lim_t T^t(x) = \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) = \left( \frac{x_1 + x_2}{2} \right) e \quad \forall x \in B.$$

Later, in Corollary 5, we will show that each robust opinion aggregator, be linear or not, is convergent in this weaker sense. Moreover, Proposition 9 will show that the Cesaro limit of a generic sequence of updates is a fixed point of  $T$ , provided  $T$  is linear.

▲

In dealing with the issue of convergence, we thus first focus on Cesaro convergence of the updates  $\{T^t(x^0)\}_{t \in \mathbb{N}}$ . We do so for two reasons:<sup>21</sup>

1. If given an initial condition  $x^0$  the updates  $T^t(x^0)$  converge, then  $\{T^t(x^0)\}_{t \in \mathbb{N}}$  converges a la Cesaro and<sup>22</sup>

$$C - \lim_t T^t(x^0) = \lim_t T^t(x^0). \quad (18)$$

Intuitively, if a sequence of updates converges, then their time averages converge too and to the same limit. Therefore, conditions on  $T$  which yield that  $C - \lim_t T^t(x)$  exists for all  $x \in B$  are conceptually the weaker counterpart of assumptions which deliver the convergence of  $T$ .

2. Example 2 illustrates the fact that the opposite cannot be true, that is, convergence on average does not yield standard convergence. It is a classic example of a more general fact: Cesaro convergence is strictly weaker than standard convergence. At the same time, Tauberian theory is the study of conditions that paired with Cesaro convergence (or other weaker forms of convergence) imply convergence.<sup>23</sup> Therefore, we first study which properties of  $T$  yield the existence of  $C - \lim_t T^t(x^0)$ , and at a later stage, we study conditions on  $T$  which play the role of Tauberian conditions.

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<sup>21</sup>The notion of Cesaro limit has been used already in the networks literature by Golub and Morris [29]. In particular, they explore the convergence of Abel averages. Under their assumptions, this convergence is equivalent to Cesaro convergence.

<sup>22</sup>See, e.g., A30 in Billingsley [8].

<sup>23</sup>We refer the interested reader to Korevaar [37].

The next result and its corollary show that nonexpansive and, in particular, robust opinion aggregators do not generate wild dynamics, but rather well-behaved ones. Indeed, for each  $x \in B$ , the time averages of the sequence of updates always converge. Moreover, such a convergence is strong, being uniform. Since convergence is uniform, the rate of convergence is independent of the initial condition  $x^0$  (cf. Theorem 4). We start by stating the result for nonexpansive opinion aggregators defined over a compact set.<sup>24</sup>

**Theorem 2** *Let  $T$  be an opinion aggregator. If  $B$  is compact and  $T$  is nonexpansive, then*

$$C - \lim_t T^t(x) \text{ exists} \quad \forall x \in B. \quad (19)$$

Moreover, if  $\bar{T} : B \rightarrow B$  is defined by

$$\bar{T}(x) = C - \lim_t T^t(x) \quad \forall x \in B, \quad (20)$$

then  $\bar{T}$  is continuous and such that  $\bar{T} \circ T = \bar{T}$  as well as

$$\lim_{\tau} \left( \sup_{x \in B} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \bar{T}(x) \right\|_{\infty} \right) = 0. \quad (21)$$

The following corollary is the consequence of combining Lemma 1 with the previous result. It shows that for robust opinion aggregators the same conclusions of Theorem 2 hold even if one dispenses with the assumption of  $B$  being compact. Nevertheless, by dropping compactness, uniform convergence will happen only if we restrict attention to bounded subsets.

**Corollary 5** *Let  $T$  be an opinion aggregator. If  $T$  is a robust opinion aggregator, then  $T$  is nonexpansive and (19) holds. Moreover, if we define  $\bar{T}$  as in (20) and  $\hat{B}$  is a bounded subset of  $B$ , then*

$$\lim_{\tau} \left( \sup_{x \in \hat{B}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \bar{T}(x) \right\|_{\infty} \right) = 0. \quad (22)$$

To sum up, Theorem 2 and Corollary 2 contain two main messages: one conceptual and one mathematical. The conceptual message is linked with our results on the wisdom of the crowd, see Section 7. There, we will give conditions under which the *Cesaro limit* of the updates converge to a true underlying parameter when the size of the society goes to infinity. If the robust opinion aggregators considered there happen to be convergent then we have the *wisdom of the crowd*, i.e., the agents are going to learn

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<sup>24</sup>In Appendix B (see Remarks 3 to 5), we discuss the relation of our results with the mathematical literature of nonlinear ergodic theory and fixed point theory.

the true parameter. Instead, if they are not, the fact that robust aggregators are always Cesaro convergent guarantees *wisdom from the crowd*, i.e., an external observer that can compute the time averages in a part of the society will extract enough information to learn the truth. From a mathematical point of view, in order to address the problem of standard convergence, all we need is a condition that paired with the convergence of time averages yields the usual convergence in norm. In light of the classic paper of Lorentz [41], we know that such a condition exists (termed asymptotic regularity; see Definition 11) and

$$\text{uniform Cesaro convergence} + \text{asymptotic regularity} = \text{convergence.} \quad (23)$$

Note that, given a sequence of vectors (the iterates  $T^t(x^0)$  in our case), uniform Cesaro convergence alone does not guarantee convergence (cf. Example 2).<sup>25</sup> At the same time, it is easy to show that the regularity condition alone does not guarantee convergence.<sup>26</sup> Thus, both conditions are essential to obtain standard convergence.

Before discussing more formally this latter intuition, we first address the issue of what are the limits of the convergent sequences of updates.

## 6.2 Equilibria

As previously argued, our strategy is simple. We first observe that robust opinion aggregators generate a weaker form of convergence. We use this weaker form of convergence as a stepping stone toward conditions that guarantee standard convergence. At the same time a natural question, paired with the issue of convergence, is about the limit itself. In other words, if  $\lim_t T^t(x^0)$  exists, what is it and what are its properties?

In order to answer these questions, fixed points of the operator  $T$  will play a fundamental role. Moreover, they have a natural interpretation: they characterize situations where an opinion distribution does not change once reached. This justifies the following definition.

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<sup>25</sup>See also Theorem 11 and Remark 4, in the appendix, for a formal statement of Lorentz's result and for a discussion of the related mathematical literature.

<sup>26</sup>Given a sequence  $\{x_t\}_{t \in \mathbb{N}} \subseteq \mathbb{R}^n$ , the regularity condition we will consider momentarily is  $\lim_t \|x_{t+1} - x_t\|_\infty = 0$ . For example, consider the sequence  $x_t$  in  $[0, 1]$  defined by

$$x_1 = 1, x_2 = \frac{1}{2}, x_{t+2} = \begin{cases} x_{t+1} + \frac{1}{t+2} & \text{if } x_t < x_{t+1} \text{ and } x_{t+1} + \frac{1}{t+2} \leq 1 \\ x_{t+1} - \frac{1}{t+2} & \text{if } x_t < x_{t+1} \text{ and } x_{t+1} + \frac{1}{t+2} > 1 \\ x_{t+1} - \frac{1}{t+2} & \text{if } x_t \geq x_{t+1} \text{ and } x_{t+1} - \frac{1}{t+2} \geq 0 \\ x_{t+1} + \frac{1}{t+2} & \text{if } x_t \geq x_{t+1} \text{ and } x_{t+1} - \frac{1}{t+2} < 0 \end{cases} \quad \forall t \in \mathbb{N}.$$

It is immediate to see that  $\{x_t\}_{t \in \mathbb{N}}$  satisfies the regularity condition, but does not converge, since it oscillates in  $[0, 1]$ .

**Definition 9** Let  $T$  be an opinion aggregator. The point  $\bar{x} \in B$  is an equilibrium of  $T$  if and only if  $T(\bar{x}) = \bar{x}$ . The set of equilibria is denoted by  $E$ .

The notions of equilibrium and convergence are tied to each other. If a sequence of updates converges, then it necessarily converges to an equilibrium.

**Lemma 2** Let  $T$  be an opinion aggregator and  $x^0 \in B$ . If  $T$  is continuous and  $\{T^t(x^0)\}_{t \in \mathbb{N}}$  converges to  $\bar{x}$ , then  $\bar{x} \in E$ . Moreover, if  $T$  is robust and convergent and  $\bar{T}$  is defined as in (20), then  $\bar{T}(x) = \lim_t T^t(x) \in E$  for all  $x \in B$ .

This simple lemma clarifies the role played by the operator  $\bar{T} : B \rightarrow B$  as defined in (20). Indeed, given  $x^0 \in B$ , if  $T$  is robust, then  $\bar{T}(x^0)$  is the opinion to which the time averages of the updates converge. If  $T$  further happens to be convergent, then  $\bar{T}(x^0)$  is the equilibrium opinion to which the sequence of updates converges to. It is a simple mathematical result, yet it stresses how strong is the property of convergence. Indeed, in contrast, if  $C - \lim_t T^t(x)$  exists, but  $\lim_t T^t(x)$  does not, then it might not be true that  $C - \lim_t T^t(x)$  is an equilibrium.

**Example 3** Assume that  $T : [0, 1]^3 \rightarrow [0, 1]^3$  is defined by

$$T(x) = (\max\{x_2, x_3\}, x_3, x_2) \quad \forall x \in \mathbb{R}^3.$$

It is immediate to check that the operator  $T$  is robust. Set  $x^0 = (1, 0, 1)$  and  $\hat{x} = (1, \frac{1}{2}, \frac{1}{2})$ . It follows that  $C - \lim_t T^t(x^0) = (1, \frac{1}{2}, \frac{1}{2}) = \hat{x}$  and  $T(\hat{x}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \neq \hat{x}$ .  $\blacktriangle$

The next result generalizes Example 2 and shows that  $C - \lim_t T^t(x)$  might not be an equilibrium only for nonlinear opinion aggregators.

**Proposition 9** If  $T$  is a linear opinion aggregator, then  $\bar{T}(x) \in E$  for all  $x \in B$ .

An important subset of equilibria that are always present in our framework is the *consensus* subset. Formally, these vectors are the constant ones, and they represent a situation in which all agents share the same opinion.

**Definition 10** Let  $T$  be an opinion aggregator. We say that  $T$  is a consensus operator if and only if the only equilibria  $T$  might have are vectors of the type  $ke$  with  $k \in I$ .

The properties of convergence and being a consensus operator are separate and independent. To wit, the opinion aggregator  $T$  in Example 2 is a consensus operator, but not a convergent one, while the identity operator is convergent, but not a consensus operator. Following result provides a necessary and sufficient condition for an opinion aggregator to be a consensus operator. This condition is easily checked to be implied

by several types of convergent opinion aggregators we study: namely, the ones that satisfy either the uniform or the pairwise common influencer property (see Sections 6.3.1 and 6.4). Before stating our result, we need to introduce an ancillary object: the map  $\delta : B \rightarrow [0, \infty)$  defined by

$$\delta(x) = \begin{cases} \min_{i,j:x_i \neq x_j} |x_i - x_j| & x \notin D \\ 0 & x \in D \end{cases} \quad \forall x \in B.$$

Intuitively,  $\delta(x)$  is a minimal measure of heterogeneity of opinions within  $x$ .

**Proposition 10** *Let  $T$  be an opinion aggregator.  $T$  is a consensus operator if and only if for each  $x \in B$  there exists  $\varepsilon(x) \in (0, 1)$  such that*

$$\|T(x) - x\|_\infty \geq \varepsilon(x) \delta(x). \quad (24)$$

We conclude by discussing three properties which yield that  $T$  is a consensus operator. The properties will be formally defined momentarily since they are also sufficient conditions for the convergence of  $T$ . Intuitively, the uniform common influencer property amounts to say that there exists an agent  $k$  whose opinion influences all the agents. The pairwise common influencer property is a weakening where each pair of agents  $(i, j) \in N \times N$  is influenced by a pair dependent third agent  $k(i, j) \in N$ . Finally, strong connectedness boils down to saying that, at least in the long run, agents will influence one another.

**Proposition 11** *Let  $T$  be a normalized and monotone opinion aggregator.  $T$  is a consensus operator provided one of the following holds:*

- a.  *$T$  has the pairwise common influencer property;*
- b.  *$T$  has the uniform common influencer property;*
- c.  *$T$  is strongly connected.*

### 6.3 Standard convergence

In this section, we focus on properties that allow us to conclude that an opinion aggregator  $T$  is convergent at  $x^0$  or is convergent tout-court. Most of our global conditions will turn out to be monotonicity properties of  $T$  which are connected to the notion of asymptotic regularity.

**Definition 11** *Let  $T$  be an opinion aggregator. We say that  $T$  is asymptotically regular if and only if for each  $x \in B$*

$$\lim_t \|T^{t+1}(x) - T^t(x)\|_\infty = 0. \quad (25)$$

At first sight, condition (25) might be mistaken for the Cauchy property. In other words, it might seem obvious that if  $\{T^t(x^0)\}_{t \in \mathbb{N}}$  satisfies (25), then  $T^t(x^0)$  converges (being  $\mathbb{R}^n$  complete) and to a point of  $B$  (being  $B$  closed). On the one hand, (25) is much weaker than the Cauchy property.<sup>27</sup> On the other hand, it is a Tauberian condition. In other words, if  $T$  is a nonexpansive opinion aggregator, then the time averages of  $\{T^t(x)\}_{t \in \mathbb{N}}$  converge uniformly (see Theorem 2). If  $\{T^t(x)\}_{t \in \mathbb{N}}$  further satisfies (25), then this is enough to show that  $C - \lim_t T^t(x) = \lim_t T^t(x)$  (see also the equality in (23) and the discussion there). The following two results contain the above observations.

**Theorem 3** *Let  $T$  be a nonexpansive opinion aggregator and  $x^0 \in B$  where  $B$  is compact. The following statements are equivalent:*

- (i)  $T$  is convergent at  $x^0$ ;
- (ii)  $T$  is convergent at  $x^0$  and  $\lim_t T^t(x^0)$  is an equilibrium;
- (iii)  $\lim_t \|T^{t+1}(x^0) - T^t(x^0)\|_\infty = 0$ .

Before moving on, we discuss an example in order to illustrate the local nature of the previous result. We will do so in the linear case. Proposition 12 will show that the observations below apply all the more for the nonlinear case.

**Example 4** Consider the case in which  $B = [0, 1]^4$  and the aggregator  $T$  is linear with matrix

$$W = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Consider  $x^0 \in [0, 1]^4$ . If  $x^0 = (\alpha, \beta, \gamma, \gamma)$  with  $\alpha, \beta, \gamma \in [0, 1]$ , then one can show by induction that

$$\|T^{t+1}(x^0) - T^t(x^0)\|_\infty \leq 2(1 - \varepsilon)^t \|x^0\|_\infty$$

where  $\varepsilon$  can be chosen to be  $1/2$ . If  $x^0 = (\alpha, \beta, \gamma, \delta)$  with  $\alpha, \beta, \gamma, \delta \in [0, 1]$  and  $\gamma \neq \delta$ , then

$$\|T^{t+1}(x^0) - T^t(x^0)\|_\infty \geq |\gamma - \delta| > 0 \quad \forall t \in \mathbb{N}.$$

By Theorem 3, in the first case,  $T$  is convergent at  $x^0$ . In the second case, it is not.  $\blacktriangle$

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<sup>27</sup>See also Footnote 25.

As the example above clarifies, the property  $\lim_t \|T^{t+1}(x^0) - T^t(x^0)\|_\infty = 0$  is a local one. However, an important difference between our aggregators and the linear one is the relation between local and global conditions in the case of convergence to consensus, as captured by the following result.

**Proposition 12** *Let  $T$  be a linear aggregator. Either*

(i)  *$T$  is a convergent consensus operator,*

(ii) *or the set*

$$\left\{ x \in B : \lim_{t \rightarrow \infty} T^t(x) = ke \text{ for some } k \in I \right\}$$

*has (Lebesgue)-measure 0.*

*This is not the case for general robust aggregators.*

The previous proposition is telling us that generically, in the linear case local conditions for the convergence to consensus imply global ones. This is not generally true for robust opinion aggregators. At the same time, if we require the condition for convergence (i.e.,  $\lim_t \|T^{t+1}(x^0) - T^t(x^0)\|_\infty = 0$ ) to hold globally (this is equivalent to require  $T$  being asymptotically regular), then it yields a remarkable form of convergence, provided  $T$  is also constant affine.

**Theorem 4** *Let  $T$  be a robust opinion aggregator. The following are equivalent facts:*

(i)  *$T$  is asymptotically regular;*

(ii)  *$T$  is convergent.*

*Moreover, if  $T$  is constant affine, then they are also equivalent to the following:*

(iii) *There exists  $\{c_t\}_{t \in \mathbb{N}} \subseteq [0, \infty)$  such that  $c_t \rightarrow 0$  and*

$$\|\bar{T}(x) - T^t(x)\|_\infty \leq c_t \|x\|_\infty \quad \forall t \in \mathbb{N}, \forall x \in B. \quad (26)$$

The previous result shows that asymptotic regularity characterizes convergence for robust opinion aggregators over  $B$ . In this case, the condition of compactness of  $B$  can be dropped. Moreover, under constant affinity (which is satisfied in several relevant cases, cf. Sections 3.1 and 6.5) the condition (26) allows us to conclude a remarkable feature of constant affine robust opinion aggregators. Indeed, if  $T$  is convergent, then the rate of convergence, captured by the sequence  $\{c_t\}_{t \in \mathbb{N}}$ , is *independent* of the initial condition  $x$ . That said, asymptotic regularity is hard to interpret in terms of economic intuition. In what follows, we discuss different properties, whose economic interpretation seems more immediate, which yield asymptotic regularity. By being stronger than asymptotic regularity, we will also be able to say more about the rate of convergence and the limits of the updates.

### 6.3.1 Monotonicity, convergence, and rate of convergence

The next part of the section is devoted to discussing elementary notions of monotonicity of  $T$ . Their economic interpretation is rather intuitive, while mathematically they turn out to imply asymptotic regularity, hence convergence. Moreover, as the next section will illustrate they also have a natural network interpretation.

**Definition 12** *Let  $T$  be a robust opinion aggregator. We say that  $T$  is self-influential if and only if there exists  $\varepsilon \in (0, 1)$  such that for each  $i \in N$ ,  $y \in B$ , and  $h \in \mathbb{R}$  such that  $y + he_i \in B$*

$$T_i(y + he_i) - T_i(y) \geq \varepsilon h. \quad (27)$$

From an economic point of view, this property characterizes a situation where the opinion of each agent has a form of own history dependence. Indeed, if we focus on a generic agent  $i$ , given two instances  $x$  and  $y$ , if the only difference is the agent's opinion, say  $x_i > y_i$ , then her revised opinion is strictly higher in the first instance than in the second one. In a repeated setting, information gathered in the past is not entirely dismissed in light of new evidence.

**Theorem 5** *Let  $T$  be a robust opinion aggregator. If  $T$  is self-influential, then  $T$  is asymptotically regular and, in particular, convergent.*

The previous result is quite strong since a very intuitive and weak condition yields dynamics which are rather well behaved. So one might be left to wonder what type of aggregators might fail to be self-influential. The next example discusses two important cases.

**Example 5** Assume  $T$  is linear with matrix  $W$ . Clearly,  $T$  is *not* self-influential if and only if there exists an agent  $i$  such that  $w_{ii} = 0$ , that is, an agent whose own opinion never enters in her updating rule. Another important example of robust opinion aggregators which may fail to be self-influential are those such that each  $T_i$  corresponds to a quantile functional (see Section 6.5 for a characterization of the dynamics induced by these aggregators). The intuition, in this case, is simple. Quantiles tend to disregard outliers, be those the opinions of the agent or not. In this case, an aggregator of this kind is self-influential if and only if it is the identity (see Proposition 15).  $\blacktriangle$

We proceed by studying conditions which guarantee the convergence of  $T$ . The following two conditions have both a common mathematical and economic intuition. From a mathematical point of view, they are monotonicity conditions as before. From an economic point of view, they correspond to the idea that each pair of different agents  $(i, j)$  shares a joint third agent  $k$  whose opinion matters, where  $k$  might be shared or not across pairs (Definitions 13 and 14 respectively).

**Definition 13** Let  $T$  be an opinion aggregator. We say that  $T$  has the uniform common influencer property if and only if there exist  $k \in N$  and  $\varepsilon \in (0, 1)$  such that for each  $i \in N$ ,  $y \in B$ , and  $h \in \mathbb{R}$  such that  $y + he_k \in B$

$$T_i(y + he_k) - T_i(y) \geq \varepsilon h. \quad (28)$$

This property is better understood in light of the network interpretation of a robust opinion aggregator. In Section 6.4, we argue that every opinion aggregator induces a network. The existence of a uniform common influencer  $k$  requires an extreme centrality for that agent. Indeed, all the standard centrality measures used in the Social Network literature are maximal for  $k$ .

Following result shows that the uniform common influencer property yields asymptotic regularity. Moreover, it delivers a very strong form of convergence. Indeed, the sequence of updates always converges to consensus (see Proposition 11) and exponentially fast: thus, in (26) we can choose  $c_t = 2(1 - \varepsilon)^t$  and we can dispense with constant affinity.

**Theorem 6** Let  $T$  be a robust opinion aggregator. If  $T$  has the uniform common influencer property, then  $\bar{T}(x) = \lim_t T^t(x) \in D$  for all  $x \in B$  and

$$\|\bar{T}(x) - T^t(x)\|_\infty \leq 2(1 - \varepsilon)^t \|x\|_\infty \quad \forall t \in \mathbb{N}, \forall x \in B. \quad (29)$$

A version of the previous result holds even if a weaker form of monotonicity holds.

**Definition 14** Let  $T$  be an opinion aggregator. We say that  $T$  has the pairwise common influencer property if and only if for each  $i, j \in N$  with  $j \neq i$  there exists  $k \in N$  and  $\varepsilon_{ij} \in (0, 1)$  such that for each  $y \in B$  and  $h \in \mathbb{R}$  such that  $y + he_k \in B$

$$T_i(y + he_k) - T_i(y) \geq \varepsilon_{ij} h \text{ and } T_j(y + he_k) - T_j(y) \geq \varepsilon_{ij} h. \quad (30)$$

Under this weaker property, we do not require the existence of an extremely central agent  $k$ . However, for every pair of agents, there must be an individual whose opinion is relevant for both of them. Intuitively, this is a minimal requirement about the presence of a source of information trusted by both agents. A typical situation where we expect the pairwise common influencer property to hold is one of asymmetric networks with a bunch of media listened by the other agents. If there is a minimal overlapping in the media listened by the agents, the property holds.

The next result shows that the pairwise common influencer property yields a strong form of convergence. Indeed, the rate of convergence is almost exponential: thus, in (26) we can choose  $c_t = 2(1 - \varepsilon)^{\lfloor \frac{t}{\hat{t}} \rfloor}$  for some  $\varepsilon \in (0, 1)$  and  $\hat{t} \in \mathbb{N}$  and also here we can dispense with constant affinity.<sup>28</sup>

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<sup>28</sup>Recall that, given  $s \in (0, \infty)$ ,  $\lfloor s \rfloor$  is the integer part of  $s$ , that is, the greatest integer  $l$  such that  $s \geq l$ .

**Theorem 7** *Let  $T$  be a robust opinion aggregator. If  $T$  has the pairwise common influencer property, then  $\bar{T}(x) = \lim_t T^t(x) \in D$  for all  $x \in B$ . Moreover, there exist  $\varepsilon \in (0, 1)$  and  $\hat{t} \in \mathbb{N}$  such that  $\hat{t} \leq n^2 - 3n + 3$  and*

$$\|\bar{T}(x) - T^{\hat{t}}(x)\|_{\infty} \leq 2(1 - \varepsilon)^{\lfloor \frac{\hat{t}}{t} \rfloor} \|x\|_{\infty} \quad \forall t \in \mathbb{N}, \forall x \in B. \quad (31)$$

## 6.4 A network interpretation

In the standard DeGroot's linear model and its variations (see, e.g., Golub and Jackson [27, Proposition 1] and DeMarzo, Vayanos, and Zwiebel [17, Theorem 1]), convergence to consensus is characterized in terms of the connectedness of an underlying network structure. In what follows, we extend the notion of the underlying network structure to the nonlinear case, and we prove that the characterization of convergence to consensus for the linear case is a sufficient condition for convergence to consensus for robust opinion aggregators.

Consider the population  $N$ . We will say that agent  $j$  strongly influences  $i$  if and only if there exists  $\varepsilon_{ij} \in (0, 1)$  such that for each  $y \in B$ , and  $h \in \mathbb{R}$  such that  $y + he_j \in B$

$$T_i(y + he_j) - T_i(y) \geq \varepsilon_{ij}h. \quad (32)$$

The interpretation of (32) is simple: if the opinion of  $j$  increases by  $h$  the update of  $i$  increases at least by  $\varepsilon_{ij}h$ . Consider now the directed network given by the adjacency matrix  $A(T)$  such that:<sup>29</sup>

$$\begin{aligned} a_{ij} = 1 &\iff j \text{ strongly influences } i \\ a_{ij} = 0 &\iff j \text{ does not strongly influence } i. \end{aligned}$$

This graph is the *minimal* network underlying the operator, since the condition for  $a_{ij} = 1$  is strong, it requires that  $i$  is influenced by  $j$  starting from every vector of opinions  $y$ . The fact that this network is a lower bound to the connections in  $N$  will be evident in Example 6 and the study of Choquet aggregators in Section 6.5.

Next, we translate some standard Social Network properties in the framework of our aggregators.

**Definition 15** *Let  $T$  be an opinion aggregator. We say that:*

1.  $N' \subseteq N$  is closed under  $T$  if and only if  $i \in N'$  and  $j \notin N'$  implies  $a_{ij} = 0$ ;
2.  $N' \subseteq N$  is strongly connected under  $T$  if and only if  $A(T)$  restricted to  $N'$  is an irreducible matrix. We say that  $T$  is strongly connected if and only if  $N$  is strongly connected under  $T$ ;

---

<sup>29</sup>Note that, differently from the linear case, it is completely plausible that given an agent  $i$ , there is no agent  $j$  strongly influencing her.

3.  $N' \subseteq N$  is aperiodic under  $T$  if and only if  $A(T)$  is aperiodic when restricted to  $N'$ . We say that  $T$  is aperiodic if and only if  $N$  is aperiodic under  $T$ .

Notice that when  $T$  is linear, the definitions above coincide with the usual ones. We conclude by studying the convergence properties of robust opinion aggregators whose induced network satisfies the conditions for convergence to consensus that have been established for linear operators. Recall that a linear operator  $T$  is convergent to a consensus if and only if there exists a unique closed class  $N' \subseteq N$  such that  $N'$  is strongly connected and aperiodic under  $T$  (see, e.g., Jackson [35, Corollary 8.3.1]).<sup>30</sup> Next proposition shows that this condition, plus nontriviality of the network, is sufficient for convergence to consensus of robust opinion aggregators too.

**Theorem 8** *Let  $T$  be a robust opinion aggregator. If*

- a. *no row of  $A(T)$  is null,*
- b. *there exists a unique closed and strongly connected class  $N' \subseteq N$  under  $T$ , and  $N'$  is aperiodic under  $T$ ,*

*then  $\bar{T}(x) = \lim_t T^t(x) \in D$  for all  $x \in B$ .*

*Moreover, there exists  $\varepsilon \in (0, 1)$  and  $\hat{t} \in \mathbb{N}$  such that*

$$\|\bar{T}(x) - T^t(x)\|_\infty \leq 2(1 - \varepsilon)^{\lfloor \frac{t}{\hat{t}} \rfloor} \|x\|_\infty \quad \forall t \in \mathbb{N}, \forall x \in B. \quad (33)$$

*Also, we have that  $\hat{t} \leq |N'|^2 - 4|N'| + 3 + n$ , and if  $T$  is also self-influential, then  $\hat{t} \leq n - 1$ .*

In many applications of the linear model, convergence to consensus is obtained by assuming that the matrix  $W$  is irreducible. Irreducibility of  $W$  is equivalent to assume that the entire underlying network is strongly connected. A similar observation holds for the nonlinear case as an immediate corollary of our previous result.

**Corollary 6** *Let  $T$  be a robust opinion aggregator. If  $T$  is strongly connected and aperiodic, then  $\bar{T}(x) = \lim_t T^t(x) \in D$  for all  $x \in B$ . Moreover, there exist  $\varepsilon \in (0, 1)$  and  $\hat{t} \in \mathbb{N}$  such that*

$$\|\bar{T}(x) - T^t(x)\|_\infty \leq 2(1 - \varepsilon)^{\lfloor \frac{t}{\hat{t}} \rfloor} \|x\|_\infty \quad \forall t \in \mathbb{N}, \forall x \in B. \quad (34)$$

*Also, we have that  $\hat{t} \leq n^2 - 3n + 3$  and if  $T$  is also self-influential, then  $\hat{t} \leq n - 1$ .*

---

<sup>30</sup>In this case, without loss of generality, we will always think of  $N'$  as being equal to the first  $|N'|$  agents, that is,  $N' = \{1, \dots, |N'|\}$ .

Given the above results, one is left to wonder whether strong connectedness and aperiodicity are also necessary for convergence to consensus. Indeed, in the linear case, given the existence of a unique strongly connected and closed class, aperiodicity is both sufficient and necessary. Interestingly, this fails to be the case for robust aggregators, as the following example illustrates.

**Example 6** Let  $B = \mathbb{R}^2$  and let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined for each  $x \in \mathbb{R}^2$  as

$$T_1(x) = \begin{cases} x_2 & \text{if } x_1 > x_2 \\ \frac{x_1}{2} + \frac{x_2}{2} & \text{if } x_2 \geq x_1 \end{cases}$$

$$T_2(x) = x_1$$

It is easy to check that  $T$  is a robust opinion aggregator such that.<sup>31</sup>

$$A(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus,  $T$  is strongly connected, but it fails to be aperiodic. Nevertheless, it is easy to show that

$$\lim_t T^t(x) = \begin{pmatrix} \min\{x_1, x_2\} \\ \min\{x_1, x_2\} \end{pmatrix} \quad \forall x \in \mathbb{R}^2$$

▲

The fact that aperiodicity might fail to be necessary is the consequence of our definition of network underlying an operator  $T$ . For, the definition of  $A(T)$  is conceptually the definition of a minimal network. Loosely speaking,  $a_{ij} > 0$  if and only if  $i$  is influenced by  $j$  at *every* possible initial opinion. In the linear case, local conditions are global: if  $i$  is influenced by  $j$  when the vector of opinion is  $x$ , she is influenced by  $j$  for every other possible opinion. This is not the case for general robust aggregators. In the previous example, 1 is influenced by herself if and only if  $x_2 \geq x_1$ , and therefore  $a_{11} = 0$ . However, it turns out that regardless of the starting opinions, she will be influenced by herself infinitely often, and this is enough to have convergence to consensus.

On a similar note, we should be cautious in concluding that, since the conditions inducing consensus for robust aggregators resemble the one for the linear case, we can follow the insights from the latter to study whether polarization or consensus will be reached in the limit. Indeed, when the aggregator is linear an analyst interested in consensus may be tempted by the following procedure: postulate a network structure  $W \in \mathcal{W}$  such that  $W$  is convergent to consensus, and check whether

$$\forall (i, j) : w_{ij} > 0, \text{ there exists } x \in B, h > 0 \text{ such that } T_i(x + he_j) - T_i(x) > 0. \quad (35)$$

---

<sup>31</sup>Actually,  $T$  is a Choquet aggregator (cf. Section 6.5).

Having this guarantees to the analyst that consensus is reached in the limit, since  $W$  will be absolutely continuous with respect to the real social network  $\bar{W}$ , so  $T = \bar{W}(\cdot)$  is *always* convergent to consensus, too. However, such a conclusion is *always* unwarranted in the case of robust aggregators.

**Proposition 13** *Suppose that  $n \geq 2$ , and let  $W \in \mathcal{W}$  be a convergent consensus operator. Then, there exists  $T$  robust satisfying (35) such that consensus is never reached in the limit unless the initial opinion is already a consensus.*

Despite this negative result, when we know the loss function originating our aggregator, it is possible to use it to check whether agent  $j$  strongly influences agent  $i$ , see Proposition 7 and Corollary 4. However, these results do not cover an important case analyzed in Section 3.1: the aggregators derived from  $L_p$ -seminorms, since they usually have  $a_{ij} = 0$  for all  $i$  and  $j$  in  $N$ . However, a different proof technique shows that, whenever the underlying network structure  $W \in \mathcal{W}$  satisfies the pairwise common influencer property and  $p > 2$ , consensus is always reached in the limit.

**Proposition 14** *Suppose that the profile of loss functions is given by  $\phi$ , and for all  $i \in N$*

$$\phi_i(z) = \sum_{j \in N} w_{ij} |z_j|^{p_i}$$

*with  $p_i \geq 2$  and  $W$  scrambling. Then, the robust opinion aggregator  $T$  obtained as the unique best reply function to  $\phi$  is a convergent consensus operator.*

The following example shows that by relaxing linearity, our aggregators are able to capture the idea that the weights agents assign each other are not entirely separable from the differences in what they think. In particular, robust opinion aggregators also arise naturally when the standard DeGroot model is modified to allow for confirmatory bias on top of the social network structure proposed above.

**Example 7 (Confirmatory bias)** It is often argued that in some societies individuals tend to trust more those people whose opinion confirms their original prior. This phenomenon can be captured by the modification of DeGroot's linear model proposed in Jackson [35]. As in the linear model, the society is represented by a stochastic matrix  $W$  where  $w_{ij}$  is the weighted assigned by individual  $i$  to agent  $j$ . However, to aggregate opinions every individual downweight the agents who disagree the most with her:

$$T(x) = W(x)x$$

where

$$w_{ij}(x) = \frac{e^{-\gamma_{ij}|x_i - x_j|}}{\sum_{k=1}^n e^{-\gamma_{ik}|x_i - x_k|}}$$

with  $\gamma_{ij} \in (0, 1]$ . Here,  $\frac{1}{\gamma_{ij}}$  captures the relative strength of the weight assigned by individual  $i$  to agent  $j$  net of the difference in their opinions. Let  $I = [0, 1]$ . It is easy to see that the aggregator  $T$  is robust and satisfy some of the monotonicity conditions that guarantee convergence, e.g., self influence.  $\blacktriangle$

## 6.5 Choquet aggregators

In this section, we consider a particular example of robust opinion aggregators. This class encompasses linear opinion aggregators, as well as those aggregators whose components are either any quantile functional (e.g., the median) or any order statistics.

**Definition 16** *Let  $T$  be an opinion aggregator. We say that  $T$  is a Choquet aggregator if and only if  $T$  is normalized, monotone, and comonotonic additive.*

It is routine to show that Choquet aggregators are constant affine and, in particular, robust. Right below, we offer a characterization which turns out to be useful in exploring the behavior of aggregators such as the one in Example 1. Moreover, it justifies our terminology. Indeed, if  $T$  is a Choquet aggregator, it follows that each  $T_i$  is normalized, monotone, and comonotonic additive. Given Schmeidler [56, p. 256], it is well known that a map  $T_i$  has these properties if and only if there exists a unique capacity  $\nu_i : 2^N \rightarrow [0, 1]$  such that

$$T_i(x) = \mathbb{E}_{\nu_i}(x) \quad \forall x \in B \quad (36)$$

where the latter is a Choquet expectation. On the one hand, a capacity is a set function with the following properties:

1.  $\nu_i(\emptyset) = 0$  and  $\nu_i(N) = 1$ ;
2.  $A \supseteq B$  implies  $\nu_i(A) \geq \nu_i(B)$ .

On the other hand, the Choquet expectation for a function defined over a finite set of points (i.e., a vector) is a rather simple object. First, in words, one should order the components of  $x$  from the highest to the lowest. Formally, this is done with the help of a permutation  $\pi$  over  $N$  such that  $x_{\pi(1)} \geq \dots \geq x_{\pi(n)}$ . Then, one computes the weight given to  $x_{\pi(l)}$  in terms of  $\nu_i$  which is

$$p_l \stackrel{def}{=} \nu_i(\cup_{j=1}^l \{\pi(j)\}) - \nu_i(\cup_{j=1}^{l-1} \{\pi(j)\}) \quad \forall l \in N \quad (37)$$

with the assumption that  $\nu_i(\cup_{j=1}^0 \{\pi(j)\}) = 0$ . If the values taken by  $x$  were pairwise distinct and we interpret  $\nu$  as a measure of likelihood, (37) is exactly the likelihood of observing a value greater than or equal to  $x_{\pi(l)}$  minus the likelihood of observing

strictly higher values. The Choquet expectation is nothing else than the average of the ordered values of  $x$  using the probability vector  $p$ :

$$\mathbb{E}_{\nu_i}(x) = \sum_{l=1}^n x_{\pi(l)} [\nu_i(\cup_{j=1}^l \{\pi(j)\}) - \nu_i(\cup_{j=1}^{l-1} \{\pi(j)\})] = \sum_{l=1}^n x_{\pi(l)} p_l \quad \forall x \in \mathbb{R}^n. \quad (38)$$

Note that if  $\nu$  is a standard additive probability, then the Choquet expectation coincides with the standard notion of expectation. If  $T$  is a Choquet aggregator, then we denote by  $\nu_i$  the capacity that represents  $T_i$  as in (36).

Next proposition shows that for comonotonic aggregators with  $\{0, 1\}$ -valued capacities, if convergence happens, then it happens in a finite number of periods. These aggregators are important since they encompass aggregators in which each agent aggregates opinions, for example, according to one of the following criteria: the median, any more general quantile, max, min, or any more general order statistic.

**Theorem 9** *Let  $T$  be a Choquet aggregator such that  $\nu_i$  is a  $\{0, 1\}$ -valued capacity for all  $i \in N$ . If  $x \in B$ , then either  $\{T^t(x)\}_{t \in \mathbb{N}}$  converges or it is eventually periodic, that is, there exists  $\bar{t}, p \leq n^n$  such that*

$$T^{t+p}(x) = T^t(x) \quad \forall t \geq \bar{t}. \quad (39)$$

*Moreover,  $\{T^t(x)\}_{t \in \mathbb{N}}$  converges if and only if it becomes constant after at most  $n^n$  periods.*

**Remark 1** The previous result provides an easy criterion to discern the behavior of the sequence of updates  $\{T^t(x)\}_{t \in \mathbb{N}}$ . Set  $t = n^n$  where  $n$  is the number of agents in the population, and so the maximum value of distinct components  $x$  can have. If  $T^t(x) = T^{t+1}(x)$ , then  $\{T^t(x)\}_{t \in \mathbb{N}}$  converges. If  $T^t(x) \neq T^{t+1}(x)$ , then  $\{T^t(x)\}_{t \in \mathbb{N}}$  is eventually periodic with period smaller than or equal to  $n^n$ .  $\blacktriangle$

One might wonder what additional restrictions the monotonicity conditions we studied above impose on the Choquet aggregators studied in Theorem 9. The result below is a negative one.

**Proposition 15** *Let  $T$  be a Choquet aggregator such that  $\nu_i$  is a  $\{0, 1\}$ -valued capacity for all  $i \in N$ . The following statements are equivalent:*

- (i)  $j$  strongly influences  $i$ ;
- (ii)  $T_i(x) = x_j$  for all  $x \in B$ .

In particular,  $T$  is self influential if and only if  $T(x) = x$  for all  $x \in B$  and  $T$  has the pairwise common influencer property if and only if there exists  $k \in N$  with  $T(x) = J_k x$  for all  $x \in B$ .

We have noticed already how convergence is implied by the seemingly natural assumption of self-influence (see Definition 12). However, for Choquet aggregators which are represented by  $\{0, 1\}$ -valued capacities, self-influence is too strong of an assumption, since the only Choquet aggregator of this type which is self-influential is the one that coincides with the identity. More generally, each agent  $i$  trusts only one other individual  $j$  who is independent on the initial opinion distribution. Self-influence yields that  $j$  coincides with  $i$ , both common influencer properties yield that  $j$  is a uniform common influencer  $k$ .

It is important not to overstate the reach of this negative result. Indeed, a parallel with Decision Theory suggests a natural way in which Choquet aggregators satisfy the properties of Section 6.3.1 can arise. Suppose that a stochastic matrix  $W \in \mathcal{W}$  captures the network of agents as in DeGroot's model. Still, it is possible that, as in Cumulative Prospect Theory for choice under risk, agent  $i$  does not linearly compute the average, but he uses a probability distortion function  $f_i$  instead. Formally, in this case

$$T_i(x) = \sum_{l=1}^n x_{\pi(l)} \left[ f_i \left( \sum_{j=1}^l w_{i\pi(j)} \right) - f_i \left( \sum_{j=1}^{l-1} w_{i\pi(j)} \right) \right]. \quad (40)$$

Since a probability distortion function is a strictly increasing function mapping  $[0, 1]$  into itself, with  $f_i(0) = 0 = 1 - f_i(1)$ , it is not difficult to show that whenever  $W$  satisfies the condition for convergence in the linear model,  $T$  is convergent, too.

**Proposition 16** *If  $T$  is a Choquet aggregator defined as in (40) then  $i$  strongly influences  $j$  if and only if  $w_{ij} > 0$ .*

However, such a specification is still able to explore economically relevant phenomena that are precluded by linear aggregators, possibly using the tools developed in Decision Theory. As an example, if  $f_i$  is set equal to the prominent Prelec's probability weighting function [52], (i.e.,  $f_i(p) = \exp(-(-\ln(p))^\alpha)$ ) we obtain a one parameter function with a clear psychological foundation. Indeed, such a functional specification characterizes an agent who is particularly sensitive to the *range* of opinions in the distribution, and that assigns disproportionately high weight to extreme stances, with the size of the distortion decreasing in  $\alpha \in (0, 1)$ , see Figure 2. More generally, using an  $f$  different from the identity map is a way to introduce a *perception bias* a la Banerjee and Fudenberg [5] in a model of naive and nonequilibrium learning.

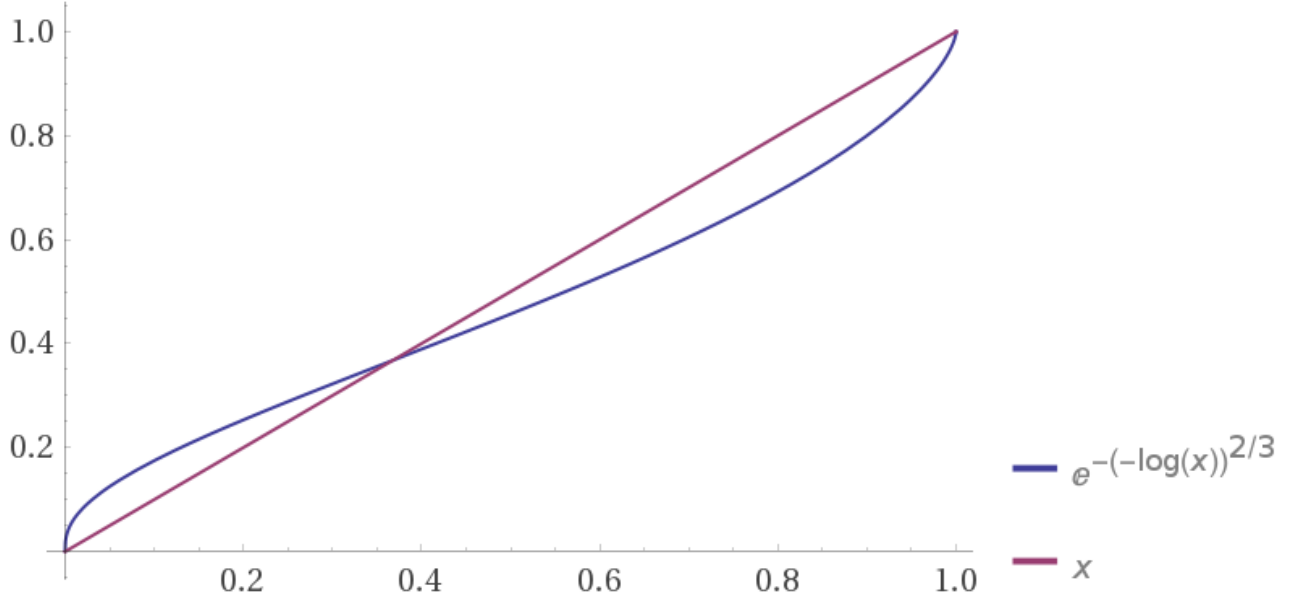


Figure 2: Probability Distortion Function

We conclude the section on Choquet aggregators by discussing another subclass for which convergence is rather easy to obtain. Before doing so, observe that for any permutation  $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  we could consider

$$B_\pi = \{x \in B : x_{\pi(1)} \geq x_{\pi(2)} \geq \dots \geq x_{\pi(n)}\}.$$

In other words,  $x \in B_\pi$  if and only if the  $i$ -th highest opinion belongs to agent  $\pi(i)$  for all  $i \in N$ . If we denote by  $\Pi$  the collection of all permutations, the set  $B$  is the union of these subsets:  $B = \cup_{\pi \in \Pi} B_\pi$ . Given (38), it follows that for each  $\pi \in \Pi$  there exists a stochastic matrix  $W_\pi \in \mathcal{W}$  such that

$$T(x) = W_\pi x \quad \forall x \in B_\pi. \quad (41)$$

Thus, if  $x \in B_\pi$ , then  $T(x) = W_\pi x$  and the update  $T(x)$  belongs to a set  $B_{\pi'}$ . A priori,  $B_{\pi'} \neq B_\pi$ , yielding that  $T^2(x) = W_{\pi'} W_\pi x$ . More in general, at each round, the updating is done via a stochastic matrix that might change, but comes from the finite set  $\{W_\pi\}_{\pi \in \Pi}$ . The next condition guarantees that only the first matrix  $W_\pi$  depends on  $x \in B_\pi$  and then  $T^t(x) = W_\pi^t x$  for all  $t \in \mathbb{N}$ .

**Definition 17** *Let  $T$  be an opinion aggregator. We say that  $T$  is assortative if and only if for each  $\pi \in \Pi$ ,  $x \in B_\pi$ ,  $i, j \in N$  such that  $i \geq j$ , and for each  $k \in N$*

$$T_{\pi(j)}(x^k) - T_{\pi(j)}(x) \geq T_{\pi(i)}(x^k) - T_{\pi(i)}(x).$$

where

$$x_{\pi(l)}^k = \begin{cases} x_{\pi(l)} + \varepsilon & \pi(l) \leq k \\ x_{\pi(l)} & \pi(l) > k \end{cases}$$

and  $\varepsilon > 0$  is such that  $x_{\pi(l)} + \varepsilon \in I$  for all  $l \in N$  such that  $\pi(l) \leq k$ .

In words, being assortative means that, given an initial vector of opinions  $x$ , individuals with higher opinions assign a higher weight to individuals with high opinions. The idea behind the definition is that, in an assortative society, a change in the stance of the individuals with a higher opinion (i.e., those with  $\pi(l) \leq k$ ) affects more the individuals with a higher opinion, because they assign to them more weight.

Assortative societies naturally arise when agents are allowed to endogenously select their network and try to trade-off the benefit of interactions with dislike to disagreement (see, e.g., Bolletta and Pin [9] and Frick, Ijima, and Ishi [22]). It turns out that when a Choquet aggregator  $T$  is assortative, given the starting point  $x$ , the evolution of the system is described by iteration of a stochastic matrix  $W$  as in the linear case.

**Proposition 17** *Let  $T$  be a Choquet aggregator. If  $T$  is assortative, then there exists  $W \in \{W_\pi\}_{\pi \in \Pi}$  such that if  $x \in B_\pi$  then*

$$T^t(x) = W_\pi^t x \quad \forall t \in \mathbb{N}.$$

As a consequence, in assortative societies, it is straightforward to compute the long-run dynamics of opinion. Indeed, for every initial opinion  $x \in B_\pi$ , the results in the linear case (see Golub and Jackson [27]) applied to  $W_\pi$  characterize the limit weight of each agent in the society. However, these weights are dependent on the initial distribution: starting from  $y \in B_{\pi'}$ ,  $\pi \neq \pi'$  there is no guarantee that limit influence of each agent is the same.

**Example 8 (Assortativeness)** An example of assortative Choquet aggregator is the one where each agent  $i \in N$  aggregates opinions using the capacity

$$\nu_i(A) = \begin{cases} 1 & A = N \\ \frac{1}{2} & A \neq N \text{ and } i \in A \\ 0 & \text{otherwise.} \end{cases}$$

▲

## 6.6 Alternative updating rules

In DeGroot's *linear* model, given  $x^0$ , the updates' dynamics is of the type

$$x^t = T(x^{t-1}) \quad \forall t \in \mathbb{N}$$

where  $T$  is linear. Insofar, what we proposed was the study of the same type of dynamics where the aggregator  $T$  was only assumed to be robust, thus not necessarily linear. At the same time, despite keeping the assumption that  $T$  is linear, other types of opinion evolution have been studied in the literature (see, e.g., Jackson [35, Chapter 8]). In this section, we focus on two particular examples: the one of Friedkin and Johnsen [23] as well as the one of DeMarzo, Vayanos, and Zwiebel [17].

We start by considering the procedure of [17]. In this case, DeMarzo, Vayanos, and Zwiebel have agents revise an opinion  $x$  at each round with a linear operator  $T$ , but they also allow agents to varying the weight they give to their own beliefs resulting in the following revision dynamic:

$$x^t = T_t(x^{t-1}) \text{ and } T_t = (1 - \lambda_t)I + \lambda_t T \quad \forall t \in \mathbb{N}. \quad (42)$$

They further assume that  $\{\lambda_t\}_{t \in \mathbb{N}} \subseteq (0, 1]$  and  $\sum_{t=1}^{\infty} \lambda_t = \infty$ . Moreover, given their foundation (see Section 3.1),  $T$  is linear and self-influential. The above condition on the weights  $\lambda_t$  intuitively captures the idea that agents cannot get fixed on their own opinion too quickly. By definition, since  $T$  is linear, there exists a stochastic matrix  $W \in \mathcal{W}$  such that  $T(x) = Wx$  for all  $x \in B$ . Theorem 1 of [17] shows that if  $W$  is irreducible, then  $\{x^t\}_{t \in \mathbb{N}}$  converges and to an equilibrium point  $\bar{x}$  of  $T$ . It turns out that  $\bar{x}$  is also a consensus opinion. In what follows, we generalize this result in two directions. First, we show that if  $T$  is robust and self-influential, then the sequence of updates defined as in (42) still converges and to a fixed point of  $T$ . Second, we can also offer a version where  $T$  is not necessarily assumed to be self-influential. This comes at the cost of requiring  $\lambda_t$  to be bounded away from 1, that is,  $\lambda_t \leq b < 1$  for all  $t \in \mathbb{N}$  and for some  $b \in (0, 1)$ . Intuitively, this means that agents, at each round, are stuck on their own opinion for at least a factor of  $1 - b$ , which can be small, but must also be strictly positive.

**Proposition 18** *Let  $T$  be a robust opinion aggregator,  $x^0 \in B$ , and  $\{\lambda_t\}_{t \in \mathbb{N}} \subseteq (0, 1]$  such that  $\sum_{t=1}^{\infty} \lambda_t = \infty$ . The following statements are true:*

1. *If  $T$  is self-influential and  $\{x^t\}_{t \in \mathbb{N}}$  is defined as in (42), then  $\lim_t x^t$  exists and it is an equilibrium of  $T$ .*
2. *If there exists  $b \in (0, 1)$  such that  $\lambda_t \leq b$  for all  $t \in \mathbb{N}$  and  $\{x^t\}_{t \in \mathbb{N}}$  is defined as in (42), then  $\lim_t x^t$  exists and it is an equilibrium of  $T$ .*

Note that for both results consensus is the only possible limit, provided  $T$  is a consensus operator (cf. Proposition 11). Note also that point 2 generalizes [17] in the linear case too. If each agent has a minimal stickiness to her own opinion (i.e.,  $b > 0$ ),

then convergence happens. Moreover, if the conditions in Proposition 10 are met, the limit point is a consensus equilibrium. This fact is remarkable because convergence to a consensus here does not rely on the usual aperiodicity and self-influentiality properties.

We next consider the procedure of Friedkin and Johnsen [23] (see also [30, p. 16]), which is popular in sociology. In this case, agents are assumed to aggregate an opinion  $x$  at each round with a linear operator  $T$  (represented by a matrix  $W$ ), but they are also allowed to hold onto their original opinion:

$$x^t = \alpha T(x^{t-1}) + (1 - \alpha)x^0 \quad \forall t \in \mathbb{N}. \quad (43)$$

Friedkin and Johnsen interpret  $\alpha$  as a measure of agents' susceptibilities to personal influence. In other words, the  $t$ -th update of agent  $i$  is a mixture of her linear  $t - 1$ -th update,  $x^{t-1}$ , and her initial opinion  $x_i^0$  where the mixture weight is given by  $\alpha$ . The convergence of the corresponding different updating process is derived under the assumption  $\alpha \in [0, 1)$  for all  $i \in N$ .<sup>32</sup>

In what follows, we study the Friedkin and Johnsen updating process in (43) when  $T$  is only assumed to be robust, but not necessarily linear. Following result shows that if  $T$  is also self-influential, then the procedure in (43) yields a convergent updating dynamics.

**Proposition 19** *Let  $T$  be a robust opinion aggregator,  $x^0 \in B$ . If  $T$  is self-influential,  $\alpha \in (0, 1)$ , and  $\{x^t\}_{t \in \mathbb{N}}$  is defined as in (43), then  $\bar{x} = \lim_t x^t \in B$  exists and it is such that*

$$\bar{x} = \alpha T(\bar{x}) + (1 - \alpha)x^0. \quad (44)$$

## 6.7 Representation and differential approach

In this section, we explore a representation for our opinion aggregators. For this purpose, consider an opinion aggregator  $T$  which is only required to be normalized, monotone, and continuous. Thus, it might fail to be nonexpansive and, a fortiori, robust. In particular, we begin by fixing  $i \in N$  and focusing our attention on the  $i$ -th component of  $T$ :  $T_i$ .

By Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi [12, Corollary 3],<sup>33</sup> we have that there exists a closed and convex set of probability vectors,

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<sup>32</sup>For example, in discussing convergence, they require  $\alpha^{-1}$  not to be an eigenvalue of  $W$ , thus  $\alpha \neq 1$ .

<sup>33</sup>Note that for each  $i \in N$ ,  $T_i : B \rightarrow I$  is such that:

1.  $T_i(ke) = k$  for all  $k \in I$ ;

$C_i \subseteq \Delta$ , and a function  $\alpha_i : B \rightarrow [0, 1]$  such that

$$T_i(x) = \alpha_i(x) \min_{p \in C_i} p \cdot x + [1 - \alpha_i(x)] \max_{p \in C_i} p \cdot x \quad \forall x \in B. \quad (45)$$

This representation turns out to be useful in different ways for our results and our aggregators. Before illustrating them, we elaborate a bit more on the nature of the set  $C_i$  and how it can be retrieved from  $T_i$ . Given  $i \in N$ , consider the following binary relation  $\succsim_i^*$  on  $B$ :

$$x \succsim_i^* y \iff T_i(\lambda x + (1 - \lambda)z) \geq T_i(\lambda y + (1 - \lambda)z) \quad \forall \lambda \in (0, 1], \forall z \in B.$$

By [12, Proposition 2], it follows that

$$x \succsim_i^* y \iff p \cdot x \geq p \cdot y \quad \forall p \in C_i. \quad (46)$$

The representations contained in (45) and (46) turn out to be useful in at least two different ways. First, they allow treating each round of updating as linear with the caveat that the weight matrix might depend on the previous update itself. Second, it allows to provide conditions on  $T$  which guarantee convergence (see, e.g., Theorem 10). Below, we expand on these two points.

From (45) and since  $C_i$  is convex, observe that for each  $x \in B$ , we have that

$$T_i(x) = \alpha(x) (q^{*,i}(x) \cdot x) + [1 - \alpha(x)] (p^{*,i}(x) \cdot x) = p^i(x) \cdot x$$

where  $q^{*,i}(x) \in \arg \min_{p \in C_i} p \cdot x$  (resp.,  $p^{*,i}(x) \in \arg \max_{p \in C_i} p \cdot x$ ) and  $p^i(x) \in C_i$  is such that  $p^i(x) = \alpha(x) q^{*,i}(x) + [1 - \alpha(x)] p^{*,i}(x)$ . Since  $i$  was arbitrarily chosen, if we construct the matrix  $W(x)$  so that each row coincides with  $p^i(x)$ , we have that for each  $x \in B$

$$T(x) = W(x)x, \quad W(x) \in \mathcal{W} \text{ and the } i\text{-th row of } W(x) \text{ belongs to } C_i \quad \forall i \in N. \quad (47)$$

In words, opinion aggregators which are normalized, monotone, and continuous (a fortiori, the robust ones) coincide to a form of “local” linear aggregation in that each update  $T(x)$  can be written as  $W(x)x$  where the matrix of weights  $W$  depends on

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2. for each  $x, y \in B$

$$x \geq y \implies T_i(x) \geq T_i(y).$$

3.  $T$  is continuous with respect to the topology generated by  $\|\cdot\|_\infty$ .

In light of this in [12, Proposition 2 and Corollary 3], set  $S = N$ ,  $\Sigma$  the power set of  $N$ ,  $X = I$ , and  $u = \text{id}_I$ . Moreover, let  $\succcurlyeq$  be the binary relation on  $B$  induced by  $T_i$  as utility function. Same observations hold for obtaining (46).

$x$ . Thus, for each opinion aggregator  $T : B \rightarrow B$  which is normalized, monotone, and continuous, we can define

$$\mathcal{W}(T) = \{W \in \mathcal{W} : W \text{ satisfies (47) for some } x \in B\}.$$

The set  $\mathcal{W}(T)$  is the collection of stochastic matrices which replicate  $T$ .<sup>34</sup> The next result shows that the *uniform common influencer* property and the *pairwise common influencer* property have a direct counterpart in terms of the set  $\mathcal{W}(T)$ . This latter fact allows us to elaborate on the second reason why (45) and (46) might turn out to be useful. From a mathematical point of view, in discussing the two properties mentioned above, we interpreted them as monotonicity properties. Intuitively, monotonicity is often characterized by the positivity of the derivative. So one might wonder if they can be checked by computing the derivatives of  $T$ . Unfortunately, in the robust case,  $T$  is typically nondifferentiable. As the representation in (45) suggests, our aggregators might have many kinks, in particular at the points of consensus. Luckily, robust opinion aggregators are also nonexpansive, therefore they are Clarke's differentiable (see, Clarke [14, Proposition 2.1.2]). Given Ghirardato, Maccheroni, and Marinacci [25, Theorem 14], if  $T$  is also constant affine, then this allows to verify the properties of  $C_i$ , and so  $\mathcal{W}(T)$ , via the Clarke's differential  $\partial T_i$ , given that  $C_i = \partial T_i(ke)$  where  $ke$  is a point in the interior of  $B$ .

**Proposition 20** *Let  $T$  be a robust opinion aggregator. The following statements are true:*

1. *If  $T$  is self-influential, then there exists  $\varepsilon \in (0, 1)$  such that  $p_i \geq \varepsilon$  for all  $p \in C_i$  for all  $i \in N$ . In particular, there exists  $\varepsilon \in (0, 1)$  such that  $w_{ii} \geq \varepsilon$  for all  $i \in N$  and for all  $W \in \mathcal{W}(T)$ .*
2. *If  $T$  has the uniform common influencer property, then there exist  $k \in N$  and  $\varepsilon \in (0, 1)$  such that  $p_k \geq \varepsilon$  for all  $p \in C_i$  for all  $i \in N$ . In particular,  $\mathcal{W}(T) \subseteq \mathcal{W}_{\varepsilon, k}$  for some  $k \in N$  and  $\varepsilon \in (0, 1)$ .*

The above result suggests the next one which is a convergence result based on the representation of  $T$ .

**Theorem 10** *Let  $T$  be an opinion aggregator which is normalized, monotone, and continuous. The following statements are true:*

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<sup>34</sup>Note that  $\mathcal{W}(T)$  is contained in the set of stochastic matrices whose only requirement is that the  $i$ -th row belongs to  $C_i$ . Note that this inclusion might be strict.

1. If  $\mathcal{W}(T) \subseteq \mathcal{W}_\varepsilon$  for some  $\varepsilon \in (0, 1)$ , then  $T$  is convergent. Moreover, if  $\bar{T} : B \rightarrow B$  is defined by

$$\bar{T}(x) = \lim_t T^t(x) \quad \forall x \in B, \quad (48)$$

then  $\bar{T}(x) \in D$  for all  $x \in B$  and

$$\|\bar{T}(x) - T^t(x)\|_\infty \leq 2(1 - \varepsilon)^t \|x\|_\infty \quad \forall t \in \mathbb{N}, \forall x \in B. \quad (49)$$

2. If each  $W \in \mathcal{W}(T)$  is scrambling, then  $T$  is convergent.

Point 1 of the previous result (paired with Proposition 20) generalizes Theorem 6 while point 2 partially generalizes Theorem 7. As for point 1, we simply dropped the assumption of the robustness of  $T$ , and we weakened the uniform common influencer property by looking at the characterization of this property in terms of the representation of  $T$  (namely, the condition  $\mathcal{W}(T) \subseteq \mathcal{W}_{\varepsilon, k}$  for some  $k \in N$  and  $\varepsilon \in (0, 1)$ ). The conclusion remains the same: Convergence to consensus is always attained and at an exponential rate. Compared to Theorem 6, the only drawback is the interpretability of the following condition:  $\mathcal{W}(T) \subseteq \mathcal{W}_\varepsilon$ . A similar discussion holds for point 2, with the additional caveat that we are also mute in terms of rate of convergence.<sup>35</sup>

## 7 Vox populi, vox dei

Motivated by our robust statistics foundation, we next study if the updating procedure  $\{T^t(x^0)\}$  leads to estimates which allow either the agents in the network or an external observer to learn the truth as the size of the population becomes larger and larger. More formally, we consider the same setup as in Section 3.1. There exists a true parameter  $\mu \in I$  and each agent  $i \in N = \{1, \dots, n\}$  observes a signal  $X_i(n)$ . We assume that the signals are defined over a probability space  $(\Omega, \mathcal{F}, P)$ . In this section, we make two assumptions which we maintain throughout.

**Assumption 1**  $I = \mathbb{R}$ .

**Assumption 2** For each  $n \in \mathbb{N}$  we assume that  $X_i(n) = \mu + \varepsilon_i(n)$  for all  $i \in N$  where  $\{\varepsilon_i(n)\}_{i \in N, n \in \mathbb{N}}$  is a collection of uniformly bounded, symmetric, and independent random variables. We denote by  $\hat{I}$  the compact interval such that  $X_i(n)(\omega) \in \hat{I}$  for all  $\omega \in \Omega$ , for all  $i \in N$ , and for all  $n \in \mathbb{N}$ . We denote by  $\ell$  the length of  $\hat{I}$ .<sup>36</sup>

<sup>35</sup>It escaped us if one could provide any meaningful bound on the rate of convergence.

<sup>36</sup>Formally, the property of symmetry means that for each  $i \in N$  and for each  $n \in \mathbb{N}$

$$P(\{\omega \in \Omega : \varepsilon_i(n) \in B\}) = P(\{\omega \in \Omega : \varepsilon_i(n) \in -B\})$$

for all Borel sets  $B \subseteq \mathbb{R}$ . The length of the interval  $\hat{I}$  is the quantity  $\ell = \max \hat{I} - \min \hat{I}$ .

Let us fix the population size  $n$ . Momentarily, we will let  $n$  increase. If agents update their estimates via a convergent robust opinion aggregator  $T(n)$ , then each agent  $i$  will reach a final estimate  $\bar{T}_i(n)(X_1(n)(\omega), \dots, X_n(n)(\omega))$ . It is then natural to ask whether or not  $\bar{T}_i(n)(X_1(n)(\omega), \dots, X_n(n)(\omega))$  gets arbitrarily close to the true parameter  $\mu$  as the population size  $n$  increases. Of course, for this to be the case, it seems natural to require that  $T(n)$  is also a consensus operator – this can be achieved in several natural ways we discussed above (cf. Proposition 10). Even though the assumption of  $T$  being a consensus operator is not strictly necessary, having it facilitates our reasoning and it seems to be also conceptually relevant. In fact, in order for the entire set of agents to learn the true parameter  $\mu$ , it seems natural to consider a situation in which the result of the updating procedure is common across agents.

To better understand the nonlinear case, it might be useful to recall what happens in the linear one. If  $T(n)$  is linear with representing matrix  $W(n)$ , then  $\bar{T}(n)$  is also linear with representing matrix  $\bar{W}$ . Since  $T$  is a consensus operator, then all the rows of  $\bar{W}(n)$  coincide with the left Perron-Frobenius vector  $s(n)$  associated to the eigenvalue 1. Golub and Jackson [27] call such a vector  $s(n)$  the *vector of influence weights* and show that if  $\lim_{n \rightarrow \infty} \max_{k \in N} s_k(n) \rightarrow 0$ , then

$$\bar{T}_i(n)(X_1, \dots, X_n) \xrightarrow{P} \mu \quad \forall i \in N.$$

In generalizing this result in the nonlinear case, one faces two main difficulties: one mathematical and one interpretative. From a mathematical point of view, it is not obvious how to generalize to the nonlinear case the notion of eigenvector. From an interpretative point of view, the conceptual relevance of the vector  $s(n)$  comes from the immediate computability in terms of the primitive matrix  $W$ . But, other than specific cases, it is not immediately evident in terms of primitives what is its significance. To wit, it is not immediate by a mere inspection of the components of  $W$ , if one can obtain useful bounds on  $\max_{i \in N} s_i(n)$ .

For such a reason, we make the following trivial, yet useful observation: In the linear case, the vector  $s(n)$  coincides with the gradient  $\nabla \bar{T}_i(n)$  of  $\bar{T}_i$  at any point  $x \in \mathbb{R}^n$ . This observation will reveal useful in two dimensions. For starters, it allows us to overcome the difficulty of not having a useful notion of eigenvector for nonlinear operators and it clarifies how the influence vector  $s(n)$  and its  $i$ -th component capture the idea of “marginal contribution of agent  $i$ ” to the final opinion  $\bar{T}_i(n)$ . Finally, via the chain rule, being  $\bar{T}(n)$  the pointwise limit of  $T^t(n)$  as  $t$  runs to infinity, it allows us to bound these marginal contributions via the marginal contributions of each agent at each round, that is, via the gradient of  $T(n)$ . Unfortunately, in proceeding this way, we might face some technical complication. Our opinion aggregators might well be nondifferentiable. Nevertheless, the fact that they are Lipschitz continuous

allows us to resort to the notion of Clarke differentiability and a generalized notion of directional/partial derivative.

**Definition 18** *Let  $T$  be a robust opinion aggregator such that  $\bar{T}$  is a consensus operator and  $\hat{B} \subseteq \mathbb{R}^n$ . We say that  $s(T, \hat{B}) \in \mathbb{R}^n$  is the influence vector of  $T$  on  $\hat{B}$  if and only if*

$$s(T, \hat{B})_i = \sup_{x \in \hat{B}} \left( \limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{\bar{T}_1(y + te^i) - \bar{T}_1(y)}{t} \right) < \infty \quad \forall i \in N.$$

As we mentioned, the above definition of influence vector coincides with the one of Golub and Jackson whenever  $T$  is also linear since  $s(T, \hat{B}) = s$  for all  $\hat{B} \subseteq \mathbb{R}^n$ . For, in this case, one has that

$$\limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{\bar{T}_1(y + te^i) - \bar{T}_1(y)}{t} = \frac{\partial \bar{T}_1}{\partial x_i}(x) = s_i \quad \forall i \in N, \forall x \in \mathbb{R}^n. \quad (50)$$

Also in the nonlinear case, the right-hand side of (50) captures the (maximal) marginal contribution of a change of the opinion  $i$  on the final consensus estimate. Intuitively, the next result shows that under the assumptions of the current section, if the influence weight of each agent goes to zero, then the estimates of the network become more and more accurate.

**Proposition 21** *Let  $\{T(n)\}_{n \in \mathbb{N}}$  be a sequence of odd robust opinion aggregators. If  $\bar{T}(n)$  is a consensus operator for all  $n \in \mathbb{N}$  and there exist sequences  $\{c(n)\}_{n \in \mathbb{N}}$  and  $\{w(n)\}_{n \in \mathbb{N}}$  such that  $c(n) \in \mathbb{R}$ ,  $w(n) \in \Delta_n$  and for each  $i \in N$*

$$s(T(n), \hat{I}^n)_i \leq c(n) w_i(n) \quad \text{and} \quad c(n)^2 \max_{k \in N} w_k(n) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (51)$$

then

$$\bar{T}_j(n)(X_1, \dots, X_n) \xrightarrow{P} \mu \quad \forall j \in N. \quad (52)$$

Compared to the linear case, we must observe that Proposition 21 differs only in one central aspect: Our result relies on signals which are symmetric around  $\mu$ . Given (50), note in fact that our conditions trivially imply the one of Golub and Jackson (namely,  $\lim_{n \rightarrow \infty} \max_{k \in N} s_k(n) \rightarrow 0$ ) while their condition implies ours given that, in the linear case, we can always set  $c(n) = 1$  and  $w(n) = s(n)$  for all  $n \in \mathbb{N}$ .

Notice that symmetry of the errors guarantees wisdom of the crowd when paired with an odd opinion aggregator. Odd aggregators naturally arise when the loss function is symmetric, see Proposition 1.

**Remark 2** We can also provide bounds on both the variance of  $\bar{T}_j(n)(X_1, \dots, X_n)$  and the probability of

$$|\bar{T}_j(n)(X_1, \dots, X_n) - \mu| \geq \varepsilon.$$

This helps in elucidating the convergence result contained in Proposition 21. If  $T(n)$  is an odd robust opinion aggregator such that  $s(T(n), \hat{I}^n) \leq cw(n)$  for some  $c \in (0, \infty)$  and  $w(n) \in \Delta_n$ , we have that  $\mathbb{E}(\bar{T}_j(n)(X_1, \dots, X_n)) = \mu$  and for each  $\varepsilon \in [0, \ell]$

$$P(\{\omega \in \Omega : |\bar{T}_j(n)(X_1(\omega), \dots, X_n(\omega)) - \mu| \geq \varepsilon\}) \leq 2 \exp\left(-\frac{2}{c^2 \max_{k \in N} w_k(n)}\right). \quad (53)$$

Since our random variables take values in a bounded interval  $\hat{I}$  the difference

$$|\bar{T}_j(n)(X_1(\omega), \dots, X_n(\omega)) - \mu|$$

can be at most the length of the interval, that is,  $\ell$ . Thus, the previous inequality provides a useful bound in terms of controlling for deviations from the true parameter. Mathematically, (53) is a consequence of McDiarmid's inequality. In turn, this allows us to control the variance of  $\bar{T}_j(n)(X_1, \dots, X_n)$ . Indeed, we have that

$$\text{Var}(\bar{T}_j(n)(X_1, \dots, X_n)) \leq 2\ell^2 \exp\left(-\frac{2}{c^2 \max_{k \in N} w_k(n)}\right) \quad \forall j \in N. \quad (54)$$

Thus, if  $\max_{k \in N} w_k(n)$  gets smaller, then the variance of  $\bar{T}_j(n)(X_1, \dots, X_n)$  gets smaller.

▲

Before moving on, we note that the above proposition and remark apply even if  $T(n)$  is not a convergent operator (which we never assumed in the formal results of this section). In such a case, recall that  $\bar{T}(n)$  is the limit of the time averages  $\{T^t\}_{t \in \mathbb{N}}$ . This generalization is interesting if we think about the following question: Can an external observer learn the true parameter by observing part of the updating dynamics? More formally, assume that the external observer from a specific point in time, say  $m$ , gets to see the updating process  $\{T^{t+m}(n)(X_1(n)(\omega), \dots, X_n(n)(\omega))\}_{t=1}^\tau$ . By the previous part of the paper, due to uniform convergence, we know that as  $\tau \rightarrow \infty$

$$\frac{1}{\tau} \sum_{t=1}^{\tau} T^{t+m}(n)(X_1(n)(\omega), \dots, X_n(n)(\omega)) \rightarrow \bar{T}(n)(X_1(n)(\omega), \dots, X_n(n)(\omega)) \quad \forall \omega \in \Omega.$$

Our results show that  $\bar{T}(n)$  can be useful to extract information, even if  $T$  might not be convergent.

The next obvious question we tackle relates on the possibility to check condition (51) in terms of the original operator  $T$ .

**Proposition 22** Let  $\{T(n)\}_{n \in \mathbb{N}}$  be a sequence of robust opinion aggregators. If  $\bar{T}(n)$  is a consensus operator for all  $n \in \mathbb{N}$  and there exist sequences  $\{c(n)\}_{n \in \mathbb{N}}$  and  $\{w(n)\}_{n \in \mathbb{N}}$  such that  $c(n) \in \mathbb{R}$ ,  $w(n) \in \Delta_n$  and for each  $i \in N$ , for each  $h \in N$ , and for each  $x \in \hat{I}^n$

$$\limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{T_h(n)(y + te^i) - T_h(n)(y)}{t} \leq c(n) w_i(n) \quad \text{and} \quad c(n)^2 \max_{k \in N} w_k(n) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (55)$$

then  $\{T(n)\}_{n \in \mathbb{N}}$  satisfies (51).

We next illustrate how our previous result can become handy in checking condition (51) in the context of our statistical foundation. As a by-product, we will obtain that, under the assumptions of this section, the wisdom of the crowd can be achieved as long as the minimum degree of connections gets larger and larger.

**Example 9** Assume that each agent processes signals via Huber's robust statistical estimation (see Section 3.1). Suppose that the conditions of Proposition 8 are satisfied and there exists  $\bar{c}$  such that

$$\frac{\rho_j''(x)}{\rho_j''(y)} \leq \bar{c} \quad \text{for all } j \in \mathbb{N}, \quad x, y \in [-2\ell, 2\ell].$$

Then  $T(n)$  is differentiable and

$$\limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{T_h(n)(y + te^i) - T_h(n)(y)}{t} = (\nabla T_h(n)(y))_i \leq \bar{c} \frac{n}{\min_{k \in \{1, \dots, n\}} |N_k|} \frac{1}{n} = c(n) w_i(n)$$

where

$$c(n) = \bar{c} \frac{n}{\min_{k \in \{1, \dots, n\}} |N_k|} \quad \text{and} \quad w(n) = \frac{e}{n}.$$

Therefore, Proposition 22 guarantees that wisdom is reached at the limit if the minimal degree in the society is growing sufficiently fast, that is,

$$\lim_{n \rightarrow \infty} \frac{n}{\left(\min_{k \in \{1, \dots, n\}} |N_k|\right)^2} = 0$$

or

$$\frac{1}{\min_{k \in \{1, \dots, n\}} |N_k|} = o\left(\frac{1}{\sqrt{n}}\right). \quad (56)$$

Notice that Equation (56) allows each agent to be connected to a vanishing fraction of the society. Inspection of the proof of Proposition 22 and the results in [27] shows that even in the case of a linear aggregator the minimal degree has to explode to guarantee wisdom, but Equation (56) can be relaxed to

$$\frac{1}{\min_{k \in \{1, \dots, n\}} |N_k|} = o(1).$$

▲

However, condition (51) allows checking that the crowd becomes wise even in some situation where the maximal degree in the society remains bounded.

**Example 10** Assume that each agent processes signals via Huber's robust statistical estimation (see Section 3.1). Suppose that the conditions of Proposition 8 are satisfied, and there exists  $\bar{c}$  such that

$$\frac{\rho_j''(x)}{\rho_j''(y)} \leq \bar{c} \text{ for all } j \in \mathbb{N}.$$

Suppose we have a circle in the society, i.e.,

$$w_{ij} = \begin{cases} w_i & \text{if } i = j, \\ (1 - w_i) & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

with the understanding that agent  $n + 1$  is identified with agent 1, and such that  $0 < \underline{w} < w_i < \bar{w} < 1$ . This is the case where everyone is connected only to the individual that immediately follows her in the society, and the weight assigned to own opinion is parametrized by  $w_i$ . Let,  $\hat{w} = \max\{(1 - \underline{w}), \bar{w}\}$ . If

$$\sup_{j, (x, y) \in [-2\ell, 2\ell]} \frac{\hat{w} \rho_j''(x)}{\hat{w} \rho_j''(y) + (1 - \hat{w}) \rho_j''(x)} < \frac{1}{\sqrt{2}}$$

then (51) holds.<sup>37</sup> ▲

Finally, we conclude with a word of caution about the possibility of obtaining the wisdom of the crowd in the realm of robust opinion aggregators. For the next result, recall that a linear aggregator  $T(x) = Wx$  is obtained in our foundations as a particular case. Here, we show that for *every* sequence of linear opinion aggregators that are wise in the sense of Golub and Jackson [27], it is possible to find an arbitrarily small perturbation of the loss functions used by the agents such that the limit opinion is bounded away from the truth *with probability 1*.

**Proposition 23** *Suppose that  $\{W(n)\}_{n \in \mathbb{N}}$  is a sequence of matrices that are wise as by Definition 3 in Golub and Jackson. Then, for all  $\delta > 0$ , there exist a  $k_\delta$  and a sequence of profiles of loss functions  $\{\phi(n)\}_{n \in \mathbb{N}}$  such that each  $\phi(n)$  is  $\delta$ -close to the loss function originating  $W(n)$ ,<sup>38</sup> the induced aggregators  $\{T(n)\}_{n \in \mathbb{N}}$  are robust, and*

$$P(|\bar{T}_j(n)(X_1, \dots, X_n) - \mu| > k_\delta) \rightarrow 1 \quad \forall j \in N.$$

<sup>37</sup>The tedious computations are available upon request.

<sup>38</sup>In the sense that

$$\sup_{i \in N, m \in \mathbb{N}, z \in [-2\ell, 2\ell]} \left| \phi_i(m)(z) - \sum_{j \in N} w_{ij}(m) z_j^2 \right| < \delta.$$

Moreover, the size of the bias can be chosen to be nonnegligible with respect to the distance between the loss functions:

$$\lim_{\delta \rightarrow 0} \frac{k_\delta}{\delta} > 0.$$

Some remarks are in order. First, the perturbation of the loss function used in the proof of Proposition 23 is very particular, but it corresponds to one case that may be relevant in applications: Extreme realizations of the signals in one direction are salient, and therefore observed by all the agents (think of extremely adverse reactions to vaccination). Second, we had to use a particular loss function because we want failure for *all* the possible networks structure and *all* the possible distribution of error terms.

The best way to interpret the result is that it underscores a role of symmetry that is absent in the case of linear aggregators. Indeed, Proposition 23 shows that minimal asymmetries in the way in which information is incorporated by the agents lead to failures of wisdom. Moreover, even if the aggregator is symmetric (e.g., the median) unbiased but skewed error terms may prevent wisdom.

More generally, one positive message of the wisdom of the crowd result in Golub and Jackson is the following. Even if the society is partitioned in disconnected components, when such building blocks are large, the existence of an “objective truth  $\mu$ ” leads to an agreement between the different components of the society. Instead, when each component features some homophily in the kind of behavioral bias allowed by the more general class of robust opinion aggregators this may not happen, and differences in beliefs may persist in the limit.

## A Appendix: A robust foundation

Before proving the main result of Section 3.1, we need to introduce additional definitions and preliminary results. We say that the correspondence  $\mathbf{T} : B \rightrightarrows B$  is *internal* whenever

$$\mathbf{T}(x) \subseteq \left[ \min_{j \in N} x_j, \max_{j \in N} x_j \right]^n \quad \forall x \in B,$$

and *translation invariant* whenever, for all  $x \in B$  and  $k \in \mathbb{R}$  such that  $x + ke \in B$ ,

$$y \in \mathbf{T}(x) \iff y + ke \in \mathbf{T}(x + ke).$$

Note that if a correspondence is internal, then each of its selections is necessarily normalized.

**Lemma 3** Consider a function  $\phi_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$ . For all  $(c, x) \in I \times B$  and  $k \in \mathbb{R}$  with  $(c + k, x + ke) \in I \times B$ , it holds

$$c \in \mathbf{T}_i(x) \Leftrightarrow c + k \in \mathbf{T}_i(x + ke).$$

**Proof.** Let  $(c, x) \in I \times B$  and consider  $k \in \mathbb{R}$  such that  $(c + k, x + ke) \in I \times B$ . If  $c \in \mathbf{T}_i(x)$ , then, for all  $d \in \mathbb{R}$  such that  $d + k \in I$ ,

$$\begin{aligned} \phi_i(x + ke - (c + k)e) &= \phi_i(x - ce) \\ &\leq \phi_i(x - de) \\ &= \phi_i(x + ke - (d + k)e). \end{aligned}$$

Given that  $d$  was arbitrarily chosen, it follows that

$$\phi_i(x + ke - (c + k)e) \leq \phi_i(x + ke - (d' + k)e) \quad \forall d' \in I,$$

that is  $c + k \in \mathbf{T}_i(x + ke)$ . ■

**Definition 19** The profile of loss functions  $\phi$  is distance monotone if, for all  $i \in N$  and  $z \in \mathbb{R}^n$ ,

$$z \gg 0 \implies \phi_i(z) > \phi_i\left(z - \min_j z_j e\right),$$

and

$$0 \gg z \implies \phi_i(z) > \phi_i\left(z - \max_j z_j e\right).$$

Note that, if  $\phi$  is distance monotone then it is sensitive. Indeed, for all  $i \in N$  and  $h \in \mathbb{R} \setminus \{0\}$  distance monotonicity implies that either

$$h > 0 \implies he \gg 0 \implies \phi_i(he) > \phi_i\left(he - \min_j ((he)_j) e\right) = \phi_i(0)$$

or

$$h < 0 \implies 0 \gg he \implies \phi_i(he) > \phi_i\left(he - \max_j ((he)_j) e\right) = \phi_i(0).$$

The following lemma shows that, under increasing shifts, these two notions are equivalent.

**Lemma 4** If the profile of loss functions  $\phi$  has strictly increasing shifts then it is sensitive if and only if it is distance monotone.

**Proof.** We only need to show that if  $\phi$  satisfies increasing shifts and sensitivity then is distance monotone. Consider  $z \in \mathbb{R}^n$  such that  $z \gg 0$ . Also, define  $v = z - \min_j z_j e$ ,  $v' = 0$ ,  $h = \min_j z_j$  and note that  $v \geq v'$ . Increasing shifts implies

$$\phi_i(v + he) - \phi_i(v) \geq \phi_i(v' + he) - \phi_i(v')$$

that is

$$\phi_i(z) - \phi_i\left(z - \min_j z_j e\right) \geq \phi_i\left(\min_j z_j e\right) - \phi_i(0) > 0$$

where the last strict inequality follows from sensitivity and the fact that  $\min_j z_j > 0$ . The case for  $z \ll 0$  is symmetric.  $\blacksquare$

**Definition 20** *The profile of loss functions  $\phi$  is separately convex if and only if, for every  $i \in N$ ,  $\phi_i$  is coordinatewise convex.*

**Definition 21** *The profile of loss functions  $\phi$  is ultramodular if and only if, for all  $i \in N$ ,  $z, v \in \mathbb{R}^n$  and  $q \in \mathbb{R}_+^n$ , with  $z \geq v$ ,*

$$\phi_i(z + q) - \phi_i(z) \geq \phi_i(v + q) - \phi_i(v).$$

Note that if  $\phi$  is ultramodular, then it satisfies increasing shifts. Indeed, it is enough to set  $q = he$  for all  $h \in \mathbb{R}_+$

**Lemma 5** (*Marinacci and Montrucchio [44, Corollary 4.1]*) *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ . If  $f$  is supermodular and separately convex then it is ultramodular. The converse holds provided that  $f$  is bounded on every bounded subinterval  $[v, z] \subseteq \mathbb{R}^n$ .*

**Lemma 6** *If  $\phi \in \Phi_R$  then, for every  $i \in N$ ,*

$$f_z(c) = \phi_i(z - ce)$$

*is convex for all  $z \in \mathbb{R}^n$ . If also  $\phi$  satisfies strictly increasing shifts, then each  $f_z(c)$  is strictly convex for all  $z \in \mathbb{R}^n$ .*

**Proof.** Given that  $\phi_i$  is lower semicontinuous, it is measurable. Therefore, by [53, Theorem C, page 221 and Theorem A, page 212] it is enough to show that  $f_z$  is midpoint convex, that is,

$$\frac{1}{2}(f_z(a) + f_z(b)) \geq f_z\left(\frac{a+b}{2}\right) \quad \forall a, b \in \mathbb{R}. \quad (57)$$

Fix  $a, b \in \mathbb{R}$  and define  $c = \frac{a+b}{2}$ . If  $a = b$  then (57) is trivially satisfied. Without loss of generality, assume that  $a > b$ . With this, we have  $a > c > b$ . Next, define  $h = c - b = a - c \in \mathbb{R}_+$ . Increasing shifts of  $\phi_i$  implies that

$$\phi_i(z - be) - \phi_i(z - ce) = \phi_i(z - ce + he) - \phi_i(z - ae + he) \geq \phi_i(z - ce) - \phi_i(z - ae),$$

that is

$$f_z(b) - f_z(c) \geq f_z(c) - f_z(a)$$

or

$$\frac{1}{2}(f_z(a) + f_z(b)) \geq f_z\left(\frac{a+b}{2}\right)$$

which shows that  $f_z$  is midpoint convex. A completely similar argument that replaces the weak with strict inequality shows that  $f_z$  is strictly midpoint convex when  $\phi_i$  satisfies strictly increasing shifts, and therefore it is convex, too. Now, it remains to show that a convex function that is strictly midpoint convex is strictly convex. Let  $a, b \in \mathbb{R}$ ,  $\alpha \in (1/2, 1)$ . The case in which  $\alpha > 1/2$  can be proved analogously. Therefore,  $\alpha a + (1 - \alpha)b = \beta a + (1 - \beta)\frac{a+b}{2}$  for  $\beta = 2\alpha - 1 \in (0, 1)$ . However, then

$$\begin{aligned} f_z(\alpha a + (1 - \alpha)b) &= f_z\left(\beta a + (1 - \beta)\frac{a+b}{2}\right) \\ &\leq \beta f_z(\beta a) + (1 - \beta)f_z\left(\frac{a+b}{2}\right) \\ &< \beta f_z(a) + \frac{(1 - \beta)}{2}(f_z(a) + f_z(b)) \\ &= \alpha f_z(a) + (1 - \alpha)f_z(b) \end{aligned}$$

where the weak inequality is because  $f_z$  is convex, and the second inequality because  $f_z$  is strictly midpoint convex. ■

**Proof of Theorem 1.** (1) Let  $T$  be a robust opinion aggregator. By Lemma 9 in the Online Appendix there exists an extension  $\hat{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  of  $T$  that is also robust. By Lemma 1, for every  $i \in N$ ,  $\hat{T}_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is nonexpansive, hence almost everywhere differentiable and everywhere Clarke differentiable. Next, fix  $i \in N$ . By Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi [12, Corollary 3], we have that there exists a closed and convex set of probability vectors,  $C_i \subseteq \Delta$ , and a function  $\alpha_i : B \rightarrow [0, 1]$  such that

$$\hat{T}_i(z) = \alpha_i(z) \min_{p \in C_i} p \cdot z + [1 - \alpha_i(z)] \max_{p \in C_i} p \cdot z \quad \forall z \in \mathbb{R}^n.$$

From (45) and since  $C_i$  is convex, observe that for each  $z \in \mathbb{R}^n$ , we have

$$\hat{T}_i(z) = \alpha(z) (q^{*,i}(z) \cdot z) + [1 - \alpha(z)] (p^{*,i}(z) \cdot z) = p^i(z) \cdot z \quad (58)$$

where  $q^{*,i}(z) \in \arg \min_{p \in C_i} p \cdot z$  (resp.,  $p^{*,i}(z) \in \arg \max_{p \in C_i} p \cdot z$ ) and  $p^i(z) \in C_i$  is such that  $p^i(z) = \alpha(z)q^{*,i}(z) + [1 - \alpha(z)]p^{*,i}(z)$ . Next, define the correspondence  $\Gamma_i : B \rightarrow 2^\Delta$  by

$$\Gamma_i(z) = \left\{ p^i \in C_i : p^i \cdot z = \hat{T}_i(z) \right\} \quad \forall z \in \mathbb{R}^n.$$

We first prove the following ancillary claim.

*Claim:*  $\Gamma_i$  is nonempty compact valued, upper hemicontinuous and translation invariant, that is,

$$\Gamma_i(z) = \Gamma_i(z + he) \quad \forall z \in \mathbb{R}^n, h \in \mathbb{R}.$$

*Proof of claim:* Fix  $z \in \mathbb{R}^n$ . From (58), it follows that there exists  $p^i \in C_i$  such that  $p^i \cdot z = \hat{T}_i(z)$ . Next, let  $\{p_n^i\}_{n \in \mathbb{N}} \subseteq \Gamma_i(z)$  such that  $p_n^i \rightarrow \bar{p}^i \in \Delta$ . Given that  $C_i$  is closed, we have  $\bar{p}^i \in C_i$ . Moreover, for all  $n \in \mathbb{N}$ ,  $p_n^i \cdot z = \hat{T}_i(z)$ . Thus,  $\bar{p}^i \cdot z = \hat{T}_i(z)$  as well. This shows that  $\bar{p}^i \in \Gamma_i(z)$ , hence, that  $\Gamma_i(z)$  is closed and in particular compact given that it is a subset of  $\Delta$ . Next, let

$$\{(z_n, p_n^i)\}_{n \in \mathbb{N}} \subseteq Gr(\Gamma) = \{(z, p^i) \in \mathbb{R}^n \times \Delta, p^i \in \Gamma_i(z)\},$$

such that  $(z_n, p_n^i) \rightarrow (\bar{z}, \bar{p}^i) \in \mathbb{R}^n \times \Delta$ . Similarly as before, the fact that  $\{p_n^i\}_{n \in \mathbb{N}} \subseteq C_i$  implies that  $\bar{p}^i \in C_i$ . Moreover, for all  $n \in \mathbb{N}$ ,

$$p_n^i \cdot z_n = \hat{T}_i(z_n).$$

By continuity of  $\hat{T}_i$ , it follows that  $\bar{p}^i \cdot \bar{z} = \hat{T}_i(\bar{z})$ , that is,  $(\bar{z}, \bar{p}^i) \in Gr(\Gamma)$ . This shows that  $Gr(\Gamma)$  is closed, hence that  $\Gamma$  is upper hemicontinuous. Finally, let  $z \in \mathbb{R}^n$  and  $h \in \mathbb{R}$ . Then, for all  $p^i \in \Gamma_i(z)$ , we have

$$p^i \cdot (z + he) = p^i \cdot z + p^i \cdot he = p^i \cdot z + h = \hat{T}_i(z) + h = \hat{T}_i(z + he),$$

that is  $p^i \in \Gamma_i(z + he)$ . This implies that  $\Gamma_i(z) \subseteq \Gamma_i(z + he)$ . The converse inclusion is analogously shown.  $\square$

Define

$$\phi_i^T(z) = \inf_{w_i \in \Gamma_i(z)} \sum_{j \in N} w_{ij} z_j^2 \quad \forall z \in \mathbb{R}^n.$$

Given that the function  $(w_i, z) \mapsto \sum_{j \in N} w_{ij} z_j^2$  is continuous over  $Gr(\Gamma)$ , by [1, Lemma 17.30] it follows that  $\phi_i^T$  is lower semicontinuous. Next, we show that  $\phi_i^T$  is distance monotone, hence sensitive. Fix  $z \gg 0$  and note that the previous claim implies

$$\phi_i^T\left(z - \min_{\ell} z_{\ell} e\right) = \min_{w_i \in \Gamma_i(z - \min_{\ell} z_{\ell} e)} \sum_{j \in N} w_{ij} \left(z_j - \min_{\ell} z_{\ell}\right)^2 = \min_{w_i \in \Gamma_i(z)} \sum_{j \in N} w_{ij} \left(z_j - \min_{\ell} z_{\ell}\right)^2.$$

Also, fix  $w_i \in \Gamma_i(z)$ . We have

$$\begin{aligned} \sum_{j \in N} w_{ij} \left(z_j - \min_{\ell} z_{\ell}\right)^2 &= \sum_{j \in N} w_{ij} z_j^2 + \left(\min_{\ell} z_{\ell}\right)^2 - 2 \left(\min_{\ell} z_{\ell}\right) \sum_{j \in N} w_{ij} z_j \\ &\leq \sum_{j \in N} w_{ij} z_j^2 - \left(\min_{\ell} z_{\ell}\right)^2 \\ &< \sum_{j \in N} w_{ij} z_j^2. \end{aligned}$$

Let  $w_i^* \in \Gamma_i(z)$  be such that

$$\sum_{j \in N} w_{ij}^* z_j^2 = \min_{w_i \in \Gamma_i(z)} \sum_{j \in N} w_{ij} z_j^2.$$

Then,

$$\sum_{j \in N} w_{ij}^* \left( z_j - \min_{\ell} z_{\ell} \right)^2 < \sum_{j \in N} w_{ij}^* z_j^2,$$

which implies

$$\phi_i^T \left( z - \min_{\ell} z_{\ell} e \right) = \min_{w_i \in \Gamma_i(z)} \sum_{j \in N} w_{ij} \left( z_j - \min_{\ell} z_{\ell} \right)^2 \leq \sum_{j \in N} w_{ij}^* \left( z_j - \min_{\ell} z_{\ell} \right)^2 < \sum_{j \in N} w_{ij}^* z_j^2 = \phi_i^T(z).$$

We can prove the case for  $0 \gg z$  with analogous passages. This shows that  $\phi_i^T$  is distance monotone. Next, we show that  $\phi_i^T$  has strictly increasing shifts. First, note that, for all  $z \in \mathbb{R}^n$  and  $h \in \mathbb{R}_{++}$ , we have

$$\begin{aligned} \phi_i^T(z + he) - \phi_i^T(z) &= \min_{w_i \in \Gamma_i(z+he)} \sum_{j \in N} w_{ij} (z_j + h)^2 - \min_{w_i \in \Gamma_i(z)} \sum_{j \in N} w_{ij} z_j^2 \\ &= \min_{w_i \in \Gamma_i(z)} \left[ \sum_{j \in N} w_{ij} z_j^2 + h^2 + 2h \sum_{j \in N} w_{ij} z_j \right] - \min_{w_i \in \Gamma_i(v)(z)} \sum_{j \in N} w_{ij} z_j^2 \\ &= h^2 + 2hT_i(z) + \min_{w_i \in \Gamma_i(z)} \left[ \sum_{j \in N} w_{ij} z_j^2 \right] - \min_{w_i \in \Gamma_i(z)} \sum_{j \in N} w_{ij} z_j^2 \\ &= h^2 + 2hT_i(z). \end{aligned}$$

Monotonicity of  $T_i$  immediately yields that  $\phi_i$  has increasing shifts. Next, let  $z, v \in \mathbb{R}^n$  be such that  $z \gg v$  and fix  $h \in \mathbb{R}_{++}$ . It follows that

$$\begin{aligned} \phi_i^T(z + he) - \phi_i^T(z) &= h^2 + 2hT_i(z) = h^2 + 2hT_i \left( \left( z - \max_j (z_j - v_j) e \right) + \max_j (z_j - v_j) e \right) \\ &= h^2 + 2hT_i \left( z - \min_j (z_j - v_j) e \right) + \min_j (z_j - v_j) \\ &> h^2 + 2hT_i \left( z - \min_j (z_j - v_j) e \right) \geq h^2 + 2hT_i(v) \\ &= \phi_i^T(v + he) - \phi_i^T(v). \end{aligned}$$

Given that  $v, z$  and  $h$  were arbitrarily chosen, it follows that  $\phi_i^T$  has strictly increasing shifts. Finally, we show that for every  $x \in B$ ,

$$\arg \min_{c \in I} \phi_i^T(x - ce) = T_i(x).$$

By the previous claim, for all  $x \in B$  and  $c \in I$ , we have

$$\phi_i^T(x - ce) = \min_{w_i \in \Gamma_i(x-ce)} \sum_{j \in N} w_{ij} (x_j - c)^2 = \min_{w_i \in \Gamma_i(x)} \sum_{j \in N} w_{ij} (x_j - c)^2.$$

Next, fix  $x \in B$  and note that, for every  $w_i \in \Gamma_i(x)$ , the problem

$$\min_{c \in I} \sum_{j \in N} w_{ij} (x_j - c)^2$$

uniquely attains its minimum at  $c^* = w_i \cdot x = \hat{T}_i(x) = T_i(x)$ , where the last two equalities follow by the definition of  $\Gamma_i$  and  $\hat{T}_{i|B} = T_i$ . Also, for every  $c \in I \setminus \{T_i(x)\}$  let  $w_i^c \in \Gamma_i(x)$  be such that

$$\sum_{j \in N} w_{ij}^c (x_j - c)^2 = \min_{w_i \in \Gamma_i(x)} \sum_{j \in N} w_{ij} (x_j - c)^2.$$

Therefore, for every  $c \in I \setminus \{T_i(x)\}$ , we have

$$\begin{aligned} \phi_i^T(x - T_i(x)e) &= \min_{w_i \in \Gamma_i(x)} \sum_{j \in N} w_{ij} (x_j - T_i(x))^2 \leq \sum_{j \in N} w_{ij}^c (x_j - T_i(x))^2 \\ &< \sum_{j \in N} w_{ij}^c (x_j - c)^2 = \min_{w_i \in \Gamma_i(x)} \sum_{j \in N} w_{ij} (x_j - c)^2 = \phi_i^T(x - ce), \end{aligned}$$

showing that  $\phi_i^T$  uniquely attains its minimum at  $T_i(x)$ . This concludes the proof of point 1.

(2) Let  $\phi \in \Phi_R$  and fix  $i \in N$ . Also recall that, by Lemma 4,  $\phi$  is distance monotone as well. First, we show that the problem

$$\min_{c \in \mathbb{R}} \phi_i(x - ce)$$

in (1) is equivalent to

$$\min_{c \in I} \phi_i(x - ce) \quad \text{sub to } c \in \left[ \min_{j \in N} x_j, \max_{j \in N} x_j \right] \quad (59)$$

for all  $x \in B$ . Fix  $x \in B$  and assume by contradiction that there exists  $c^* \in I \setminus [\min_{j \in N} x_j, \max_{j \in N} x_j]$  such that  $\phi_i(x - c^*e) < \phi_i(x - ce)$  for all  $c \in [\min_{j \in N} x_j, \max_{j \in N} x_j]$ . If  $c^* < \underline{x} = \min_{j \in N} x_j$ , then  $(x - c^*e) \gg 0$ , and distance monotonicity implies

$$\phi_i(x - c^*e) > \phi_i(x - c^*e - (\underline{x} - c^*)e) = \phi_i(x - \underline{x}e)$$

which is a contradiction. If  $c^* > \bar{x} = \max_{j \in N} x_j$ , then  $0 \gg (x - c^*e)$ , and distance monotonicity implies

$$\phi_i(x - c^*e) > \phi_i(x - c^*e - (\bar{x} - c^*)e) = \phi_i(x - \bar{x}e)$$

which is a contradiction. With this,

$$\arg \min_{c \in I} \phi_i(x - ce) = \arg \min_{c \in [\min_{j \in N} x_j, \max_{j \in N} x_j]} \phi_i(x - ce) \subseteq \left[ \min_{j \in N} x_j, \max_{j \in N} x_j \right] \quad \forall x \in B.$$

From this, it follows that the set of solutions of (59), that is a nonempty and compact set by the Weierstrass Theorem for lower semicontinuous functions (see Section D.3.3 in [50]), coincides with  $\mathbf{T}_i(x)$ . Translation invariance of  $\mathbf{T}_i$  follows from Lemma 3. Next, define  $f : I \times B \rightarrow \mathbb{R}_+$  as

$$f(c, x) = -\phi_i(x - ce) \quad \forall (c, x) \in I \times B,$$

and note that

$$\mathbf{T}_i(x) = \arg \max_{c \in I} f(c, x).$$

We want to show that  $f$  has increasing differences in  $(c, x)$ . Fix  $x, y \in B$  and  $c, d \in I$  such that  $x \geq y$  and  $c \geq d$ . Define  $h = (c - d)$ ,  $v = y - ce$  and  $z = x - ce$ . Given that  $z \geq v$  and  $h \in \mathbb{R}_+$ , it follows

$$\begin{aligned} f(c, y) - f(d, y) &= \phi_i(y - de) - \phi_i(y - ce) \\ &= \phi_i(v + he) - \phi_i(v) \\ &\leq \phi_i(z + he) - \phi_i(z) \\ &= \phi_i(x - de) - \phi_i(x - ce) \\ &= f(c, x) - f(d, x), \end{aligned}$$

where the weak inequality follows directly from the definition of increasing shifts. This shows that  $f$  satisfies increasing differences in  $(c, x)$  and by [46, Theorem 5],  $\mathbf{T}_i$  is monotone nondecreasing in the strong set order. Given that  $i$  was arbitrarily chosen, each  $\mathbf{T}_i$  is internal, translation invariant, upper hemicontinuous, nonempty compact valued and monotone in the strong set order. All the previous properties are clearly inherited by  $\mathbf{T} = \prod_{i \in N} \mathbf{T}_i$ . For the second part of the result, define for each  $i \in N$ ,

$$\underline{T}_i(x) = \min \mathbf{T}_i(x) \quad \forall x \in B,$$

and  $\underline{T} = (\underline{T}_i)_{i \in N}$ . Clearly,  $\underline{T}$  is a selection of the correspondence  $\mathbf{T}$  which is internal. Hence  $\underline{T}$  is normalized. Next, fix  $i \in N$  and consider  $k \in \mathbb{R}$  such that  $x + ke \in B$ . From Lemma 3 it follows that  $\underline{T}_i(x) + k \in \mathbf{T}_i(x + ke)$ . Assume there exists  $c^* \in \mathbf{T}_i(x + ke)$  such that  $c^* < \underline{T}_i(x) + k$ . Again, by Lemma 3 it follows that  $c^* - k \in \mathbf{T}_i(x)$ . However, we notice that  $c^* - k < \underline{T}_i(x)$  is a contradiction with  $\underline{T}_i(x) = \min \mathbf{T}_i(x)$ . Therefore, we get  $\underline{T}_i(x) + k = \min \mathbf{T}_i(x + ke)$ , which shows that  $\underline{T}_i$  is a translation invariant selection of  $\mathbf{T}_i$ . Moreover, given that  $\mathbf{T}_i$  is monotone nondecreasing in the strong set order, we have that  $\underline{T}_i$  is monotone. In particular, given that  $i$  was arbitrarily chosen, each  $\underline{T}_i$  is translation invariant and monotone, which implies that  $\underline{T}$  translation invariant and monotone. Finally, if  $\phi$  has strictly increasing shifts, Lemma 6 tells us that, for all  $i \in N$  and  $x \in B$ , the map  $c \mapsto \phi_i(x - ce)$  is strictly convex in  $c$ . With this, each

$\mathbf{T}_i$  is single valued and, in particular,  $\mathbf{T}_i(x) = \{\underline{T}_i(x)\}$  for all  $x \in B$ . Moreover,  $\mathbf{T}(x) = \{\underline{T}(x)\}$  for all  $x \in B$ , implying that  $\mathbf{T} = \underline{T}$  is a robust aggregator. ■

**Proof of Proposition 5.** Consider an arbitrary  $i \in N$ . Suppose that  $i$  is using a weighted average of order statistics with weights  $(q_1, \dots, q_{N_i})$  where  $q_i$  is the weight assigned to the  $i$ -th higher observation in her neighborhood  $N_i$ . Clearly,  $T_i(ce) = c$ . Moreover, if  $x \geq y$  also the  $i$ -th higher observation in  $N_i$  under  $x$  will be larger than the  $i$ -th higher observation in  $N_i$  under  $y$ . Finally, let  $x, y$  be comonotonic additive. Consider the subsample of observations  $\bar{x}$  and  $\bar{y}$  in  $\mathbb{R}^{N_i}$  that only includes the opinion in  $N_i$  ordered in an arbitrary way. Using the notation of Section 6.5, notice that

$$(\bar{x} + \bar{y})_{\pi(i)} = \bar{x}_{\pi(i)} + \bar{y}_{\pi(i)}$$

and

$$T_i(x + y) = \sum_{i=1}^{N_i} q_i (\bar{x} + \bar{y})_{\pi(i)} = \sum_{i=1}^{N_i} q_i \bar{x}_i + \sum_{i=1}^{N_i} q_i \bar{y}_{\pi(i)} = T_i(x) + T_i(y). \quad \blacksquare$$

**Proof of Proposition 6.**

1. Since the loss function is strictly convex in  $c$  and twice continuously differentiable, a necessary and sufficient condition for the solution function  $T_i^\lambda$  is

$$\begin{aligned} 0 &= \frac{\partial \phi_i^\lambda(x - ce)}{\partial c} \\ 0 &= \sum_{j \in N} w_{ij} (-\lambda \exp(\lambda(x_j - T_i^\lambda(x)))) + \lambda \\ 1 &= \sum_{j \in N} w_{ij} \exp(\lambda x_j) \exp(-\lambda T_i^\lambda(x)) \\ \exp(\lambda T_i^\lambda(x)) &= \sum_{j \in N} w_{ij} \exp(\lambda x_j) \\ T_i^\lambda(x) &= \frac{1}{\lambda} \log \left( \sum_{j \in N} w_{ij} \exp(\lambda x_j) \right). \end{aligned}$$

2. Standard.

3. We prove by induction on  $t \in \mathbb{N}$  that

$$(T_i^\lambda(x))^t = T_i^\lambda(x) = \frac{1}{\lambda} \ln(W^t \exp(\lambda x_j)).$$

The basis step follows by 1. Suppose that it holds for  $t$ . Then

$$\begin{aligned}
(T^\lambda(x))_i^{t+1} &= \frac{1}{\lambda} \ln \left( \sum_{j=1}^n w_{ij} \exp \left( \lambda (T^\lambda(x))_j^t \right) \right) \\
&= \frac{1}{\lambda} \ln \left( \sum_{j=1}^n w_{ij} \exp \left( \lambda \frac{1}{\lambda} \ln \left( \sum_{l=1}^n (W^t)_{jl} \exp((\lambda x_l)) \right) \right) \right) \\
&= \frac{1}{\lambda} \ln \left( \sum_{j=1}^n w_{ij} \sum_{l=1}^n (W^t)_{jl} \exp((\lambda x_l)) \right) \\
&= \frac{1}{\lambda} \ln (W^t \exp(\lambda x_j))
\end{aligned}$$

proving the inductive step. By taking the limit for  $t \rightarrow \infty$ , since

$$\lim_{t \rightarrow \infty} W_{ij}^t = s_j \quad \forall i, j \in N$$

and  $\ln$  is a continuous function, the first part of the result holds. The influence vector immediately follows from taking derivatives. To conclude, consider  $x$  such that  $x_i \geq x_j$ . Let  $k \in \mathbb{R}^+$  be such that  $x_i + k \in I$  and  $x_j - k \in I$ . Define

$$y_l = \begin{cases} x_i + k & l = i \\ x_j - k & l = j \\ x_l & \text{otherwise.} \end{cases}$$

We want to prove that

$$\lim_t (T^\lambda(y))^t - \lim_t (T^\lambda(x))^t \geq 0.$$

However, by the Gradient Theorem, and since  $\nabla \left( \lim_t (T^\lambda(z))^t \right)_i \geq \nabla \left( \lim_t (T^\lambda(z))^t \right)_i$  for all  $z \in \text{co}(x, y)$ ,

$$\lim_t (T^\lambda(y))^t - \lim_t (T^\lambda(x))^t = \int_x^y \nabla \left( \lim_t (T^\lambda(z))^t \right) dz \geq 0$$

concluding the proof. ■

**Proof of Corollary 1.** Let  $\phi \in \Phi_R^*$  and fix  $i \in N$ . It follows that  $\phi_i$  is continuous, convex (hence separately convex), sensitive and supermodular. It is clear that  $\phi_i$  is lower semicontinuous. Moreover, by Lemma 5,  $\phi_i$  is also ultramodular, hence it has increasing shifts. Given that  $i$  and  $\phi$  were arbitrarily chosen, we have that  $\Phi_R^* \subseteq \Phi_R$ . Upper hemicontinuity of each  $\mathbf{T}_i$  (and consequently of  $\mathbf{T}$ ) follows by an application of Berge Maximum Theorem. Finally, whenever each  $\phi_i$  is strictly convex, then the set of minimizers  $\mathbf{T}_i$  is a singleton for every  $x \in B$ , hence  $\mathbf{T} = T$  is a robust aggregator. ■

Before proving Proposition 1 state and prove a preliminary Lemma of independent interest.

**Lemma 7** Let  $\phi \in \Phi_R$ . The following facts are true:

1. If  $\phi$  is positive homogeneous, then there exists a robust constant affine selection  $T$  of  $\mathbf{T}$ ;
2. If  $\phi$  is symmetric, then there exists a robust odd selection  $T$  of  $\mathbf{T}$ .

**Proof.** Let  $\phi \in \Phi_R$ . By point (2) of Theorem 1, we already know that there exists a robust selection  $T$  of  $\mathbf{T}$ .

(1) By inspection of the proof of Theorem 1, a robust selection of  $\mathbf{T}$  is given by  $\underline{T} = (\underline{T}_i = \min \mathbf{T}_i)_{i \in N}$ . Moreover, for every  $i \in N$ , we have that

$$\mathbf{T}_i(x) = \arg \min_{c \in \mathbb{R}} \phi_i(x - ce) \quad \forall x \in B. \quad (60)$$

Let  $\phi$  be positive homogeneous and fix  $i \in N$ ,  $k \in I$  and  $\lambda \in [0, 1]$ . Next, note that equation (60) implies that for every  $c' \in \mathbb{R}$ ,

$$\begin{aligned} \phi_i(\lambda x + (1 - \lambda)ke - (\lambda \underline{T}_i(x) + (1 - \lambda)k)e) &= \eta_i(\lambda) \phi_i(x - \underline{T}_i(x)e) \\ &\leq \eta_i(\lambda) \phi_i(x - c'e) \\ &= \phi_i(\lambda x + (1 - \lambda)ke - (\lambda c' + (1 - \lambda)k)e). \end{aligned}$$

Also, given that, for every  $c \in I$ , there exists  $c' \in \mathbb{R}$  such that  $c = \lambda c' + (1 - \lambda)k$ , it follows that

$$\phi_i(\lambda x + (1 - \lambda)ke - (\lambda \underline{T}_i(x) + (1 - \lambda)k)e) \leq \phi_i(\lambda x + (1 - \lambda)ke - ce) \quad \forall c \in I.$$

Finally, we need to show that  $\lambda \underline{T}_i(x) + (1 - \lambda)k = \underline{T}_i(\lambda x + (1 - \lambda)ke)$ . Assume by contradiction that there exists  $c^* \in \mathbf{T}_i(\lambda x + (1 - \lambda)ke)$  with  $c^* < \lambda \underline{T}_i(x) + (1 - \lambda)k$ . This would imply that  $c^{**} = \frac{c^*}{\lambda} - \frac{(1 - \lambda)}{\lambda}k < \underline{T}_i(x)$  and

$$\begin{aligned} \phi_i(x - c^{**}e) &= \phi_i\left(x - \left(\frac{c^*}{\lambda} - \frac{(1 - \lambda)}{\lambda}k\right)e\right) \\ &= \phi_i\left(\frac{\lambda x + (1 - \lambda)ke}{\lambda} - \frac{(1 - \lambda)}{\lambda}ke - \left(\frac{c^*}{\lambda} - \frac{(1 - \lambda)}{\lambda}k\right)e\right) \\ &= \phi_i\left(\frac{\lambda x + (1 - \lambda)ke}{\lambda} - \frac{c^*}{\lambda}e\right) \\ &= \eta_i\left(\frac{1}{\lambda}\right) \phi_i(\lambda x + (1 - \lambda)ke - c^*e) = \eta_i\left(\frac{1}{\lambda}\right) \phi_i(\lambda x + (1 - \lambda)ke - c^*e) \\ &\leq \eta_i\left(\frac{1}{\lambda}\right) \phi_i(\lambda x + (1 - \lambda)ke - \lambda \underline{T}_i(x)e + (1 - \lambda)ke) \\ &= \eta_i\left(\frac{1}{\lambda}\right) \phi_i(\lambda x - \lambda \underline{T}_i(x)e) = \phi_i(x - \underline{T}_i(x)e), \end{aligned}$$

that is,  $c^{**} \in \mathbf{T}_i(x)$ . However, this is a contradiction with  $\underline{T}_i(x) = \min \mathbf{T}_i(x)$ , hence  $\lambda \underline{T}_i(x) + (1 - \lambda)k = \underline{T}_i(\lambda x + (1 - \lambda)ke)$ .

(2) Fix an  $i \in N$ . By inspection of the proof of Theorem 1, we know that the selection  $\underline{T}_i = \min \mathbf{T}_i$  is robust. A completely analogous argument shows that the selection  $\bar{T}_i = \max \mathbf{T}_i$  is also robust. Thus, the selection  $T_i$  of  $\mathbf{T}_i$  defined as

$$T_i(x) = \frac{1}{2} (\min \mathbf{T}_i(x) + \max \mathbf{T}_i(x)) \quad x \in B,$$

is a robust opinion aggregator since it is a convex linear combination of two robust aggregators. We next show that  $T_i$  is odd. Let  $x \in B$  be such that  $-x \in B$ . Let  $x_l = \min_{i \in N} x_i$  and  $x_h = \max_{i \in N} x_i$ , then  $-x \in B$  implies  $[-x_h, -x_l] \subseteq I$ . By inspection of the proof of Theorem 1 the problems

$$\min_{c \in I} \phi_i((-x) - ce) \quad \text{and} \quad \min_{c \in [-x_h, -x_l]} \phi_i((-x) - ce) \quad (61)$$

are equivalent. Fix  $c^* \in \mathbf{T}_i(x)$  and  $c' \in [-x_h, -x_l]$ . Since  $I$  is an interval, we have  $-c' \in I$ . By definition of  $\mathbf{T}_i$

$$\phi_i(x - c^*e) \leq \phi_i(x - (-c')e).$$

Moreover, given that  $c^* \in [x_l, x_h]$ , we also have  $-c^* \in [-x_h, -x_l] \subseteq I$ . But then, by symmetry,

$$\phi_i((-x) - (-c^*)e) \leq \phi_i((-x) - c'e)$$

Given that  $c'$  was arbitrarily chosen, we have  $-c^* \in \mathbf{T}_i(-x)$ . Given that  $c^* \in \mathbf{T}_i(x)$  was arbitrarily chosen, we have

$$-\mathbf{T}_i(x) \subseteq \mathbf{T}_i(-x).$$

Next, let  $\hat{x} = -x$  and note that, by construction,  $-\hat{x} \in B$ . Therefore, by repeating the same argument above, we have

$$-\mathbf{T}_i(\hat{x}) \subseteq \mathbf{T}_i(-\hat{x}) \implies -\mathbf{T}_i(-x) \subseteq \mathbf{T}_i(x) \implies \mathbf{T}_i(-x) \subseteq -\mathbf{T}_i(x),$$

hence  $-\mathbf{T}_i(x) = \mathbf{T}_i(-x)$ . With this,

$$-\min \mathbf{T}_i(x) = \max \mathbf{T}_i(-x) \quad \text{and} \quad -\max \mathbf{T}_i(x) = \min \mathbf{T}_i(-x).$$

Finally,

$$T_i(-x) = \frac{1}{2} (\min \mathbf{T}_i(-x) + \max \mathbf{T}_i(-x)) = -\frac{1}{2} (\min \mathbf{T}_i(x) + \max \mathbf{T}_i(x)) = -T_i(x).$$

Given that  $i$  and  $x$  were arbitrarily chosen, the result holds. ■

**Proof of Proposition 1.** (1) Let  $T$  be a constant affine robust aggregator. By Theorem 1, we know that there exists a  $\phi^T \in \Phi_R$  with strictly increasing shifts such that, for all  $i \in N$ ,  $T_i(x) = \arg \min_{c \in I} \phi_i^T(x - ce)$  for every  $x \in B$ . In particular, each  $\phi_i^T$  can be defined as

$$\phi_i^T(z) = \inf_{w_i \in \Gamma_i(z)} \sum_{j \in N} w_{ij} z_j^2 \quad \forall z \in \mathbb{R}^n, \quad (62)$$

where

$$\Gamma_i(z) = \left\{ p^i \in C_i : p^i \cdot z = \hat{T}_i(z) \right\}$$

and  $\hat{T}_i$  is an extension of  $T_i$  to  $\mathbb{R}^n$ . Also, note that Lemma 9 in the Online Appendix yields that  $\hat{T}_i$  is the unique extension of  $T_i$  being the latter constant affine. Moreover, for all  $\lambda > 0$  and  $z \in \mathbb{R}^n$

$$\Gamma_i(\lambda z) = \left\{ p^i \in C_i : p^i \cdot (\lambda z) = \hat{T}_i(\lambda z) \right\} = \left\{ p^i \in C_i : p^i \cdot z = \hat{T}_i(z) \right\} = \Gamma_i(z).$$

Therefore, for all  $\lambda > 0$  and  $z \in \mathbb{R}^n$ ,

$$\phi_i^T(\lambda z) = \inf_{w_i \in \Gamma_i(\lambda z)} \sum_{j \in N} w_{ij} (\lambda z_j)^2 = \lambda^2 \inf_{w_i \in \Gamma_i(z)} \sum_{j \in N} w_{ij} z_j^2 = \lambda^2 \phi_i^T(z),$$

showing that  $\phi_i^T$  is positive homogeneous. Conversely, consider a constant affine  $\phi \in \Phi_R$  with strictly increasing shifts such that  $T_i(x) = \arg \min_{c \in I} \phi_i^T(x - ce)$  for every  $x \in B$ . By Lemma 7, there exists a constant affine robust selection  $T'$  from  $\mathbf{T}$ . Given that  $\mathbf{T}$  is single valued, it follows that  $T = T'$ , showing that  $T$  is constant affine and robust.

(2) By Lemma 7, there exists a symmetric selection  $T'$  from  $\mathbf{T}$ . Given that  $\mathbf{T}$  is single-valued, it follows that  $T = T'$ , showing that  $T$  is symmetric and robust.  $\blacksquare$

**Proof of Proposition 2.** Let  $W \in \mathcal{W}$  and  $\rho = (\rho_i : \mathbb{R} \rightarrow \mathbb{R}_+)_{i \in N}$ .

(ii  $\implies$  i) It is obvious.

(i  $\implies$  iii) Let  $(W, \rho) \in \Phi_A$  and fix  $i \in N$ . First, we show that  $\rho_i$  is convex. Given that  $\rho_i$  is lower semicontinuous, it is measurable. Therefore, by [53, Theorem C, page 221 and Theorem A, page 212] it is enough to show that  $\rho_i$  is midpoint convex, that is,

$$\frac{1}{2}(\rho(a) + \rho(b)) \geq \rho\left(\frac{a+b}{2}\right) \quad \forall a, b \in \mathbb{R}. \quad (63)$$

Fix  $a, b \in \mathbb{R}$  and define  $c = \frac{a+b}{2}$ . If  $a = b$  then (63) is trivially satisfied. Without loss of generality, assume that  $a > b$ . With this, we have  $a > c > b$ . Next, define  $h = c - b \in \mathbb{R}_+$ ,  $v = be$  and  $z = ce$ . It is clear that  $v, z \in \mathbb{R}^n$  and  $z \geq v$ . Increasing shifts of  $(W, \rho)$  imply that

$$\phi_i(z + he) - \phi_i(z) \geq \phi_i(v + he) - \phi_i(v),$$

that is

$$\sum_{j \in N} w_{ij} (\rho_i(c+h) - \rho_i(c)) \geq \sum_{j \in N} w_{ij} (\rho_i(b+h) - \rho_i(b))$$

that is

$$\rho_i(c+h) + \rho_i(b) \geq \rho_i(b+h) + \rho_i(c)$$

that is

$$\rho_i(a) + \rho_i(b) \geq 2\rho_i(c) = 2\rho_i\left(\frac{a+b}{2}\right)$$

which shows that  $\rho_i$  is midpoint convex, hence convex. Next, we show that  $\rho_i$  is strictly decreasing on  $\mathbb{R}_-$  and strictly increasing on  $\mathbb{R}_+$ . First, fix  $c < d \leq 0$  and define  $z \in \mathbb{R}^n$  such that

$$z_\ell = \begin{cases} c & \text{if } w_{i\ell} > 0 \\ c-d & \text{else} \end{cases} \quad \forall \ell \in N.$$

With this, we have  $z \ll 0$ . By Lemma 4 sensitivity of  $(W, \rho)$  implies distance monotonicity of  $(W, \rho)$ , hence that

$$\phi_i(z) > \phi_i\left(z - \max_{\ell} z_{\ell} e\right)$$

that is  $\rho_i(c) > \rho_i(d)$ . The case for  $c > d \geq 0$  is analogously shown.

(iii  $\implies$  ii) For every  $i \in N$ , let  $\rho_i$  be convex, strictly decreasing on  $\mathbb{R}_-$  and strictly increasing on  $\mathbb{R}_+$ . Since  $\rho_i$  is convex, it is continuous. Therefore  $\phi_i$  is continuous, too. Moreover, it follows that

$$\phi_i(z) = \sum_{j \in N} w_{ij} \rho_i(z_j) \quad z \in \mathbb{R}^n$$

is separately convex. Since  $\phi_i$  is additive separable, it follows that it is modular, hence supermodular. By Lemma 5,  $\phi_i$  is ultramodular, hence it has increasing shifts. Next, we show that  $\phi_i$  satisfies distance monotonicity. Let  $z \gg 0$  and note that

$$\phi_i(z) = \sum_{j \in N} w_{ij} \rho_i(z_j) > \sum_{j \in N} w_{ij} \rho_i\left(z_j - \min_{\ell} z_{\ell} e\right) = \phi_i\left(z - \min_{\ell} z_{\ell} e\right),$$

because  $\rho_i$  is strictly increasing on  $\mathbb{R}_+$ . The case for  $z \ll 0$  is analogous and exploits the fact that  $\rho_i$  is strictly decreasing on  $\mathbb{R}_-$ . This shows that  $\phi = (W, \rho) \in \Phi_{R^*}$ .  $\blacksquare$

**Proof of Corollary 2.** If  $(W, \rho)$  is additive separable and, for all  $i \in N$ ,  $\rho_i$  is convex, strictly decreasing on  $\mathbb{R}_-$  and strictly increasing on  $\mathbb{R}_+$ , then, by Proposition 2,  $(W, \rho) \in \Phi_A \subseteq \Phi_R$ . By Theorem 1,  $\mathbf{T}$  admits a selection  $T$  which is robust. Finally, if each  $\rho_i$  is strictly convex, then each

$$\phi_i(z) = \sum_{j \in N} w_{ij} \rho_i(z_j) \quad z \in \mathbb{R}^n$$

is strictly convex, implying that the updating correspondence  $\mathbf{T} = T$  is a well defined robust opinion aggregator.  $\blacksquare$

## B Appendix: Convergence

**Proof of Lemma 1.** Since  $T$  is a robust opinion aggregator,  $T_i$  is normalized, monotone, and translation invariant for all  $i \in N$ . By [11, Theorem 4], it follows that  $T_i$  is a niveloid for all  $i \in N$ . By [11, p. 346] (cf. also [15, Proposition 2]), it follows that  $|T_i(x) - T_i(y)| \leq \|x - y\|_\infty$  for all  $x, y \in B$  and for all  $i \in N$ . It follows that

$$\|T(x) - T(y)\|_\infty = \max_{i \in N} |T_i(x) - T_i(y)| \leq \|x - y\|_\infty \quad \forall x, y \in B,$$

proving that  $T$  is nonexpansive.

By induction, we show that  $T^t$  is nonexpansive for all  $t \in \mathbb{N}$ . Since we have shown that  $T$  is nonexpansive,  $T^t$  is nonexpansive for  $t = 1$ , proving the initial step. By inductive step, assume that  $T^t$  is nonexpansive, we have that

$$\begin{aligned} \|T^{t+1}(x) - T^{t+1}(y)\|_\infty &= \|T(T^t(x)) - T(T^t(y))\|_\infty \leq \|T^t(x) - T^t(y)\|_\infty \\ &\leq \|x - y\|_\infty \quad \forall x, y \in B, \end{aligned}$$

proving the inductive step. The statement follows by induction.

Finally, assume that  $T$  satisfies (17). Let  $x \in B$ . Choose  $\varepsilon = r/2$ . Let  $y \in B$  be a generic element such that  $\|x - y\|_\infty < \varepsilon$ . From the previous part, it follows that

$$\|T^t(x) - T^t(y)\|_\infty < \varepsilon = \frac{r}{2} \quad \forall t \in \mathbb{N}.$$

Since  $T$  satisfies (17), it follows that there exists  $\bar{y} \in B$  and  $\bar{t} \in \mathbb{N}$  such that  $\|x - \bar{y}\|_\infty < \varepsilon$  and  $\frac{r}{2} > \|T^{\bar{t}}(x) - T^{\bar{t}}(\bar{y})\|_\infty \geq r > 0$ , a contradiction. ■

We next prove an ancillary lemma that will play an important role later on in our proofs. It highlights the properties of the limiting operator  $\bar{T}$ .

**Lemma 8** *Let  $T$  be an opinion aggregator. If  $T$  is such that*

$$C - \lim_t T^t(x) \text{ exists} \quad \forall x \in B,$$

*then  $\bar{T} : B \rightarrow B$ , defined by  $\bar{T}(x) = C - \lim_t T^t(x)$  for all  $x \in B$ , is well defined and  $\bar{T} \circ T = \bar{T}$ . Moreover,*

1. *If  $T$  is nonexpansive, so is  $\bar{T}$ . In particular,  $\bar{T}$  is continuous.*
2. *If  $T$  is normalized and monotone, so is  $\bar{T}$ .*
3. *If  $T$  is constant affine, so is  $\bar{T}$ .*
4. *If  $T$  is robust, so is  $\bar{T}$ .*
5. *If  $T$  is odd, so is  $\bar{T}$ .*

**Proof.** Let  $x \in B$ . Since  $T$  is a selfmap, we have that  $\{T^t(x)\}_{t \in \mathbb{N}} \subseteq B$ . Since  $B$  is convex, we have that

$$\frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) \in B \quad \forall \tau \in \mathbb{N}.$$

Since  $B$  is closed, we have that  $\bar{T}(x) = \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) \in B$ . Consider now  $T(x)$ . We have that for each  $\tau \in \mathbb{N}$

$$\frac{1}{\tau} \sum_{t=1}^{\tau} T^t(T(x)) = \frac{1}{\tau} \sum_{t=1}^{\tau} T^{t+1}(x) = \frac{1}{\tau} \sum_{t=2}^{\tau+1} T^t(x) = \frac{\tau+1}{\tau} \frac{1}{\tau+1} \sum_{t=1}^{\tau+1} T^t(x) - \frac{1}{\tau} T(x).$$

This implies that

$$\bar{T}(T(x)) = \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(T(x)) = \lim_{\tau} \frac{\tau+1}{\tau} \lim_{\tau} \frac{1}{\tau+1} \sum_{t=1}^{\tau+1} T^t(x) - \lim_{\tau} \frac{1}{\tau} T(x) = \bar{T}(x).$$

Since  $x$  was arbitrarily chosen,  $\bar{T}$  is well defined and  $\bar{T} \circ T = \bar{T}$ .

1. Since  $T$  is nonexpansive, we have that  $T^t$  is nonexpansive for all  $t \in \mathbb{N}$ . This implies that for each  $x, y \in B$

$$\left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(y) \right\|_{\infty} \leq \frac{1}{\tau} \sum_{t=1}^{\tau} \|T^t(x) - T^t(y)\|_{\infty} \leq \|x - y\|_{\infty} \quad \forall \tau \in \mathbb{N}.$$

By passing to the limit in  $\tau$ , we have that

$$\|\bar{T}(x) - \bar{T}(y)\|_{\infty} \leq \|x - y\|_{\infty} \quad \forall x, y \in B,$$

proving that  $\bar{T}$  is nonexpansive.

2. By induction, if  $T$  is normalized, then we have that  $T^t(ke) = ke$  for all  $t \in \mathbb{N}$  and for all  $k \in I$ . It follows that  $\bar{T}(ke) = \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(ke) = ke$  for all  $k \in I$ , proving that  $\bar{T}$  is normalized. Next, consider  $x, y \in B$ . By induction, if  $T$  is monotone, then we have that  $x \geq y$  implies  $T^t(x) \geq T^t(y)$  for all  $t \in \mathbb{N}$ . It follows that  $\bar{T}(x) = \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) \geq \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(y) = \bar{T}(y)$ , proving that  $\bar{T}$  is monotone.

3. Consider  $x \in B$ ,  $k \in I$ , and  $\lambda \in [0, 1]$ . By induction and since  $T$  is constant affine, we have that

$$T^t(\lambda x + (1 - \lambda) ke) = \lambda T^t(x) + (1 - \lambda) ke \quad \forall t \in \mathbb{N}.$$

It follows that

$$\begin{aligned} \bar{T}(\lambda x + (1 - \lambda) ke) &= \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(\lambda x + (1 - \lambda) ke) = \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} [\lambda T^t(x) + (1 - \lambda) ke] \\ &= \lambda \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) + (1 - \lambda) ke = \lambda \bar{T}(x) + (1 - \lambda) ke, \end{aligned}$$

proving that  $\bar{T}$  is constant affine.

4. By point 2, we are only left to show that  $\bar{T}$  is translation invariant. Let  $x \in B$  and  $k \in \mathbb{R}$  be such that  $x + ke \in B$ . By induction and since  $T$  is translation invariant, we have that

$$B \ni T^t(x + ke) = T^t(x) + ke \quad \forall t \in \mathbb{N}.$$

It follows that

$$\begin{aligned} \bar{T}(x + ke) &= \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x + ke) = \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} [T^t(x) + ke] \\ &= \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) + ke = \bar{T}(x) + ke, \end{aligned}$$

proving that  $\bar{T}$  is translation invariant.

5. Let  $x \in B$  be such that  $-x \in B$ . By induction and since  $T$  is odd, we have that

$$B \ni T^t(-x) = -T^t(x) \quad \forall t \in \mathbb{N}.$$

It follows that

$$\begin{aligned} \bar{T}(-x) &= \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(-x) = \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} [-T^t(x)] \\ &= -\lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) = \bar{T}(x), \end{aligned}$$

proving that  $\bar{T}$  is odd. ■

**Proof of Theorem 2.** By Lemma 1, we know that for each  $t \in \mathbb{N}$  the maps  $T^t : B \rightarrow B$  are nonexpansive. Consider the space of continuous functions over  $B$ :  $C(B)$ . We endow this space with the supnorm. With a small abuse of notation, we will denote by  $\|\cdot\|_{\infty}$  also the supnorm of  $C(B)$  where  $\|f\|_{\infty} = \sup_{x \in B} |f(x)|$  for all  $f \in C(B)$ . Define  $S : C(B) \rightarrow C(B)$  by

$$S(f) = f \circ T \quad \forall f \in C(B).$$

Note that  $S$  is a positive linear selfmap on  $C(B)$ . Moreover,  $S^t(f) = f \circ T^t$  for all  $f \in C(B)$  and for all  $t \in \mathbb{N}$ . Fix  $f \in C(B)$ . Since  $B$  is compact and  $f$  is continuous, it follows that  $f$  is uniformly continuous (see, e.g., [1, Corollary 3.31]), that is, for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$x, y \in B \text{ and } \|x - y\|_{\infty} < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Since  $T^t$  is nonexpansive for all  $t \in \mathbb{N}$ , this implies that for each  $t \in \mathbb{N}$  and  $x, y \in B$

$$\|x - y\|_{\infty} < \delta \implies \|T^t(x) - T^t(y)\|_{\infty} \leq \|x - y\|_{\infty} < \delta \implies |f(T^t(x)) - f(T^t(y))| < \varepsilon.$$

We have that  $\{S^t(f)\}_{t \in \mathbb{N}}$  is a sequence of equicontinuous functions. Moreover, since  $B$  is compact and  $f$  is continuous, there exists  $\bar{x} \in B$  such that  $|f(\bar{x})| = \|f\|_\infty$ . This implies that  $|S^t(f)(y)| = |f(T^t(y))| \leq |f(\bar{x})| = \|f\|_\infty$  for all  $t \in \mathbb{N}$  and for all  $y \in B$ . It follows that  $\|S^t(f)\|_\infty \leq \|f\|_\infty$  for all  $t \in \mathbb{N}$ , that is  $\{S^t(f)\}_{t \in \mathbb{N}}$  is bounded. By setting  $t = 1$  and since  $f$  was arbitrarily chosen, it also follows that  $S$  is a bounded operator. We can conclude that  $S$  is a positive equicontinuous operator. For each  $\tau \in \mathbb{N}$  also define the operator  $S_\tau : C(B) \rightarrow C(B)$  by

$$S_\tau = \frac{1}{\tau} \sum_{t=1}^{\tau} S^t.$$

By Rosenblatt's Ergodic Theorem [55, Theorem 1 p. 134] (see also [60, Section 4.1]), it follows that  $S_\tau(f) \xrightarrow{\|\cdot\|_\infty} \bar{S}(f)$  for all  $f \in C(B)$  where  $\bar{S} : C(B) \rightarrow C(B)$ . It is immediate to see that  $\bar{S}$  is linear and bounded as well (see, e.g., [1, Corollary 6.18]).

Next, for each  $i \in N$  define  $f_i : B \rightarrow \mathbb{R}$  by  $f_i(x) = x_i$ . Note that  $f_i$  is affine and  $f_i \in C(B)$  for all  $i \in N$ . By the previous part of the proof, we have that  $S_\tau(f_i) \xrightarrow{\|\cdot\|_\infty} \bar{S}(f_i)$  for all  $i \in N$ .

Define  $\bar{T} : B \rightarrow B$  by  $\bar{T}(x)_i = \bar{S}(f_i)(x)$  for all  $i \in N$  and for all  $x \in B$ . Note that  $\bar{T}$  is continuous.<sup>39</sup>

This implies that for each  $i \in N$  for each  $\varepsilon > 0$  there exists  $\tau_i(\varepsilon) \in \mathbb{N}$  such that  $\tau \geq \tau_i(\varepsilon)$  yields that

$$\begin{aligned} \sup_{x \in B} \left| f_i \left( \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) \right) - \bar{T}(x)_i \right| &= \sup_{x \in B} \left| \frac{1}{\tau} \sum_{t=1}^{\tau} f_i(T^t(x)) - \bar{S}(f_i)(x) \right| \\ &= \sup_{x \in B} \left| \frac{1}{\tau} \sum_{t=1}^{\tau} S^t(f_i)(x) - \bar{S}(f_i)(x) \right| \\ &= \sup_{x \in B} |S_\tau(f_i)(x) - \bar{S}(f_i)(x)| = \|S_\tau(f_i) - \bar{S}(f_i)\|_\infty < \varepsilon. \end{aligned}$$

For each  $\varepsilon > 0$  define  $\tau(\varepsilon) = \max_{i \in N} \tau_i(\varepsilon)$ . In particular, we have that for each  $\varepsilon > 0$  and for each  $\tau \geq \tau(\varepsilon)$

$$\begin{aligned} \varepsilon > \sup_{i \in N} \sup_{x \in B} \left| f_i \left( \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) \right) - \bar{T}(x)_i \right| &= \sup_{x \in B} \sup_{i \in N} \left| f_i \left( \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) \right) - \bar{T}(x)_i \right| \\ &= \sup_{x \in B} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \bar{T}(x) \right\|_\infty, \end{aligned}$$

proving that  $C - \lim_t T^t(x) = \bar{T}(x)$ , that is, (19) as well as (21) holds. By Lemma 8,  $\bar{T}$  is well defined, continuous, and such that  $\bar{T} \circ T = \bar{T}$ . ■

<sup>39</sup>In what follows, inter alia, we will show that  $\bar{T}$  coincides with the operator  $\bar{T}$  defined in (20).

**Remark 3** Theorem 2 could be seen as a version of the classic nonlinear ergodic theorem of Baillon (see Aubin and Ekeland [2, p. 253] as well as Krengel [40, Section 9.3]). In this literature, the assumption of finite dimensionality does not seem to make a huge difference, while the properties of the norm do. In fact, on the one hand, our selfmap is nonexpansive when  $B$  is endowed with the  $\|\cdot\|_\infty$  norm. On the other hand, in the original version of Baillon’s result,  $T$  must be nonexpansive with respect to the Euclidean norm  $\|\cdot\|_2$ .<sup>40</sup> This is not a mere technical choice, but rather a fundamental one driven by our opinion aggregators and their properties. For example, when  $T$  is as in Example 3,  $T$  is *not* nonexpansive for  $\|\cdot\|_2$  while it is so for  $\|\cdot\|_\infty$ . At the same time, generalizations of Baillon’s Theorem allow for more general norms (e.g.,  $\|\cdot\|_p$  for  $p \in (1, \infty)$ ), but to the best of our knowledge the only one that encompasses the case  $\|\cdot\|_\infty$  is the one contained in Baillon, Bruck, and Reich [3, Theorem 3.2 and Corollary 3.1]. Compared to our version, the part that would be missing is the one contained in (21). Observe that (21), not only guarantees uniform Cesaro convergence of  $\{T^t(x)\}_{t \in \mathbb{N}}$  (something also present in [3]), but also the independence of the rate of such convergence from the initial condition and might play an important role in applications (a feature which is missing in the aforementioned works).  $\blacktriangle$

**Proof of Corollary 5.** By Lemma 1 and since  $T$  is a robust opinion aggregator,  $T$  is nonexpansive. Since  $B = I^n$ , if  $I$  is bounded (thus, compact), then the statement immediately follows from Theorem 2. If  $I$  is not compact, consider  $x \in B$ . By point 4 of Lemma 9 in the Online Appendix, it follows that there exists a compact subinterval  $\tilde{I} \subseteq I$  with nonempty interior such that  $x \in \tilde{I}^n \subseteq B$ . Define  $\tilde{B} = \tilde{I}^n$ . Consider the restriction  $\tilde{T} = T|_{\tilde{B}}$  which is a nonexpansive opinion aggregator over the compact set  $\tilde{B}$ . By Theorem 2, we have that

$$\lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} \tilde{T}^t(x) \text{ exists.}$$

Since  $\tilde{T}^t(x) = T^t(x)$  for all  $t \in \mathbb{N}$  and  $x$  was arbitrarily chosen, (19) follows. By Lemma 8, if we define  $\bar{T}$  as in (20),  $\bar{T}$  is well defined. Let  $\hat{B}$  be a bounded subset of  $B$ . Note that there exists a compact interval with nonempty interior  $\tilde{I} \subseteq I$  such that  $\hat{B} \subseteq \tilde{B}$  where  $\tilde{B} = \tilde{I}^n$ . Define  $\tilde{T}$  as before. By (21) applied to  $\tilde{T}$ , we have that

$$\begin{aligned} \sup_{x \in \hat{B}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \bar{T}(x) \right\|_{\infty} &\leq \sup_{x \in \tilde{B}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \bar{T}(x) \right\|_{\infty} \\ &= \sup_{x \in \tilde{B}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} \tilde{T}^t(x) - \tilde{T}(x) \right\|_{\infty} \rightarrow 0, \end{aligned}$$

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<sup>40</sup>Recall that  $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ .

proving (22). ■

**Proof of Lemma 2.** Define  $\bar{x} = \lim_t T^t(x^0)$ . Since  $B$  is closed, note that  $\bar{x} \in B$ . Since  $T$  is continuous, it follows that

$$T(\bar{x}) = T\left(\lim_t T^t(x^0)\right) = \lim_t T(T^t(x^0)) = \lim_t T^{t+1}(x^0) = \bar{x},$$

proving the statement. By Lemma 1, if  $T$  is robust, then it is nonexpansive and, in particular, continuous. If  $T$  is convergent, then  $\lim_t T^t(x)$  exists for all  $x \in B$ . Since standard convergence implies Cesaro convergence to the same limit (see, e.g., Theorem 11 below), we can conclude that  $\bar{T}(x) = \lim_t T^t(x) \in E$  for all  $x \in B$ . ■

**Proof of Proposition 9.** Since  $T$  is linear,  $T$  is robust. By Corollary 5, we have that  $C - \lim_t T^t(x)$  exists for all  $x \in B$ . Fix  $x \in B$ . For ease of notation, define  $\hat{x} = C - \lim_t T^t(x)$ . Since  $T$  is linear and continuous, we have that

$$\begin{aligned} T(\hat{x}) &= T\left(\lim_\tau \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x)\right) = \lim_\tau T\left(\frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x)\right) = \lim_\tau \frac{1}{\tau} \sum_{t=1}^{\tau} T^{t+1}(x) \\ &= \lim_\tau \left[ \frac{\tau+1}{\tau} \frac{1}{\tau+1} \sum_{t=1}^{\tau+1} T^t(x) - \frac{1}{\tau} T(x) \right] \\ &= \lim_\tau \frac{\tau+1}{\tau} \lim_\tau \frac{1}{\tau+1} \sum_{t=1}^{\tau+1} T^t(x) - \left(\lim_\tau \frac{1}{\tau}\right) T(x) = \lim_\tau \frac{1}{\tau+1} \sum_{t=1}^{\tau+1} T^t(x) = \hat{x}, \end{aligned}$$

proving the statement. ■

**Proof of Proposition 10.** “Only if” If  $x \in D$ , then set  $\varepsilon(x) = \frac{1}{2}$ . Consider then  $x \in B \setminus D$ . We have that  $\delta(x) > 0$ . Since  $T$  is a consensus operator, we have that  $T(x) \neq x$ . It follows that  $\|T(x) - x\|_\infty > 0$ . Define  $\varepsilon(x) = \min\left\{\frac{1}{2}, \frac{\|T(x) - x\|_\infty}{\delta(x)}\right\} \in (0, 1)$ . In both cases, we have that (24) holds.

“If” Consider an equilibrium  $\bar{x}$ . Since  $\varepsilon(\bar{x}) \in (0, 1)$ , (24) yields that  $\delta(\bar{x}) = 0$ , that is,  $\bar{x} \in D$ . ■

In the next result, we prove that in the nonlinear case, the underlying network structure turns out to provide a form of monotonicity of the operator  $T$  which will help us in proving convergence. We encountered first this form of monotonicity in Lemma 10 in the Online Appendix. Recall that for each opinion aggregator  $T$  we denote by  $A(T)$  the square matrix such that for each  $i, j \in N$  we have  $a_{ij} \in \{0, 1\}$  and  $a_{ij} = 1$  if and only if  $j$  strongly influences  $i$ .

**Proposition 24** *Let  $T$  be a normalized and monotone opinion aggregator. The following statements are equivalent:*

(i) No row of  $A(T)$  is null;

(ii) There exist  $W \in \mathcal{W}$  and  $\varepsilon \in (0, 1)$  such that

$$T(x) = \varepsilon Wx + (1 - \varepsilon)S(x) \quad \forall x \in B \quad (64)$$

where  $S$  is a normalized and monotone operator.

Moreover, we have that

1.  $W$  in (ii) can be chosen to be such that  $A(W) = A(T)$ .
2.  $W$  in (ii) can be chosen to belong to  $\mathcal{W}_\delta$  for some  $\delta \in (0, 1]$  if and only if  $T$  has the uniform common influencer property.
3.  $W$  in (ii) can be chosen to be scrambling if and only if  $T$  has the pairwise common influencer property.
4.  $W$  in (ii) can be chosen to be irreducible if and only if  $T$  is strongly connected.

**Proof.** (i) implies (ii). For each  $i, j \in N$  consider  $\varepsilon_{ij} \in (0, 1)$  as in (32) if  $j$  strongly influences  $i$  and  $\varepsilon_{ij} = 1/2$  if  $j$  does not strongly influence  $i$ . Define  $\tilde{W}$  to be such that  $\tilde{w}_{ij} = a_{ij}\varepsilon_{ij}$  for all  $i, j \in N$ . Since each row of  $A(T)$  is not null, for each  $i \in N$  there exists  $j \in N$  such that  $a_{ij} = 1$  and, in particular,  $\tilde{w}_{ij} > 0$ . This implies that  $\sum_{l=1}^n \tilde{w}_{il} > 0$  for all  $i \in N$ . Define also  $\varepsilon = \min\{\min_{i \in N} \sum_{l=1}^n \tilde{w}_{il}, 1/2\} > 0$ . Define  $W \in \mathcal{W}$  to be such that  $w_{ij} = \tilde{w}_{ij} / \sum_{l=1}^n \tilde{w}_{il}$  for all  $i, j \in N$ . Clearly, we have that  $w_{ij} > 0$  if and only if  $\tilde{w}_{ij} > 0$  if and only if  $a_{ij} = 1$  for all  $i, j \in N$ . This yields that  $A(T) = A(W)$ . Next, consider  $x, y \in B$  such that  $x \geq y$ . Define  $y^0 = y$ . For each  $t \in \{1, \dots, n-1\}$  define  $y^t \in B$  to be such that  $y_i^t = x_i$  for all  $i \leq t$  and  $y_i^t = y_i$  for all  $i \geq t+1$ . Define  $y^n = x$ . Note that  $x = y^n \geq \dots y^1 \geq y^0 = y$ . It follows that

$$\begin{aligned} T_i(x) - T_i(y) &= \sum_{j=1}^n [T_i(y^j) - T_i(y^{j-1})] \geq \sum_{j=1}^n a_{ij}\varepsilon_{ij} (y_j^j - y_j^{j-1}) \\ &= \sum_{j=1}^n \tilde{w}_{ij} (x_j - y_j) = \left( \sum_{l=1}^n \tilde{w}_{il} \right) \left( \sum_{j=1}^n \frac{\tilde{w}_{ij}}{\sum_{l=1}^n \tilde{w}_{il}} (x_j - y_j) \right) \\ &= \left( \sum_{l=1}^n \tilde{w}_{il} \right) \left( \sum_{j=1}^n w_{ij} (x_j - y_j) \right) \\ &\geq \varepsilon \sum_{j=1}^n w_{ij} (x_j - y_j) \quad \forall i \in N. \end{aligned}$$

Since  $i$  was arbitrarily chosen, it follows that  $x \geq y$  implies  $T(x) - T(y) \geq \varepsilon(Wx - Wy)$ . By Lemma 10 in the Online Appendix, (64) follows.

(ii) implies (i). Consider  $i \in N$ . Since  $W$  is a stochastic matrix there exists  $j \in N$  such that  $w_{ij} > 0$ . Let  $y \in B$ ,  $h \in \mathbb{R}$  be such that  $y + he_j \in B$ . By (64), we have that

$$\begin{aligned} T_i(y + he_j) - T_i(y) &= \varepsilon w_{ij}h + (1 - \varepsilon) S_i(y + he_j) - (1 - \varepsilon) S_i(y) \\ &\geq \varepsilon w_{ij}h, \end{aligned}$$

proving that  $j$  strongly influences  $i$  and  $a_{ij} = 1$ . It follows that the  $i$ -th row of  $A(T)$  is not null. Since  $i$  was arbitrarily chosen, the statement follows.

1. By the proof of (i) implies (ii), we also have that  $W$  can be chosen to be such that  $A(T) = A(W)$ .

2. “If”. Since  $T$  has the uniform common influencer property, we have that there exists  $k \in N$  which strongly influences each  $i \in N$ . By the proof of (i) implies (ii), we also have that  $W \in \mathcal{W}$  is such that  $w_{ik} > 0$  for all  $i \in N$ . If we define  $\delta = \min_{i \in N} w_{ik}$ , then we have that  $W \in \mathcal{W}_\delta$ . “Only if”. Since  $W \in \mathcal{W}_\delta$  for some  $\delta \in (0, 1]$  satisfies (64), we have that there exists  $k \in N$  such that  $w_{ik} > 0$  for all  $i \in N$ . By the proof of (ii) implies (i), this implies that for all  $y \in B$ ,  $h \in \mathbb{R}$  such that  $y + he_k \in B$

$$T_i(y + he_k) - T_i(y) \geq \varepsilon w_{ik}h \geq \varepsilon \delta h \quad \forall i \in N,$$

proving the implication.

3. “If”. Since  $T$  has the pairwise common influencer property, we have that for each  $i, j \in N$  there exists  $k = k(i, j) \in N$  which strongly influences both  $i, j \in N$ . By the proof of (i) implies (ii), we also have that  $W \in \mathcal{W}$  is such that  $w_{ik} > 0$  and  $w_{jk} > 0$ . It follows that  $W$  is scrambling. “Only if”. Since  $W$  is scrambling and satisfies (64), we have that for each  $i, j \in N$  there exists  $k \in N$  such that  $w_{ik} > 0$  and  $w_{jk} > 0$ . Define  $\varepsilon_{ij} = \min \{w_{ik}\varepsilon, w_{jk}\varepsilon\} > 0$  for all  $i, j \in N$ . By the proof of (ii) implies (i), this implies that for all  $y \in B$ ,  $h \in \mathbb{R}$  such that  $y + he_k \in B$

$$T_i(y + he_k) - T_i(y) \geq \varepsilon w_{ik}h \geq \varepsilon_{ij}h$$

and

$$T_j(y + he_k) - T_j(y) \geq \varepsilon w_{jk}h \geq \varepsilon_{ij}h$$

proving the implication.

4. “If” By point 1 we know that  $W$  can be chosen such that  $A(W) = A(T)$ . Since  $T$  is strongly connected,  $A(T)$  is irreducible, and so is  $W$ . “Only if”. By the proof of (ii) implies (i), this implies that for all  $i, j \in N$  such that  $w_{ij} > 0$ , if we let  $\varepsilon_{ij} = \varepsilon w_{ij}$ , then for all  $y \in B$ ,  $h \in \mathbb{R}$  such that  $y + he_j \in B$

$$T_i(y + he_j) - T_i(y) \geq \varepsilon w_{ij}h \geq \varepsilon_{ij}h.$$

But then,  $A(T) \geq A(W)$ , and since  $A(W)$  is irreducible, so is  $A(T)$ , proving the implication.  $\blacksquare$

**Proof of Proposition 11.** By Proposition 24, if  $T$  is scrambling (resp., strongly connected) we have that there exist  $\varepsilon \in (0, 1)$  and  $W$  scrambling such that

$$T(x) = \varepsilon Wx + (1 - \varepsilon) S(x) \quad \forall x \in B$$

where  $S : B \rightarrow B$  is a normalized and monotone opinion aggregator (resp., strongly connected).

a. Assume that  $T$  has the pairwise common influencer property. Let  $x \in B \setminus D$ . Define  $x_i = \min_{l \in N} x_l$  and  $x_j = \max_{l \in N} x_l$ . It follows that  $x_j > x_i$  and  $i \neq j$ . Since  $W$  is scrambling, there exists  $k = k(i, j) \in N$  such that  $w_{ik} > 0$  and  $w_{jk} > 0$ . Define  $\gamma = \min_{l, m \in N: w_{lm} \neq 0} w_{lm} > 0$ . Note that  $w_{ik}, w_{jk} \geq \gamma$ . We have two cases:

1.  $x_k < x_j$ . It follows that

$$\begin{aligned} \|T(x) - x\|_\infty &\geq |T_j(x) - x_j| = \left| \varepsilon \sum_{l=1}^n w_{jl} x_l + (1 - \varepsilon) S_j(x) - x_j \right| \\ &= \left| \varepsilon \sum_{l=1}^n w_{jl} (x_l - x_j) + (1 - \varepsilon) (S_j(x) - x_j) \right| \\ &= \varepsilon \sum_{l=1}^n w_{jl} (x_j - x_l) + (1 - \varepsilon) (x_j - S_j(x)) \\ &\geq \varepsilon \gamma (x_j - x_k) = \varepsilon \gamma |x_j - x_k| \geq \varepsilon(x) \delta(x) \end{aligned}$$

where  $\varepsilon(x) = \varepsilon \gamma$ .

2.  $x_k > x_i$ . It follows that

$$\begin{aligned} \|T(x) - x\|_\infty &\geq |T_i(x) - x_i| = \left| \varepsilon \sum_{l=1}^n w_{il} x_l + (1 - \varepsilon) S_i(x) - x_i \right| \\ &= \left| \varepsilon \sum_{l=1}^n w_{il} (x_l - x_i) + (1 - \varepsilon) (S_i(x) - x_i) \right| \\ &= \varepsilon \sum_{l=1}^n w_{il} (x_l - x_i) + (1 - \varepsilon) (S_i(x) - x_i) \\ &\geq \varepsilon \gamma (x_k - x_i) = \varepsilon \gamma |x_k - x_i| \geq \varepsilon(x) \delta(x) \end{aligned}$$

where  $\varepsilon(x) = \varepsilon \gamma$ .

Case 1 and 2 prove that if  $x \in B \setminus D$ , then (24) is satisfied by setting  $\varepsilon(x) = \varepsilon \gamma$ . If  $x \in D$ , we can still set  $\varepsilon(x) = \gamma \varepsilon > 0$  and have (24) satisfied. By Proposition 10, the statement follows.

b. Assume that  $T$  has the uniform common influencer property. It follows that  $T$  has the pairwise common influencer property. Since the latter property implies that  $T$  is a consensus operator, the statement follows.

c. We start by proving an ancillary fact. Namely, that for each pair of agents  $(i, j)$  there exists a  $t \in \mathbb{N}$  such that  $j$  strongly influences  $i$  with respect to  $T^t$ , that is, formally, there exists  $\varepsilon_{ij,t} \in (0, 1)$  such that

$$x \geq y \implies T_i^t(x) - T_i^t(y) \geq \varepsilon_{ij,t}(x_j - y_j). \quad (65)$$

Thus, consider two generic agents  $i, j \in N$ . Since  $T$  is strongly connected, recall that  $A(T)$  is strongly connected. This implies that there exists  $t \in \mathbb{N}$  and a finite sequence  $\{i_1, \dots, i_{t+1}\} \subseteq N$  such that  $i_1 = i$ ,  $i_{t+1} = j$ , and  $a_{i_k i_{k+1}} = 1$  for all  $k \in \{1, \dots, t\}$ . Next, consider  $x, y \in B$  such that  $x \geq y$ . By construction, we have that  $i_{k+1}$  strongly influences  $i_k$  for all  $k \in \{1, \dots, t\}$  where  $\varepsilon_{i_k i_{k+1}} \in (0, 1)$  is as in (32). Next, fix  $s \in \{1, \dots, t\}$  and  $k \in \mathbb{N}$ . It follows that

$$T_{i_k}^s(x) - T_{i_k}^s(y) = T_{i_k}(T^{s-1}(x)) - T_{i_k}(T^{s-1}(y)) \geq \varepsilon_{i_k i_{k+1}} \left( T_{i_{k+1}}^{s-1}(x) - T_{i_{k+1}}^{s-1}(y) \right). \quad (66)$$

Next, by finite induction, we show that for each  $l \in \{1, \dots, t\}$

$$T_{i_1}^t(x) - T_{i_1}^t(y) \geq \left( \prod_{k=1}^l \varepsilon_{i_k i_{k+1}} \right) \left( T_{i_{l+1}}^{t-l}(x) - T_{i_{l+1}}^{t-l}(y) \right). \quad (67)$$

Clearly, if  $l = 1$ , then (67) follows from (66), by setting in the latter  $s = t$  and  $k = 1$ . If (67) holds for  $l \in \{1, \dots, t-1\}$ , then

$$\begin{aligned} T_{i_1}^t(x) - T_{i_1}^t(y) &\geq \left( \prod_{k=1}^l \varepsilon_{i_k i_{k+1}} \right) \left( T_{i_{l+1}}^{t-l}(x) - T_{i_{l+1}}^{t-l}(y) \right) \\ &\geq \left( \prod_{k=1}^l \varepsilon_{i_k i_{k+1}} \right) \varepsilon_{i_{l+1} i_{l+2}} \left( T_{i_{l+2}}^{t-l-1}(x) - T_{i_{l+2}}^{t-l-1}(y) \right) \\ &= \left( \prod_{k=1}^{l+1} \varepsilon_{i_k i_{k+1}} \right) \left( T_{i_{l+2}}^{t-(l+1)}(x) - T_{i_{l+2}}^{t-(l+1)}(y) \right) \end{aligned}$$

where the second inequality follows from (66), by setting in the latter  $s = t - l$  and  $k = l + 1$ . Finally, by setting  $l = t$  in (67), we obtain that

$$T_i^t(x) - T_i^t(y) = T_{i_1}^t(x) - T_{i_1}^t(y) \geq \left( \prod_{k=1}^t \varepsilon_{i_k i_{k+1}} \right) \left( T_{i_{t+1}}^0(x) - T_{i_{t+1}}^0(y) \right) = \varepsilon_{ij,t}(x_j - y_j)$$

where  $\varepsilon_{ij,t} = \prod_{k=1}^t \varepsilon_{i_k i_{k+1}} > 0$ , proving (65).

We next prove  $T$  is a consensus operator. By contradiction, assume that  $x$  is an equilibrium of  $T$ , but  $x \notin D$ . Define  $x_i = \min_{l \in N} x_l$  and  $x_j = \max_{l \in N} x_l$ . It follows that  $x_j > x_i$  and  $i \neq j$ . Consider  $t$  and  $\varepsilon_{ij,t}$  as in (65). Define  $y = x_i e$ . Clearly, we have that  $x \geq y$ . By and since  $x$  is an equilibrium of  $T$  and  $T$  is normalized, we have that

$$0 = x_i - x_i = T_i^t(x) - T_i^t(y) \geq \varepsilon_{ij,t}(x_j - y_j) = \varepsilon_{ij,t}(x_j - x_i) > 0,$$

a contradiction. ■

The next result is going to prove useful in transforming the convergence of the updates' time averages into standard convergence.

**Theorem 11 (Lorentz)** *Let  $\{x^t\}_{t \in \mathbb{N}} \subseteq \mathbb{R}^n$  be a bounded sequence. The following statements are equivalent:*

(i) *There exists  $\bar{x} \in \mathbb{R}^n$  such that*

$$\forall \varepsilon > 0 \exists \bar{\tau} \in \mathbb{N} \forall m \in \mathbb{N} \text{ s.t. } \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} - \bar{x} \right\|_{\infty} < \varepsilon \quad \forall \tau \geq \bar{\tau} \quad (68)$$

*and  $\lim_{\tau} \|x^{\tau+1} - x^{\tau}\|_{\infty} = 0$ ;*

(ii)  *$\lim_{\tau} x^{\tau} = \bar{x}$ .*

**Remark 4** To the best of our knowledge, Theorem 11 was first proved by Lorentz [41]. His result is stated for a sequence  $\{x^t\}_{t \in \mathbb{N}}$  in  $\mathbb{R}$  where each  $x^t$  is the partial sum up to  $t$  of another sequence  $\{a^s\}_{s \in \mathbb{N}}$ . In other words, Lorentz's result is a Tauberian theorem for series. Nonetheless, the techniques used to prove Theorem 11 are the same elementary ones applied by Lorentz with the extra caveat of setting  $a^1 = x^1$  and  $a^s = x^s - x^{s-1}$  for all  $s \geq 2$ , see the proof in the Online Appendix. ▲

**Proof of Theorem 3.** Before starting observe that  $T$  is nonexpansive and, in particular, continuous. By Theorem 2, we have that

$$\lim_{\tau} \left( \sup_{x \in B} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \bar{T}(x) \right\|_{\infty} \right) = 0 \quad (69)$$

where  $\bar{T} : B \rightarrow B$  is a continuous map such that  $\bar{T} \circ T = \bar{T}$ . Since  $\bar{T}(T(x)) = \bar{T}(x)$  for all  $x \in B$ , by induction, we have that  $\bar{T}(T^m(x)) = \bar{T}(x)$  for all  $m \in \mathbb{N}$  and for all  $x \in B$ .

(i) implies (ii). By Lemma 2, the implication follows.

(ii) implies (iii). Define  $y_t = T^{t+1}(x^0) - T^t(x^0)$  for all  $t \in \mathbb{N}$ . If  $\lim_t T^t(x^0)$  exists, we have that there exists  $\bar{x} \in B$  such that for each  $\varepsilon > 0$  there exists  $t_{\varepsilon} \in \mathbb{N}$  such that  $\|T^t(x^0) - \bar{x}\|_{\infty} < \varepsilon/2$  for all  $t \geq t_{\varepsilon}$ . It follows that for each  $t \geq t_{\varepsilon}$  we have that

$$\|y_t\|_{\infty} = \|T^{t+1}(x^0) - \bar{x} + \bar{x} - T^t(x^0)\|_{\infty} \leq \|T^{t+1}(x^0) - \bar{x}\|_{\infty} + \|\bar{x} - T^t(x^0)\|_{\infty} < \varepsilon,$$

proving that  $\lim_t \|y_t\|_{\infty} = 0$  and the implication.

(iii) implies (i). Define the sequence  $x^t = T^t(x^0)$  for all  $t \in \mathbb{N}$ . By point 4 of Lemma 9 in the Online Appendix, we have that  $\{x^t\}_{t \in \mathbb{N}}$  is bounded. Note that for each  $t \in \mathbb{N}$  and for each  $m \in \mathbb{N}$

$$\frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} = \frac{1}{\tau} \sum_{t=1}^{\tau} T^{m+t}(x^0) = \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(T^m(x^0)).$$

Since (69) holds, if we define  $\bar{T}(x^0) = \bar{x}$ , then we have that for each  $m \in \mathbb{N}$

$$\lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} = \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(T^m(x^0)) = \bar{T}(T^m(x^0)) = \bar{T}(x^0) = \bar{x}.$$

It follows that

$$\begin{aligned} \sup_{m \in \mathbb{N}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} - \bar{x} \right\|_{\infty} &= \sup_{m \in \mathbb{N}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(T^m(x^0)) - \bar{T}(T^m(x^0)) \right\|_{\infty} \\ &\leq \sup_{x \in B} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \bar{T}(x) \right\|_{\infty}. \end{aligned}$$

Since (69) holds, we have that  $\{x^t\}_{t \in \mathbb{N}}$  satisfies (68) in (i) of Theorem 11. By Theorem 11, we have that  $\lim_t x^t = \lim_t T^t(x^0)$  exists.  $\blacksquare$

**Proof of Proposition 12.** The result for linear aggregators follows immediately from [7, Theorem 2]. A counterexample in the case of robust opinion aggregators is given by  $N = 2$ , and  $T_1(x) = x_1$ ,  $T_2(x) = \max\{x_1, x_2\}$ , since  $T$  converges to consensus if and only if the initial opinion  $y$  is such that  $y_1 \geq y_2$ .  $\blacksquare$

**Proof of Theorem 4.** (i) implies (ii). By Lemma 1 and since  $T$  is a robust opinion aggregator, it follows that  $T$  is nonexpansive. Consider  $x^0 \in B$ . Define  $\tilde{B}$  and  $\tilde{T}$  as in point 4 of Lemma 9 in the Online Appendix. Since  $T$  is nonexpansive and asymptotically regular and  $T^t(x^0) = \tilde{T}^t(x^0)$  for all  $t \in \mathbb{N}$ , it follows that  $\tilde{T}$  is nonexpansive and such that  $\lim_t \left\| \tilde{T}^{t+1}(x^0) - \tilde{T}^t(x^0) \right\|_{\infty} = 0$ . By Theorem 3 and since  $\tilde{B}$  is compact and  $\tilde{T}$  is nonexpansive and such that  $T(x) = \tilde{T}(x)$  for all  $x \in \tilde{B}$ , we have that  $\lim_t T^t(x^0) = \lim_t \tilde{T}^t(x^0)$  exists. Since  $x^0$  was arbitrarily chosen, it follows that  $T$  is convergent.

(ii) implies (i). Let  $x \in B$ . Since  $T$  is convergent, it follows that  $\{T^t(x)\}_{t \in \mathbb{N}}$  converges for all  $x \in B$ . We can conclude that  $\lim_t \|T^{t+1}(x) - T^t(x)\|_{\infty} = 0$ . Since  $x$  was arbitrarily chosen,  $T$  is asymptotically regular.

Next, we assume that, in addition,  $T$  is also constant affine.

(ii) implies (iii). By point 2 of Lemma 9 in the Online Appendix and since  $T$  is robust and constant affine, it admits a unique extension  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  which is also

robust and constant affine. Inter alia, we have that  $S^t$  is robust for all  $t \in \mathbb{N}$  and, in particular, positive homogeneous. Moreover, by Corollary 5 and Lemma 8, we have that  $\bar{S}$  has the same properties. Let  $x = 0$ . Consider  $\tilde{B}$  as in the proof of point 4 of Lemma 9 in the Online Appendix applied to  $S$ . We can choose  $\tilde{I}$  to be  $\tilde{I} = [-1, 1]$  and  $\tilde{B} = \tilde{I}^n$ . Let  $\tilde{S}$  be the restriction to  $\tilde{B}$  of  $S$ . Consider the space  $C(\tilde{B}, \mathbb{R}^n)$ : the space of continuous functions over  $\tilde{B}$  which take values in  $\mathbb{R}^n$ . The space  $C(\tilde{B}, \mathbb{R}^n)$  is a Banach space and we endow it with the norm  $\|f\|_* = \sup_{x \in \tilde{B}} \|f(x)\|_\infty$  for all  $f \in C(\tilde{B}, \mathbb{R}^n)$ . Note that  $\{\tilde{S}^t\}_{t \in \mathbb{N}} \subseteq C(\tilde{B}, \mathbb{R}^n)$ . Since  $T$  is convergent, so is the extension  $S$  and we have that  $\lim_t \tilde{S}^t(x) = \lim_t S^t(x) = \bar{S}(x)$  for all  $x \in \tilde{B}$ . This implies that  $\{\tilde{S}^t(x)\}_{t \in \mathbb{N}} \subseteq \mathbb{R}^n$  is bounded for all  $x \in \tilde{B}$ . By Lemma 8, recall also that  $\bar{S} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function and so is its restriction to  $\tilde{B}$  which we still denote  $\bar{S}$ . By Lemma 1 and since  $\tilde{S}$  is a robust opinion aggregator, it follows that  $\tilde{S}^t$  is nonexpansive for all  $t \in \mathbb{N}$ . By [19, pp. 135–136], the sequence  $\{\tilde{S}^t\}_{t \in \mathbb{N}} \subseteq C(\tilde{B}, \mathbb{R}^n)$  is also equicontinuous. By contradiction, assume that  $\lim_t \|\tilde{S}^t - \bar{S}\|_* \neq 0$ . This implies that there exists  $\varepsilon > 0$  and a subsequence  $\{\tilde{S}^{t_m}\}_{m \in \mathbb{N}} \subseteq \{\tilde{S}^t\}_{t \in \mathbb{N}}$  such that  $\|\tilde{S}^{t_m} - \bar{S}\|_* \geq \varepsilon$  for all  $m \in \mathbb{N}$ . By Arzela-Ascoli Theorem (see, e.g., [19, Theorem 7.5.7]) and since  $\{\tilde{S}^{t_m}\}_{m \in \mathbb{N}}$  is equicontinuous and  $\{\tilde{S}^{t_m}(x)\}_{m \in \mathbb{N}} \subseteq \mathbb{R}^n$  is bounded for all  $x \in \tilde{B}$ , this implies that there exists a subsequence  $\{\tilde{S}^{t_{m(l)}}\}_{l \in \mathbb{N}}$  and a function  $\hat{S} \in C(\tilde{B}, \mathbb{R}^n)$  such that  $\lim_l \|\tilde{S}^{t_{m(l)}} - \hat{S}\|_* = 0$ . By definition of  $\|\cdot\|_*$ , it follows that  $\bar{S}(x) = \lim_l \tilde{S}^{t_{m(l)}}(x) = \hat{S}(x)$  for all  $x \in \tilde{B}$ , that is,  $\hat{S} = \bar{S}$ . This implies that  $\varepsilon \leq \lim_l \|\tilde{S}^{t_{m(l)}} - \bar{S}\|_* = 0$ , a contradiction. We can conclude that

$$\lim_t \left( \sup_{x \in \tilde{B}} \|\tilde{S}^t(x) - \bar{S}(x)\|_\infty \right) = \lim_t \|\tilde{S}^t - \bar{S}\|_* = 0.$$

By point 2 of Lemma 9 in the Online Appendix, recall that  $\tilde{S}^t(x) = S^t(x)$  for all  $x \in \tilde{B}$  and for all  $t \in \mathbb{N}$ . Note also that  $\bar{S}(x) \in \tilde{B}$  for all  $x \in \tilde{B}$ . Consequently, define  $\{c_t\}_{t \in \mathbb{N}} \subseteq [0, \infty)$  by

$$c_t = \sup_{x \in \tilde{B}} \|\tilde{S}^t(x) - \bar{S}(x)\|_\infty = \sup_{x \in \tilde{B}} \|S^t(x) - \bar{S}(x)\|_\infty \quad \forall t \in \mathbb{N}. \quad (70)$$

By the previous part of the proof, note that  $c_t \rightarrow 0$ . Consider  $y \in \mathbb{R}^n \setminus \{0\}$  and  $t \in \mathbb{N}$ . It follows that  $\frac{1}{\|y\|_\infty} S^t(y) = S^t\left(\frac{y}{\|y\|_\infty}\right)$ ,  $\frac{1}{\|y\|_\infty} \bar{S}(y) = \bar{S}\left(\frac{y}{\|y\|_\infty}\right)$ , and  $\frac{y}{\|y\|_\infty} \in \tilde{B}$ . By (70),

we have that

$$\begin{aligned} \frac{1}{\|y\|_\infty} \|S^t(y) - \bar{S}(y)\|_\infty &= \left\| \frac{1}{\|y\|_\infty} S^t(y) - \frac{1}{\|y\|_\infty} \bar{S}(y) \right\|_\infty \\ &= \left\| S^t\left(\frac{y}{\|y\|_\infty}\right) - \bar{S}\left(\frac{y}{\|y\|_\infty}\right) \right\|_\infty \leq c_t. \end{aligned}$$

Since  $y$  was arbitrarily chosen different from 0 and  $S^t(0) = 0 = \bar{S}(0)$  for all  $t \in \mathbb{N}$ , we have that

$$\|S^t(y) - \bar{S}(y)\|_\infty \leq c_t \|y\|_\infty \quad \forall t \in \mathbb{N}, \forall y \in \mathbb{R}^n.$$

Since  $S$  is the extension of  $T$ , we can conclude that

$$\|T^t(x) - \bar{T}(x)\|_\infty = \|S^t(x) - \bar{S}(x)\|_\infty \leq c_t \|x\|_\infty \quad \forall t \in \mathbb{N}, \forall x \in B,$$

proving (26).

(iii) implies (ii). Let  $x \in B$ . Since  $T$  satisfies (26), we clearly have that  $T$  is convergent. ■

**Remark 5** Proving that asymptotic regularity is equivalent to convergence can also be obtained with an alternative technique which does not resort to Lorentz's Theorem, but is a minor adjustment of Browder and Petryshyn [10, Theorem 2].<sup>41</sup> Compared to our version, the part that would be missing is the one regarding the rate of convergence which is mainly due to the property of constant affinity of  $T$ . ▲

**Proof of Theorem 5.** Let  $x, y \in B$  with  $x \geq y$ . Let  $i \in I$ . Then, since  $T$  is monotone

$$T_i(x) - T_i(y) \geq T_i(y + (x_i - y_i)e_i) - T_i(y) \geq \varepsilon(x_i - y_i).$$

Therefore,  $T$  satisfies (94) with  $W$  being the identity. By Lemma 10 in the Online Appendix,  $T = \varepsilon I + (1 - \varepsilon)S$  where  $S$  is a robust opinion aggregator. By Lemma 1,  $S$  is nonexpansive. Let  $x \in B$ . Let  $\tilde{B}$  be as in point 4 of Lemma 9 in the Online Appendix. Denote by  $\tilde{T}$  and  $\tilde{S}$  the restrictions of  $T$  and  $S$  to  $\tilde{B}$ . It follows that  $\tilde{T} = \varepsilon I + (1 - \varepsilon)\tilde{S}$  and  $\tilde{S}$  is nonexpansive. By [26, Theorem 9.4] and since  $\tilde{B}$  is compact, this implies that  $\tilde{T}$  is asymptotically regular. We can conclude that  $\lim_t \|T^{t+1}(x) - T^t(x)\|_\infty = \lim_t \left\| \tilde{T}^{t+1}(x) - \tilde{T}^t(x) \right\|_\infty = 0$ . Since  $x$  was arbitrarily chosen,  $T$  is asymptotically regular. By Theorem 4, the statement follows. ■

Our results about convergence of robust opinion aggregators can all be reduced to the following convergence result which generalizes Berger's Theorem (see, e.g., [6] and [35, Corollary 8.2]). On the one hand, Proposition 25 generalizes Berger's result in

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<sup>41</sup>Browder and Petryshyn [10, Theorem 2] is a result for self-maps from a Banach space to a Banach space. Here,  $B$  might be a proper subset of  $\mathbb{R}^n$ . A proof is available upon request.

two distinct ways: a) it allows for nonlinearities of  $T$  and b) it offers some estimate about the rate of convergence. On the other hand, it provides only a sufficient condition for convergence to consensus. In words, it says that if there exists  $\hat{t} \in \mathbb{N}$  such that  $T^{\hat{t}}$  has the uniform common influencer property, then  $T$  is a convergent opinion aggregator which always converge to consensus. From an economic point of view, if  $T^{\hat{t}}$  has the uniform common influencer property, it means that there exists an agent in the population such that a change in her opinion influences each agent after  $\hat{t}$  rounds of updating.

**Proposition 25** *Let  $T$  be a robust opinion aggregator. If there exists  $\hat{t}$  such that  $T^{\hat{t}}$  has the common influencer property, then  $\bar{T}(x) = \lim_t T^t(x) \in D$  and there exists  $\varepsilon \in (0, 1)$*

$$\|\bar{T}(x) - T^t(x)\|_{\infty} \leq 2(1 - \varepsilon)^{\lfloor \frac{t}{\hat{t}} \rfloor} \|x\|_{\infty} \quad \forall t \in \mathbb{N}, \forall x \in B. \quad (71)$$

Moreover, if  $\hat{t} = 1$ , then  $\varepsilon$  can be chosen to be the one that satisfies (28) and for each  $m, t \in \mathbb{N}$  such that  $m \geq t$

$$\|T^m(x) - T^t(x)\|_{\infty} \leq 2(1 - \varepsilon)^t \|x\|_{\infty} \quad \forall x \in B. \quad (72)$$

**Proof.** Before proving the main statement, we need to state and prove an ancillary claim. We just introduce some notation. Given a sequence of stochastic matrices,  $\{W_l\}_{l \in \mathbb{N}} \subseteq \mathcal{W}$ , we denote by  $\Pi_{l=1}^{t+1} W_l$  the backward product of the first  $t+1$  elements, that is,  $\Pi_{l=1}^{t+1} W_l = W_{t+1} \Pi_{l=1}^t W_l = W_{t+1} \dots W_1$  for all  $t \in \mathbb{N}$ .

*Claim:* If  $\{W_l\}_{l \in \mathbb{N}} \subseteq \mathcal{W}_{\varepsilon}$  for some  $\varepsilon \in (0, 1)$ , then for each  $t, m \in \mathbb{N}$  such that  $m \geq t$

$$\|(\Pi_{l=1}^m W_l)x - (\Pi_{l=1}^t W_l)x\|_{\infty} \leq 2(1 - \varepsilon)^t \|x\|_{\infty} \quad \forall x \in B$$

*Proof of the Claim.* Recall that the product of stochastic matrices is a stochastic matrix, thus  $\Pi_{l=1}^t W_l \in \mathcal{W}$  for all  $t \in \mathbb{N}$ . Next, define  $V_0 = \{y \in \mathbb{R}^n : \sum_{i=1}^n y_i = 0\}$ . By [58, p. 28], note that for each  $y \in V_0$  and  $l \in \mathbb{N}$

$$y^T W_l \in V_0 \text{ and } \|y^T W_l\|_1 \leq (1 - \varepsilon) \|y\|_1$$

where  $\|y\|_1 = \sum_{i=1}^n |y_i|$ . By induction, this yields that

$$\|y^T \Pi_{l=1}^t W_l\|_1 \leq (1 - \varepsilon)^t \|y\|_1 \quad \forall y \in V_0. \quad (73)$$

Finally, consider  $x \in B$  and  $m > t$ . By the definition of backward product, it follows that  $(\Pi_{l=1}^m W_l)x = (\Pi_{l=t+1}^m W_l)((\Pi_{l=1}^t W_l)x)$  and  $(\Pi_{l=1}^t W_l)x$ . Observe that

$$(\Pi_{l=t+1}^m W_l)((\Pi_{l=1}^t W_l)x) - (\Pi_{l=1}^t W_l)x = ((\Pi_{l=t+1}^m W_l) - I)((\Pi_{l=1}^t W_l)x) = Z(Rx)$$

where  $Z = \Pi_{l=t+1}^m W_l - I$  and  $R = \Pi_{l=1}^t W_l$ . Note that  $R, \Pi_{l=t+1}^m W_l \in \mathcal{W}$  and, in particular, the entries of each row of  $Z$  sum up to 0. Denote by  $z^i$  the column vector whose transpose is exactly the  $i$ -th row of  $Z$ . It is immediate to see that it is the difference of two probability vectors, thus  $z^i \in V_0$  and  $\|z^i\|_1 \leq 2$ . The quantities  $Z(Rx)$  and  $Rx$  are vectors of  $\mathbb{R}^n$  and the  $i$ -th component of the former is exactly  $(z^i)^\top Rx$ . By (73) and since  $z^i \in V_0$ , we can conclude that

$$\begin{aligned} \left| (z^i)^\top (Rx) \right| &= \left| \left( (z^i)^\top R \right) x \right| \leq \left\| (z^i)^\top R \right\|_1 \|x\|_\infty \leq (1 - \varepsilon)^t \|z^i\|_1 \|x\|_\infty \\ &\leq 2(1 - \varepsilon)^t \|x\|_\infty. \end{aligned}$$

Since  $(\Pi_{l=1}^m W_l)x - (\Pi_{l=1}^t W_l)x = Z(Rx)$  and  $t, m, x$ , and  $i$  were arbitrarily chosen, we have that

$$\left| \left( (\Pi_{l=1}^m W_l)x - (\Pi_{l=1}^t W_l)x \right)_i \right| = \left| (z^i)^\top (Rx) \right| \leq 2(1 - \varepsilon)^t \|x\|_\infty \quad \forall x \in B, \forall i \in N, \forall m > t.$$

We can conclude that

$$\left\| (\Pi_{l=1}^m W_l)x - (\Pi_{l=1}^t W_l)x \right\|_\infty \leq 2(1 - \varepsilon)^t \|x\|_\infty \quad \forall x \in B, \forall t, m \in \mathbb{N} \text{ s.t. } m \geq t, \quad (74)$$

proving the claim.  $\square$

We start by proving the case  $\hat{t} = 1$ . By Proposition 20 and its proof and since  $\hat{t} = 1$ ,  $T$  has the uniform common influencer property. It follows that for each  $x \in B$  there exists a matrix  $W(x) \in \mathcal{W}_\varepsilon$  such that  $T(x) = W(x)x$  for all  $x \in B$ . Fix  $x \in B$ . Define  $W_t = W(T^{t-1}(x))$  for all  $t \in \mathbb{N}$ .<sup>42</sup> We have that for each  $m \in \mathbb{N}$

$$T^m(x) = W(T^{m-1}(x))T^{m-1}(x) = (\Pi_{l=1}^m W_l)x.$$

By the previous claim, we have that

$$\left\| T^m(x) - T^t(x) \right\|_\infty = \left\| (\Pi_{l=1}^m W_l)x - (\Pi_{l=1}^t W_l)x \right\|_\infty \leq 2(1 - \varepsilon)^t \|x\|_\infty \quad \forall t, m \in \mathbb{N} \text{ s.t. } m \geq t.$$

Since  $x$  was arbitrarily chosen, (72) holds with  $\varepsilon$  where  $\varepsilon$  is the one that satisfies (28). This implies that  $\{T^t(x)\}_{t \in \mathbb{N}}$  is a Cauchy sequence in  $B$ . Since  $B$  is closed and  $x$  was arbitrarily chosen, it follows that  $\lim_t T^t(x)$  exists and belongs to  $B$  for all  $x \in B$ . By Lemma 2 and since  $T$  is convergent, we have that  $\bar{T}(x) = \lim_t T^t(x)$  for all  $x \in B$ . By taking the limit in  $m$  in (72), (71) immediately follows for the case  $\hat{t} = 1$ . By Lemma 2 and since  $\bar{T}(x) = \lim_t T^t(x)$ , we have that  $\bar{T}(x)$  is an equilibrium of  $T$  for all  $x \in B$ . Proposition 11 and since  $T$  has the uniform common influencer property,  $T$  is a consensus operator and  $\bar{T}(x) = \lim_t T^t(x) \in D$  for all  $x \in B$ . If  $\hat{t} > 1$ , define  $U : B \rightarrow B$  to be such that  $U = T^{\hat{t}}$ . Note that  $U$  is a robust opinion aggregator.

<sup>42</sup>As usual,  $T^0(x) = x$  for all  $x \in B$ .

By assumption, it follows that  $U$  satisfies the uniform common influencer property for some  $\varepsilon \in (0, 1)$ . By Theorem 6, we have that

$$\|\bar{U}(x) - U^t(x)\|_\infty \leq 2(1 - \varepsilon)^t \|x\|_\infty \quad \forall t \in \mathbb{N}, \forall x \in B \quad (75)$$

and  $\bar{U}(x) = \lim_t U^t(x) \in D$  for all  $x \in B$ . Consider  $x \in B$  and, for ease of notation, we denote  $\bar{x} = \bar{U}(x)$ . Let  $t \in \mathbb{N}$ . We have two cases:

1.  $t < \hat{t}$ . In this case, it follows that  $\lfloor \frac{t}{\hat{t}} \rfloor = 0$  and

$$\|\bar{x} - T^t(x)\|_\infty \leq \|\bar{U}(x)\|_\infty + \|T^t(x)\|_\infty \leq \|x\|_\infty + \|x\|_\infty = 2(1 - \varepsilon)^{\lfloor \frac{t}{\hat{t}} \rfloor} \|x\|_\infty.$$

2.  $t \geq \hat{t}$ . In this case, it follows that there exists  $n \in \mathbb{N}$  such that  $n\hat{t} \leq t < (n+1)\hat{t}$ . This implies that  $n \leq \frac{t}{\hat{t}} < (n+1)$  and  $n = \lfloor \frac{t}{\hat{t}} \rfloor$ . We have two subcases:

(a)  $n = \frac{t}{\hat{t}}$ . In this case, we have that

$$\|\bar{x} - T^t(x)\|_\infty = \|\bar{x} - T^{n\hat{t}}(x)\|_\infty = \|\bar{x} - U^n(x)\|_\infty \leq 2(1 - \varepsilon)^n \|x\|_\infty.$$

(b)  $n < \frac{t}{\hat{t}}$ . In this case, define  $j = t - n\hat{t}$ . Since  $\bar{x} \in D$ , we have that

$$\begin{aligned} \|\bar{x} - T^t(x)\|_\infty &= \|T(\bar{x}) - T^t(x)\|_\infty \leq \|\bar{x} - T^{t-1}(x)\|_\infty \\ &= \|T(\bar{x}) - T^{t-1}(x)\|_\infty \leq \dots \\ &\leq \|\bar{x} - T^{t-j}(x)\|_\infty = \|\bar{x} - T^{n\hat{t}}(x)\|_\infty \\ &= \|\bar{x} - U^n(x)\|_\infty \leq 2(1 - \varepsilon)^n \|x\|_\infty. \end{aligned}$$

Since  $t$  and  $x$  were arbitrarily chosen, points 1 and 2 prove that

$$\|\bar{U}(x) - T^t(x)\|_\infty \leq 2(1 - \varepsilon)^{\lfloor \frac{t}{\hat{t}} \rfloor} \|x\|_\infty \quad \forall t \in \mathbb{N}, \forall x \in B. \quad (76)$$

It follows that  $\{T^t(x)\}_{t \in \mathbb{N}}$  converges for all  $x \in B$ . By taking the limit in  $t$ , this implies that  $\bar{T}(x) = \lim_t T^t(x) = \bar{U}(x) \in D$  for all  $x \in B$ . Thus, (76) implies (71).  $\blacksquare$

**Proof of Theorem 6.** By hypothesis, in Proposition 25, we can set  $\hat{t} = 1$ . By (72), it follows that

$$\|T^m(x) - T^t(x)\|_\infty \leq 2(1 - \varepsilon)^t \|x\|_\infty \quad \forall x \in B \quad (77)$$

holds. By Proposition 25, we also have that  $\bar{T}(x) = \lim_t T^t(x) \in D$  for all  $x \in B$ . By taking the limit in  $m$  in (77), this implies that (29) follows. Finally, by (77) and setting  $m = t + 1$  and taking the limit in  $t$ , we have that for each  $x \in B$

$$0 \leq \limsup_t \|T^{t+1}(x) - T^t(x)\|_\infty \leq \lim_t 2(1 - \varepsilon)^t \|x\|_\infty \leq 0,$$

proving that  $T$  is asymptotically regular. ■

**Proof of Theorem 7.** By point 3 of Proposition 24 and since  $T$  has the pairwise common influencer property, we have that there exist  $\tilde{\varepsilon} \in (0, 1)$  and a scrambling matrix  $\tilde{W} \in \mathcal{W}$  such that

$$T(x) = \tilde{\varepsilon}\tilde{W}x + (1 - \tilde{\varepsilon})\tilde{S}(x) \quad \forall x \in B$$

where  $\tilde{S}$  is a normalized and monotone opinion aggregator. By induction, this implies that for each  $t \in \mathbb{N}$

$$x \geq y \implies T^t(x) - T^t(y) \geq \tilde{\varepsilon}^t \tilde{W}^t(x - y).$$

By [57, Lemma 4.5] and since  $\tilde{W}$  is scrambling, we have that there exists  $\hat{t} \in \mathbb{N}$  such that  $\tilde{W}^{\hat{t}}$  has a column whose entries are all strictly positive. By [33, Theorem 2], it follows that  $\hat{t}$  can be chosen to be such that  $\hat{t} \leq n^2 - 3n + 3$ . Define  $W = \tilde{W}^{\hat{t}}$ . Since  $W$  has a column whose entries are all strictly positive, note that  $W \in \mathcal{W}_\delta$  for some  $\delta \in (0, 1)$ . It follows that

$$x \geq y \implies T^{\hat{t}}(x) - T^{\hat{t}}(y) \geq \tilde{\varepsilon}^{\hat{t}}W(x - y) = \tilde{\varepsilon}^{\hat{t}}(Wx - Wy).$$

By Lemma 10 in the Online Appendix, we have that there exists a robust opinion aggregator  $S : B \rightarrow B$  such that  $T^{\hat{t}}(x) = \tilde{\varepsilon}^{\hat{t}}Wx + (1 - \tilde{\varepsilon}^{\hat{t}})S(x)$  for all  $x \in B$ . By point 2 of Proposition 24, we can conclude that  $T^{\hat{t}}$  has the common influencer property. By Proposition 25,  $\bar{T}(x) = \lim_t T^t(x) \in D$  for all  $x \in B$  and there exists  $\varepsilon \in (0, 1)$  such that (31) holds. ■

**Proof of Theorem 8.** By Proposition 24 and since no row of  $A(T)$  is null,  $T$  admits a decomposition

$$T(x) = \varepsilon Wx + (1 - \varepsilon)S(x) \quad \forall x \in B$$

where  $S$  is a robust aggregator,  $\varepsilon \in (0, 1)$ ,  $A(W) = A(T)$ , and therefore  $N'$  is the unique closed and strongly connected class under  $W$ , and  $N'$  is aperiodic under  $W$ . By induction, note also that

$$T^t(x) = \varepsilon^t W^t x + (1 - \varepsilon^t)\tilde{S}_t(x) \quad \forall x \in B, \forall t \in \mathbb{N} \quad (78)$$

where  $\tilde{S}_t$  is a normalized and monotone opinion aggregator for all  $t \in \mathbb{N}$ . By [35, Corollaries 8.2 and 8.3], we have that there exists  $\hat{t} \in \mathbb{N}$  and  $\bar{\varepsilon} \in (0, 1)$  such that  $W^{\hat{t}}$  has a column  $k$  whose entries are all greater than or equal to some  $\bar{\varepsilon} \in (0, 1)$ . Since  $N'$  is closed  $k \in N'$ . Define  $U : B \rightarrow B$  to be such that  $U = T^{\hat{t}}$ . Note that  $U$  is a robust opinion aggregator. By (78), it follows that  $U$  satisfies the uniform common influencer property for some  $\hat{\varepsilon} \in (0, 1)$ . By the same proof of Theorem 7, we have that

$$\|\bar{T}(x) - T^t(x)\|_\infty \leq 2(1 - \hat{\varepsilon})^{\lfloor \frac{t}{\hat{t}} \rfloor} \|x\|_\infty \quad \forall t \in \mathbb{N}, \forall x \in B.$$

Moreover, by [33, Theorem 2], and since  $N'$  is closed, strongly connected, and aperiodic under  $W$ , we have that  $w_{ij}^t > 0$  for every  $i, j \in N'$  and  $t \geq |N'|^2 - 3|N'| + 3$ . Let  $j \in N'$ . Since  $N'$  is the unique closed and strongly connected class under  $W$ , there exists a sequence of indexes such that  $w_{i_1 i_2} \cdot \dots \cdot w_{i_{l-1} i_l} > 0$ ,  $i_1 = j$ , and  $i_l \in N'$ . Since there are only  $n - |N'|$  elements in  $N \setminus N'$  the length of the shortest of these paths is at most  $n - |N'|$ . Therefore  $w_{jk}^\tau > 0$  if  $\tau \geq |N'|^2 - 3|N'| + 3 + n - |N'| = |N'|^2 - 4|N'| + 3 + n$ . This proves that  $\hat{t}$  can be chosen equal to  $|N'|^2 - 4|N'| + 3 + n$ .

If  $N'$  is also self influential, since  $W$  is self influential on  $N'$  we have  $w_{ij}^t > 0$  for every  $i, j \in N'$  and  $t \geq |N'| - 1$  and, by an argument analogous to the one above, we can choose  $\hat{t} = |N'| - 1 + n - |N'| = n - 1$ . ■

**Proof of Corollary 6.** The result immediately follows from Theorem 8 since a strongly connected and aperiodic robust opinion aggregator satisfies (i) and (ii), with  $N' = N$ . ■

**Proof of Proposition 13** The result immediately follows from considering  $T$  such that  $T_1(x) = \min_{i \in N} x_i$  and  $T_j(x) = \max_{i \in N} x_i$  for all  $j \neq 1$ . ■

**Proof of Proposition 14** By Proposition 1,  $T$  is constant affine. Let  $x \notin D$ . Since the loss function is convex and differentiable, a necessary condition for optimality of  $T_i(x)$  is given by

$$\sum_{j: x_j > T_i(x)} p_i w_{ij} |x_j - T_i(x)|^{p_i-1} + \sum_{j: x_j < T_i(x)} p_i w_{ij} |x_j - T_i(x)|^{p_i-1} = 0.$$

Since this is a differentiable function of  $x$ , the Implicit function Theorem guarantees that

$$\frac{\partial T_i(x)}{\partial x_l} = \frac{w_{il} |x_l - T_i(x)|^{p_i-2}}{\sum_{j \in N} w_{ij} |x_j - T_i(x)|^{p_i-2}}.$$

By Euler's homogeneous functions theorem,

$$T_i(x) = \sum_{l \in N} \frac{\partial T_i(x)}{\partial x_l} x_l = \frac{w_{il} |x_l - T_i(x)|^{p_i-2}}{\sum_{j \in N} w_{ij} |x_j - T_i(x)|^{p_i-2}} x_l.$$

Let  $i^* \in \arg \max_{i \in N} x_i$  and  $i_* \in \arg \min_{i \in N} x_i$ . We show that

$$|T_{i^*}(x) - T_{i_*}(x)| < x_{i^*} - x_{i_*} \quad (79)$$

and since  $i^*$  and  $i_*$  are chosen arbitrarily in the set of maximizers and minimizers, this shows that  $\max_{i,j} |T_i(x) - T_j(x)| < \max_{i,j} |x_i - x_j|$ . Since  $W$  is scrambling, there exists  $j \in N$  such that  $w_{i^*j} > 0$  and  $w_{i_*j} > 0$ . We have four possible cases:

- $T_{i^*}(x) = T_{i_*}(x) = x_j$ . Then (79) follows immediately.

- $T_{i^*}(x) = x_j \neq T_{i_*}(x)$ . Then

$$\frac{w_{i_*j} |x_j - T_{i_*}(x)|^{p_{i_*}-2}}{\sum_{h \in N} w_{ih} |x_h - T_i(x)|^{p_i-2}} > 0$$

and

$$\begin{aligned} |T_{i^*}(x) - T_{i_*}(x)| &= |x_j - T_{i_*}(x)| = \left| x_j - \frac{w_{il} |x_l - T_i(x)|^{p_i-2}}{\sum_{h \in N} w_{ih} |x_h - T_i(x)|^{p_i-2}} x_l \right| \\ &= \left| \sum_{l \neq j} \frac{w_{il} |x_l - T_i(x)|^{p_i-2}}{\sum_{h \in N} w_{ih} |x_h - T_i(x)|^{p_i-2}} (x_l - x_j) \right| \\ &\leq \left| \sum_{l \neq j} \frac{w_{il} |x_l - T_i(x)|^{p_i-2}}{\sum_{h \in N} w_{ih} |x_h - T_i(x)|^{p_i-2}} \right| (x_{i^*} - x_{i_*}) \\ &= \left( 1 - \frac{w_{i_*l} |x_l - T_{i_*}(x)|^{p_{i_*}-2}}{\sum_{h \in N} w_{i_*h} |x_h - T_{i_*}(x)|^{p_{i_*}-2}} \right) (x_{i^*} - x_{i_*}) \\ &< (x_{i^*} - x_{i_*}). \end{aligned}$$

- A similar proof covers the case  $T_{i^*}(x) \neq x_j = T_{i_*}(x)$ .
- Finally, suppose that we have  $T_{i^*}(x) \neq x_j \neq T_{i_*}(x)$ . Let

$$K = \min \left\{ \frac{w_{i_*j} |x_j - T_{i_*}(x)|^{p_{i_*}-2}}{\sum_{h \in N} w_{ih} |x_h - T_i(x)|^{p_i-2}}, \frac{w_{i^*j} |x_j - T_{i^*}(x)|^{p_{i^*}-2}}{\sum_{h \in N} w_{ih} |x_h - T_i(x)|^{p_i-2}} \right\} > 0.$$

We have

$$|T_{i^*}(x) - T_{i_*}(x)| \leq (1 - K) (x_{i^*} - x_{i_*}) \leq (x_{i^*} - x_{i_*}).$$

Then the result follows from point (iii) of Theorem 8.3.4 in [39]. ■

**Proof of Theorem 9.** Let  $x \in B$ . Call  $V$  the set of values the components of  $x$  take:  $V = \{x_1, \dots, x_n\}$ . Define  $U$  to be the subset of vectors  $y$  in  $B$  such that each component of  $y$  coincides to some component of  $x$ , formally

$$U = \{y \in B : y_i \in V \quad \forall i \in \{1, \dots, n\}\}.$$

Since the components of the vector  $x$  might not be distinct, note that the cardinality of  $U$  is at most  $n^n$ . Since  $\nu_i$  is  $\{0, 1\}$ -valued for all  $i \in \{1, \dots, n\}$ , note that  $T_i(y) \in V$  for all  $y \in U$  and for all  $i \in \{1, \dots, n\}$ . This implies that  $T(x) \in U$ . By induction, it follows that  $T^t(x) \in U$  for all  $t \in \mathbb{N}$ . This implies that the sequence  $\{T^t(x)\}_{t \in \mathbb{N}}$  can take at most a finite number of values. We have two cases:

1.  $\{T^t(x)\}_{t \in \mathbb{N}}$  converges. If  $\{T^t(x)\}_{t \in \mathbb{N}}$  converges, then the previous part implies that  $\{T^t(x)\}_{t \in \mathbb{N}}$  becomes constant, that is, there exists  $\tilde{t} \in \mathbb{N}$  such that

$$T^t(x) = T^{\tilde{t}}(x) \in U \quad \forall t \geq \tilde{t}. \quad (80)$$

Call  $\bar{x}$  the limit of  $\{T^t(x)\}_{t \in \mathbb{N}}$ . Note that  $\bar{x} = T^{\tilde{t}}(x)$  and  $T^t(\bar{x}) = \bar{x}$  for all  $t \in \mathbb{N}$ . In particular, we have that

$$T(\bar{x}) = \bar{x}. \quad (81)$$

Define now  $\bar{t} \in \mathbb{N}$  to be such that  $\bar{t} = \min \{t \in \mathbb{N} : T^t(x) = \bar{x}\}$ . By (80),  $\bar{t}$  is well defined. We next show that  $T^t(x) \neq T^m(x)$  for all  $m, t < \bar{t}$  such that  $m \neq t$ . By contradiction, assume that there exist  $m, t < \bar{t}$  such that  $m \neq t$  and  $T^t(x) = T^m(x)$ . Without loss of generality, we assume that  $m > t$ . This would imply that  $T^{t+n}(x) = T^n(T^t(x)) = T^n(T^m(x)) = T^{m+n}(x)$  for all  $n \in \mathbb{N}$ . In particular, by setting  $n = \bar{t} - m > 0$ , we would have that  $T^{t+n}(x) = T^{m+n}(x) = T^{\bar{t}}(x) = \bar{x}$ . Note that  $\hat{t} = t + n < m + n = \bar{t}$ . Thus, this would imply that

$$T^{\hat{t}}(x) = \bar{x} \text{ and } \hat{t} < \bar{t},$$

a contradiction with the minimality of  $\bar{t}$ . By definition of  $\bar{t}$ , we can also conclude that  $T^t(x) \neq \bar{x}$  for all  $t < \bar{t}$ . This implies that  $\{T^t(x)\}_{t=1}^{\bar{t}-1}$  is contained in  $U \setminus \{\bar{x}\}$ . Since  $U$  contains at most  $n^n$  elements and the elements of  $\{T^t(x)\}_{t=1}^{\bar{t}-1}$  are pairwise distinct, it follows that  $\bar{t} - 1 \leq n^n - 1$ , proving that  $\{T^t(x)\}_{t \in \mathbb{N}}$  converges only if it becomes constant after at most  $n^n$  periods.

2.  $\{T^t(x)\}_{t \in \mathbb{N}}$  does not converge. Define  $\tilde{n} = n^n$ . Recall that  $\{T^t(x)\}_{t=1}^{\tilde{n}+1} \subseteq U$  where the latter set has cardinality at most  $\tilde{n}$ . This implies that there exist  $\hat{m}, \hat{t} \leq \tilde{n} + 1$  such that  $T^{\hat{m}}(x) = T^{\hat{t}}(x)$  and  $\hat{m} \neq \hat{t}$ . Without loss of generality, we assume that  $\hat{m} > \hat{t}$ . It follows that

$$T^{\hat{t}+n}(x) = T^n(T^{\hat{t}}(x)) = T^n(T^{\hat{m}}(x)) = T^{\hat{m}+n}(x) \quad \forall n \in \mathbb{N}_0.$$

Define  $p = \hat{m} - \hat{t} > 0$ . We have that  $T^{\hat{t}+n}(x) = T^{\hat{t}+n+p}(x)$  for all  $n \in \mathbb{N}_0$ , proving that  $T^t(x) = T^{t+p}(x)$  for all  $t \geq \hat{t}$ .

Points 1 and 2 prove the first part of the statement as well as the “only if” of the second part. The “if” part is trivial. ■

**Proof of Proposition 15.** Before starting, define  $B_{\text{inj}} = \{x \in \text{int } B : x_l \neq x_k \text{ if } l \neq k\} \cap \text{int } B$ . By Proposition 26 in the Online Appendix,  $B_{\text{inj}}$  is dense in  $\text{int } B$ , and so, in  $B$ . Since  $T$  is nonexpansive and  $B_{\text{inj}}$  is dense, if  $T_i(x) = x_j$  for all  $x \in B_{\text{inj}}$ , then  $T_i(x) = x_j$  for all  $x \in B$ . Thus,  $T_i(x) \neq x_j$  for some  $x \in B$  if and only if there exists  $y \in B_{\text{inj}}$  such that  $T_i(y) \neq y_j$ .

(i) implies (ii). By assumption, there exists  $\varepsilon_{ij} \in (0, 1)$  such that

$$x \geq y \implies T_i(x) - T_i(y) \geq \varepsilon_{ij}(x_j - y_j).$$

By contradiction, assume that there exists  $y \in B_{\text{inj}}$  such that  $T_i(y) \neq y_j$ . Since  $B_{\text{inj}} \subseteq \text{int } B$ , there exists  $\bar{\varepsilon} > 0$  such that  $z \in \mathbb{R}^n$  and  $\|z - y\|_\infty < \bar{\varepsilon}$  yields that  $z \in \text{int } B$ . Since  $T_i(x) = \mathbb{E}_{\nu_i}(x)$  for all  $x \in B$  and  $\nu_i$  is  $\{0, 1\}$ -valued, it follows that there exists  $h \in N$  such that  $T_i(y) = y_h$ . It follows that  $j \neq h$ . Define  $\delta = \min\{\min_{l, h \in N: l \neq h} |y_l - y_h|, \bar{\varepsilon}\} / 2$ . Since  $y \in B_{\text{inj}}$  and  $\bar{\varepsilon} > 0$ , we have that  $\delta > 0$  and  $\delta < \bar{\varepsilon}$ . Define  $x$  to be such that  $x_h = y_h$  if  $h \neq j$  and  $x_j = y_j + \delta$ . It is immediate to see that  $x \geq y$  and  $x \in B$ . At the same time,  $T_i(x) = \int x d\nu_i = x_h = y_h$ , yielding that

$$0 = x_h - y_h = T_i(x) - T_i(y) \geq \varepsilon_{ij}(x_j - y_j) \geq \varepsilon_{ij}\delta > 0,$$

a contradiction.

(ii) implies (i) and the particular cases are trivial. ■

**Proof of Proposition 16.** (If) Let  $\varepsilon_{ij} = \min_{A \in 2^N} f_i(\sum_{l \in A \setminus \{j\}} w_{il} + w_{ij}) - f_i(\sum_{l \in A} w_{il})$ . Since  $f_i$  is strictly increasing,  $w_{ij} > 0$ , and  $N$  is finite we have  $\varepsilon_{ij} > 0$ . Given [25, Theorem 14 and Example 17]  $p_i \in \partial T_i(0) \implies p_{ij} \geq \varepsilon_{ij}$ . Now, let  $x, y \in B$  with  $x \geq y$ . By Lebourg's Mean Value Theorem, we have that there exist  $\lambda \in (0, 1)$  and  $p_i \in \partial T_i(\lambda x + (1 - \lambda)y)$  such that

$$T_i(x) - T_i(y) = p_i \cdot (x - y)$$

By [25, Theorem 14 and Example 17],  $\partial T_i(\lambda x + (1 - \lambda)y) \subseteq \partial T_i(0) \subseteq \Delta(N)$ , and therefore

$$T_i(x) - T_i(y) = p_i \cdot (x - y) \geq \varepsilon_{ij}(x_j - y_j).$$

(Only If) Suppose that  $i$  strongly influences  $j$ , and let  $x \in \text{int } B \cap B_{\text{inj}}$ . Then  $x \in B_\pi$  and there exists  $\delta > 0$  such that  $x + \delta e_j \in B_\pi$ . Then by (40)

$$f_i\left(\sum_{l=1}^{\pi^{-1}(j)} w_{il}\right) - f_i\left(\sum_{l=1}^{\pi^{-1}(j)-1} w_{il}\right) = T_i(x + \delta e_j) - T_i(x) \geq \varepsilon_{ij}\delta > 0$$

and this implies  $w_{ij} > 0$ . ■

**Proof of Proposition 17** Let  $x \in B$ . It follows that  $x \in B_\pi$  for some  $\pi \in \Pi$ . Consider  $W_\pi$  as in (41). First, we prove that if  $T$  is assortative, then  $i \geq j$  implies  $T^t(x)_{\pi(i)} \geq T^t(x)_{\pi(j)}$  for all  $t \in \mathbb{N}_0$ , that is,  $T^t(x) \in B_\pi$ .<sup>43</sup> The statement is clearly true for  $t = 0$ . Suppose it holds for  $t$ . By (41), we have that

$$T(y) = W_{\pi'} y \quad \forall y \in B_{\pi'}, \forall \pi' \in \Pi.$$

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<sup>43</sup>As usual,  $T^0$  is the identity operator.

Let  $W$  be such that  $T(x) = Wx$ . Let  $k \in \mathbb{N}$ . Since  $x$  and  $x^k$  are comonotonic, by [43] we have that

$$T(x^k) = Wx^k.$$

Therefore assortativeness implies that for all  $k \in \mathbb{N}$

$$w_i \cdot x^k - w_i \cdot x \geq w_j \cdot x^k - w_j \cdot x.$$

In turn, this implies that

$$\sum_{l \leq k} w_{il} \geq \sum_{l \leq k} w_{jl}. \quad (82)$$

But then, since by the inductive assumption  $T^{t-1}(x)$  and  $x$  are comonotonic, by [43]

$$T(T^{t-1}(x)) = WT^{t-1}(x). \quad (83)$$

This, together with (82) implies that  $T^t(x)_i \geq T^t(x)_j$ , proving the preliminary claim. But notice that (83) implies that

$$T^t(x) = W^t \cdot x$$

proving the statement. ■

**Proof of Proposition 18.** We first prove point 2, then point 1.

2. Given  $x^0 \in B$ , the sequence  $\{x^t\}_{t \in \mathbb{N}}$ , defined as in (42), is a specification of the Mann's iterates of  $T$ , using as weights  $\{\lambda_t\}_{t \in \mathbb{N}}$ . Consider  $\tilde{B}$  as in point 4 of Lemma 9 in the Online Appendix. Note that  $T(\tilde{B}) \subseteq \tilde{B}$  and  $x^t \in \tilde{B}$  for all  $t \in \mathbb{N}_0$ . By Ishikawa [34, Theorem 1], we have that  $\{x^t\}_{t \in \mathbb{N}}$  converges and its limit is a fixed point of  $T$ .

1. By Lemma 10 in the Online Appendix and since  $T$  is self-influential, there exists  $\varepsilon \in (0, 1)$  such that  $T = \varepsilon I + (1 - \varepsilon)S$  where  $S$  is a robust opinion aggregator. Note that given  $\lambda \in (0, 1]$  we have that

$$T_\lambda \stackrel{def}{=} (1 - \lambda)I + \lambda T = (1 - \lambda)I + \lambda \varepsilon I + (1 - \varepsilon)\lambda S = (1 - \gamma)I + \gamma S \stackrel{def}{=} S_\gamma$$

where  $\gamma = (1 - \varepsilon)\lambda \leq (1 - \varepsilon)$ . Given  $x^0 \in B$ , the sequence  $\{x^t\}_{t \in \mathbb{N}}$ , defined as in (42), can be rewritten as

$$x^t = S_t(x^{t-1}) \quad \text{and} \quad S_t = (1 - \gamma_t)I + \gamma_t S \quad \forall t \in \mathbb{N}$$

where  $\gamma_t = (1 - \varepsilon)\lambda_t$  for all  $t \in \mathbb{N}$ . We have that  $0 < \gamma_t \leq (1 - \varepsilon) < 1$  for all  $t \in \mathbb{N}$  as well as  $\sum_{t=1}^{\infty} \gamma_t = \infty$ . By point 2, we can conclude that  $\{x^t\}_{t \in \mathbb{N}}$  converges to a fixed point of  $S$ . It is immediate to see that the fixed points of  $T$  and  $S$  coincide, proving the statement. ■

**Remark 6** On the one hand, the updating process that we study at the very beginning of the paper generates a sequence of updates  $\{T^t(x^0)\}_{t \in \mathbb{N}}$ . If we define  $x^t = T^t(x^0)$  for all  $t \in \mathbb{N}$ , then the sequence of updates is such that

$$x^t = T(x^{t-1}) \quad \forall t \in \mathbb{N}$$

which are also known in the mathematical literature as Picard's iterates. On the other hand, the sequence of updates generated by the updating procedure of DeMarzo, Vayanos, Zwiebel [17] are known as (a version of) Mann's iterates (see Mann [42] and Ishikawa [34]). Both types of iterates are widespread in the literature of fixed points approximation which deals with the study of recursive procedures that yield existence and convergence to fixed points of selfmaps (see, e.g., Berinde [6]).  $\blacktriangle$

**Proof of Proposition 19.** Fix  $x^0 \in B$ . Consider  $\tilde{B}$  as in point 4 of Lemma 9 in the Online Appendix. Observe that  $T(\tilde{B}) \subseteq \tilde{B}$ . Define also  $R : \tilde{B} \rightarrow \tilde{B}$  to be such that  $R(x) = \alpha T(x) + (1 - \alpha)x^0$  for all  $x \in \tilde{B}$ . Since  $\alpha \in (0, 1)$ ,  $R$  is well defined. By Lemma 10 in the Online Appendix and since  $T$  is self-influential, there exists  $\varepsilon \in (0, 1)$  such that  $T = \varepsilon I + (1 - \varepsilon)S$  where  $S$  is a robust opinion aggregator. It follows that for each  $x \in \tilde{B}$

$$\begin{aligned} R(x) &= \alpha(\varepsilon I(x) + (1 - \varepsilon)S(x)) + (1 - \alpha)x^0 \\ &= \alpha\varepsilon I(x) + (1 - \varepsilon)\alpha S(x) + (1 - \alpha)x^0 \\ &= \delta I(x) + (1 - \delta)P(x). \end{aligned}$$

where  $\delta = \alpha\varepsilon \in (0, 1)$  and  $P(x) = \frac{(1-\varepsilon)\alpha}{1-\alpha\varepsilon}S(x) + \frac{(1-\alpha)}{1-\alpha\varepsilon}x^0$  for all  $x \in \tilde{B}$ . Since  $S(x), x^0 \in \tilde{B}$ , note that  $P : \tilde{B} \rightarrow \tilde{B}$ . Since  $S$  is nonexpansive, observe that for each  $x, y \in \tilde{B}$

$$\|P(x) - P(y)\|_\infty = \frac{(1 - \varepsilon)\alpha}{1 - \alpha\varepsilon} \|S(x) - S(y)\|_\infty \leq \|S(x) - S(y)\|_\infty \leq \|x - y\|_\infty.$$

By definition of  $R$ , we have that  $R(x^0) = x^1$ . If  $R^t(x^0) = x^t \in \tilde{B}$  when  $t \in \mathbb{N}$ , note also that  $R^{t+1}(x^0) = R(R^t(x^0)) = R(x^t) = x^{t+1}$  and  $x^{t+1} = R(x^t) \in \tilde{B}$ . By induction, it follows that  $\{R^t(x^0)\}_{t \in \mathbb{N}} = \{x^t\}_{t \in \mathbb{N}} \subseteq \tilde{B}$ . By [26, Corollary 9.1] and since  $R = \delta I + (1 - \delta)P$  where  $\delta \in (0, 1)$  and  $P$  is nonexpansive and  $\tilde{B}$  is compact, we have that  $\bar{x} = \lim_t x^t = \lim_t R^t(x^0)$  exists and  $\bar{x}$  is a fixed point of  $P$ , that is,

$$\bar{x} = \delta\bar{x} + (1 - \delta)\bar{x} = \delta\bar{x} + (1 - \delta)P(\bar{x}) = R(\bar{x}) = \alpha T(\bar{x}) + (1 - \alpha)x^0,$$

proving the statement.  $\blacksquare$

**Proof of Proposition 20.** 1. By assumption, there exists  $\varepsilon \in (0, 1)$  such that for each  $i \in N$  and for each  $x, y \in B$

$$x \geq y \implies T_i(x) - T_i(y) \geq \varepsilon(x_i - y_i). \quad (84)$$

Consider  $r \in \text{int } I$ . Denote by  $e^i$  the vector whose  $i$ -th component is 1, and all the others are 0. Since  $re \in \text{int } B$ , we can choose  $\gamma > 0$  such that  $re + \gamma e^i \in B$ . Define  $x = re + \gamma e^i$  and  $y = re + \tilde{\gamma}e$  where  $\tilde{\gamma} = \varepsilon\gamma > 0$ . Since  $re + \gamma e^i \in B$  and  $\varepsilon \in (0, 1)$ , note that  $r \leq r + \tilde{\gamma} \leq r + \gamma \in I$ , yielding that  $r + \tilde{\gamma} \in I$  and  $y \in B$ . By (84) and since  $T$  is robust, it follows that for each  $\lambda \in (0, 1]$  and  $z \in B$

$$\begin{aligned}
& T_i(\lambda x + (1 - \lambda)z) - T_i(\lambda y + (1 - \lambda)z) \\
&= T_i(\lambda x + (1 - \lambda)z) - T_i(\lambda(r + \tilde{\gamma})e + (1 - \lambda)z) \\
&= T_i(\lambda x + (1 - \lambda)z) - T_i(\lambda re + (1 - \lambda)z) - \lambda\tilde{\gamma} \\
&\geq \varepsilon(\lambda x_i + (1 - \lambda)z_i - \lambda r - (1 - \lambda)z_i) - \lambda\tilde{\gamma} \\
&= \varepsilon\lambda\gamma - \lambda\tilde{\gamma} = 0.
\end{aligned}$$

By definition of  $\succsim_i^*$ , it follows that  $x \succsim_i^* y$ . By (46), we have that

$$p \cdot (re + \gamma e^i) \geq p \cdot (re + \tilde{\gamma}e) \quad \forall p \in C_i,$$

that is,

$$\gamma p_i \geq \tilde{\gamma} \quad \forall p \in C_i \implies p_i \geq \varepsilon \quad \forall p \in C_i,$$

and the implication follows. Since  $i$  was arbitrarily chosen, the statement follows. It is immediate to conclude that there exists  $\varepsilon \in (0, 1)$  such that  $w_{ii} \geq \varepsilon$  for all  $i \in N$  and for all  $W$  in  $\mathcal{W}(T)$ .

2. By assumption, there exist  $k \in N$  and  $\varepsilon \in (0, 1)$  such that for each  $i \in N$  and for each  $x, y \in B$

$$x \geq y \implies T_i(x) - T_i(y) \geq \varepsilon(x_k - y_k). \quad (85)$$

Consider  $r \in \text{int } I$ . Denote by  $e^k$  the vector whose  $k$ -th component is 1 and all the others are 0. Since  $re \in \text{int } B$ , we can choose  $\gamma > 0$  such that  $re + \gamma e^k \in B$ . Define  $x = re + \gamma e^k$  and  $y = re + \tilde{\gamma}e$  where  $\tilde{\gamma} = \varepsilon\gamma > 0$ . Since  $re + \gamma e^k \in B$  and  $\varepsilon \in (0, 1)$ , note that  $r \leq r + \tilde{\gamma} \leq r + \gamma \in I$ , yielding that  $r + \tilde{\gamma} \in I$  and  $y \in B$ . By (85) and since  $T$  is robust, it follows that for each  $i \in N$ ,  $\lambda \in (0, 1]$ , and  $z \in B$

$$\begin{aligned}
& T_i(\lambda x + (1 - \lambda)z) - T_i(\lambda y + (1 - \lambda)z) \\
&= T_i(\lambda x + (1 - \lambda)z) - T_i(\lambda(r + \tilde{\gamma})e + (1 - \lambda)z) \\
&= T_i(\lambda x + (1 - \lambda)z) - T_i(\lambda re + (1 - \lambda)z) - \lambda\tilde{\gamma} \\
&\geq \varepsilon(\lambda x_k + (1 - \lambda)z_k - \lambda r - (1 - \lambda)z_k) - \lambda\tilde{\gamma} \\
&= \varepsilon\lambda\gamma - \lambda\tilde{\gamma} = 0.
\end{aligned}$$

By definition of  $\succsim_i^*$ , it follows that  $x \succsim_i^* y$  for all  $i \in N$ . By (46), we have that

$$p \cdot (re + \gamma e^k) \geq p \cdot (re + \tilde{\gamma}e) \quad \forall p \in C_i, \forall i \in N,$$

that is,

$$\gamma p_k \geq \tilde{\gamma} \quad \forall p \in C_i, \forall i \in N \implies p_k \geq \varepsilon \quad \forall p \in C_i, \forall i \in N,$$

and the implication follows. It is immediate to conclude that  $\mathcal{W}(T) \subseteq \mathcal{W}_{\varepsilon, k}$  for some  $k \in N$  and for some  $\varepsilon > 0$ .  $\blacksquare$

**Proof of Theorem 10.** Before starting, recall that for each  $y \in B$  there exists  $W(y) \in \mathcal{W}(T)$  such that

$$T(y) = W(y)y \quad (86)$$

Given  $x \in B$ , for ease of notation, define  $W_t = W(T^{t-1}(x))$  for all  $t \in \mathbb{N}$ .<sup>44</sup> We have that for each  $m \in \mathbb{N}$

$$T^m(x) = W(T^{m-1}(x))T^{m-1}(x) = (\Pi_{t=1}^m W_t)x. \quad (87)$$

1. By assumption  $\{W_t\}_{t \in \mathbb{N}} \subseteq \mathcal{W}_\varepsilon$  for some  $\varepsilon \in (0, 1)$ . By the claim in the proof of Proposition 25, we have that

$$\|T^m(x) - T^t(x)\|_\infty = \|(\Pi_{l=1}^m W_l)x - (\Pi_{l=1}^t W_l)x\|_\infty \leq 2(1 - \varepsilon)^t \|x\|_\infty \quad \forall m \geq t. \quad (88)$$

This implies that  $\{T^t(x)\}_{t \in \mathbb{N}}$  is a Cauchy sequence in  $B$ . Since  $B$  is closed and  $x$  was arbitrarily chosen, it follows that  $\lim_t T^t(x)$  exists and belongs to  $B$  for all  $x \in B$ . This implies that  $\bar{T}$ , defined as in (48), is well defined. By taking the limit in  $m$  in (88), (49) immediately follows. We are left to show that  $\bar{T}(x) \in \lim_t T^t(x) \in D$  for all  $x \in B$ . We argue by contradiction. Fix  $x \in B$  and assume that  $\bar{T}(x) \notin D$ . By Lemma 2, we know that  $\bar{T}(x)$  is a fixed point of  $T$ . For ease of notation, we denote  $\bar{x} = \bar{T}(x)$ . Since  $\bar{x} \notin D$ , it follows that there exists  $i, j \in N$  such that  $\bar{x}_i < \bar{x}_j$ . This implies that  $\min_{l \in N} \bar{x}_l \leq \bar{x}_i < \bar{x}_j = \max_{l \in N} \bar{x}_l$ . Without loss of generality assume that  $\min_{l \in N} \bar{x}_l = \bar{x}_i$  and  $\bar{x}_j = \max_{l \in N} \bar{x}_l$ . Consider  $W(\bar{x})$  as in (86). By assumption,  $W(\bar{x}) \in \mathcal{W}_\varepsilon$  for some  $\varepsilon \in (0, 1)$ . Thus, there exists a column  $k$  whose entries are greater than or equal to  $\varepsilon \in (0, 1)$ . Denote the entries of  $W(\bar{x})$  by  $\bar{w}_{ml}$  with  $l, m \in \{1, \dots, n\}$ . We have two cases either  $\bar{x}_k < \bar{x}_j$  or  $\bar{x}_k > \bar{x}_i$ . Since  $\bar{x}$  is a fixed point of  $T$ , in the first case, we have that

$$\bar{x}_j = T_j(\bar{x}) = \sum_{l=1}^n \bar{w}_{jl} \bar{x}_l = \bar{w}_{jk} \bar{x}_k + \sum_{l \neq k} \bar{w}_{jl} \bar{x}_l < \bar{w}_{jk} \bar{x}_j + \sum_{l \neq k} \bar{w}_{jl} \bar{x}_j = \bar{x}_j,$$

yielding a contradiction. In the second case, we have that

$$\bar{x}_i = T_i(\bar{x}) = \sum_{l=1}^n \bar{w}_{il} \bar{x}_l = \bar{w}_{ik} \bar{x}_k + \sum_{l \neq k} \bar{w}_{il} \bar{x}_l > \bar{w}_{ik} \bar{x}_i + \sum_{l \neq k} \bar{w}_{il} \bar{x}_i = \bar{x}_i,$$

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<sup>44</sup>As usual,  $T^0(x) = x$  for all  $x \in B$ .

yielding a contradiction, proving the statement.

2. By (87),  $\{T^m(x)\}_{m \in \mathbb{N}} = \{(\Pi_{t=1}^m W_t)x\}_{m \in \mathbb{N}}$  for all  $x \in B$ . By [13], we have that  $\{(\Pi_{t=1}^m W_t)\}_{m \in \mathbb{N}} \subseteq \mathcal{W}$  componentwise converges to a matrix  $W \in \mathcal{W}$ . It follows that  $\lim_m T^m(x) = \lim_m (\Pi_{t=1}^m W_t)x$  is well defined, proving that  $T$  is convergent. ■

## C Appendix: Location experiments and other foundations

**Proof of Proposition 3.** For each  $i \in N$ , the MLE  $T_i$  is a selection from the solution correspondence

$$x \mapsto \mathbf{T}_i(x) = \arg \min_{c \in \mathbb{R}} -f_{i,X,c}(x)$$

where  $f_{i,X,c}$  denotes the pdf of  $X$  given that the location parameter is  $c$  and the error terms have the distribution  $f_i$ , that is,

$$f_{i,X,c}(x) = f_i(x - ce).$$

Therefore,

$$\mathbf{T}_i(x) = \arg \min_{c \in \mathbb{R}} -f_i(x - ce) = \arg \min_{c \in \mathbb{R}} - \prod_{j \in N} f_{ij}(x_j - c).$$

Since the set of minimizers is not affected by a strictly increasing transformation of the objective function, we have

$$\mathbf{T}_i(x) = \arg \min_{c \in \mathbb{R}} - \sum_{j \in N} \log f_{ij}(x_j - c).$$

Then, the result follows by Corollary 2. ■

**Proof of Proposition 4.** Notice that the posterior expectation is the (unique) solution of the problem

$$\begin{aligned} \arg \min_{c \in \mathbb{R}} \int_{-\infty}^{\infty} (c - \mu)^2 d\Lambda(\mu | (x_j)_{j \in N_i}) &= \arg \min_{c \in \mathbb{R}} \int_{-\infty}^{\infty} (c - \mu)^2 \frac{\prod_{j \in N_i} f_{ij}(x_j - \mu)}{\int_{-\infty}^{\infty} \prod_{j \in N_i} f_{ij}(x_j - \mu') d\mu'} d\mu \\ &= \arg \min_{c \in \mathbb{R}} \int_{-\infty}^{\infty} (c - \mu)^2 \prod_{j \in N_i} f_{ij}(x_j - \mu) d\mu. \end{aligned} \tag{89}$$

We start by proving normalization. Suppose that the realization is such that  $x = ke$  for some  $k \in \mathbb{R}$ , and let  $\bar{c} < k$ . We will prove that if  $\bar{c} + \varepsilon < k$  then

$$\int_{-\infty}^{\infty} (\bar{c} + \varepsilon - \mu)^2 \prod_{j \in N_i} f_{ij}(k - \mu) d\mu < \int_{-\infty}^{\infty} (\bar{c} - \mu)^2 \prod_{j \in N_i} f_{ij}(k - \mu) d\mu.$$

By setting  $\delta = k - \mu$ , this is equivalent to

$$\begin{aligned}
& \int_{-\infty}^{\infty} ((\bar{c} + \varepsilon - \mu)^2 - (\bar{c} - \mu)^2) \prod_{j \in N_i} f_{ij}(k - \mu) d\mu < 0 \\
& \Leftrightarrow - \int_{-\infty}^{\infty} ((\bar{c} + \varepsilon - k + \delta)^2 - (\bar{c} - k + \delta)^2) \prod_{j \in N_i} f_{ij}(\delta) d\delta < 0 \\
& \Leftrightarrow - \int_{-\infty}^0 ((\bar{c} + \varepsilon - k + \delta)^2 - (\bar{c} - k + \delta)^2) \prod_{j \in N_i} f_{ij}(\delta) d\delta \\
& \quad - \int_0^{\infty} ((\bar{c} + \varepsilon - k + \delta)^2 - (\bar{c} - k + \delta)^2) \prod_{j \in N_i} f_{ij}(\delta) d\delta < 0 \\
& \Leftrightarrow - \int_{-\infty}^0 ((\bar{c} + \varepsilon - k + \delta)^2 - (\bar{c} - k + \delta)^2) \prod_{j \in N_i} f_{ij}(\delta) d\delta \\
& \quad + \int_{-\infty}^0 ((\bar{c} + \varepsilon - k - \delta)^2 - (\bar{c} - k - \delta)^2) \prod_{j \in N_i} f_{ij}(-\delta) d\delta < 0 \\
& \Leftrightarrow - \int_{-\infty}^0 ((\bar{c} + \varepsilon - k + \delta)^2 - (\bar{c} - k + \delta)^2) \prod_{j \in N_i} f_{ij}(\delta) d\delta \\
& \quad + \int_{-\infty}^0 ((\bar{c} + \varepsilon - k - \delta)^2 - (\bar{c} - k - \delta)^2) \prod_{j \in N_i} f_{ij}(\delta) d\delta < 0 \\
& \Leftrightarrow \int_{-\infty}^0 ((\bar{c} + \varepsilon - k - \delta)^2 - (\bar{c} - k - \delta)^2) - ((\bar{c} + \varepsilon - k + \delta)^2 - (\bar{c} - k + \delta)^2) \prod_{j \in N_i} f_{ij}(\delta) d\delta < 0
\end{aligned}$$

where the fourth equivalence is because of the symmetry of  $f_{ij}$ . However, then notice that strict convexity of the square guarantees that the last inequality holds. A similar result proves that also  $\bar{c} > k$  cannot be a solution of the problem. Monotonicity follows from [45, Proposition 1]. Finally, for translation invariance, notice that if

$$\int_{-\infty}^{\infty} (c - \mu)^2 \prod_{j \in N_i} f_{ij}(x_j - \mu) d\mu - \int_{-\infty}^{\infty} (c' - \mu)^2 \prod_{j \in N_i} f_{ij}(x_j - \mu) d\mu \geq 0$$

then by letting  $\mu' = \mu - k$

$$\begin{aligned}
& \int_{-\infty}^{\infty} (c + k - \mu)^2 \prod_{j \in N_i} f_{ij}(x_j + k - \mu) d\mu - \int_{-\infty}^{\infty} (c' + k - \mu)^2 \prod_{j \in N_i} f_{ij}(x_j + k - \mu) d\mu \\
& = \int_{-\infty}^{\infty} (c - \mu')^2 \prod_{j \in N_i} f_{ij}(x_j - \mu') d\mu' - \int_{-\infty}^{\infty} (c' - \mu')^2 \prod_{j \in N_i} f_{ij}(x_j - \mu') d\mu' \geq 0,
\end{aligned}$$

proving that the solution of the problem is translation invariant. ■

**Proof of Proposition 7.** By Theorem 1, we know that

$$\mathbf{T}_i(x) = \arg \min_{c \in \mathbb{R}} \phi_i(x - ce) \quad \forall x \in B$$

admits a robust selection  $T$ . Fix that selection as well as  $x \in B$  and  $h \in \mathbb{R}_+$  such that  $x + he_j \in B$ . By Lemma 6, for all  $z \in \mathbb{R}$ ,  $\phi_i(z - ce)$  is a convex function of  $c$ . Therefore,  $T_i$  satisfies the first order conditions (see, e.g., [50, Proposition K.4.8]):

$$\phi'_i(x - T_i(x)e; e) = 0,$$

and

$$\phi'_i(x + he_j - T_i(x + he_j)e; e) = 0.$$

Since  $\phi'_i$  is locally Lipschitz,  $\phi'_i$  is Lipschitz on every compact set. Let  $L$  be the Lipschitz constant of  $\phi'_i(\cdot; e)$  on  $[\min I - \max I, \max I - \min I]^n$ . Note that

$$\begin{aligned} \varepsilon_{ij}h &\leq \phi'_i(x - T_i(x)e + he_j; e) - \phi'_i(x - T_i(x)e; e) \\ &= \phi'_i(x - T_i(x)e + he_j; e) - \phi'_i(x + he_j - T_i(x + he_j)e; e) \\ &\leq L \|(x - T_i(x)e + he_j) - (x + he_j - T_i(x + he_j)e + he_j)\|_\infty \\ &= L \|(T_i(x + he_j) - T_i(x))e\|_\infty \\ &= L (T_i(x + he_j) - T_i(x)) \|e\|_\infty \end{aligned}$$

that is

$$T_i(x + he_j) - T_i(x) \geq \frac{\varepsilon_{ij}}{L}h.$$

■

**Proof of Corollary 4.** Since  $\rho_i$  is strongly convex,  $\rho''_i(z) > 0$  for all  $z \in [\min I - \max I, \max I - \min I]$ . Being strongly convex,  $\rho_i$  is also strictly convex, and by Proposition 2 the solution correspondence is single valued. Denote the solution function as  $T_i$ .

But then, notice that since  $\rho$  is twice continuously differentiable  $K = \min_{z \in [\min I - \max I, \max I - \min I]} \rho''_i(z) > 0$

$$\begin{aligned} &\phi'_i(x - T_i(x)e + he_j; e) - \phi'_i(x - T_i(x)e; e) \\ &= w_{ij}\rho'_i(x_j - T_i(x) + h) - w_{ij}\rho'_i(x_j - T_i(x)) \\ &\geq Kh > 0. \end{aligned}$$

■

**Proof of Proposition 8.** Fix  $x \in B$ . The problem of agent  $i$  is equivalent to

$$\arg \min_{c \in \mathbb{R}} \sum_{j \in N} a_{ij}\rho_i(x_j - c)$$

where  $a_{ij} = 1$  if  $w_{ij} > 0$  and  $a_{ij} = 0$  otherwise. Since  $\rho_i$  is strictly convex, a necessary and sufficient condition for  $T_i(x)$  to solve the problem is the first order condition

$$\sum_{j \in N} a_{ij}\rho'_i(x_j - T_i(x)) = 0.$$

Define

$$f_k(z, y) = \sum_{j \neq k} a_{ij} \rho'_i(x_j - y) + a_{ik} \rho'_i(z - y).$$

Therefore, the necessary and sufficient condition implies that for all  $k \in N$

$$f_k(x_k, T(x)) = 0.$$

Let  $k \in N$ . By the Implicit Function Theorem

$$\frac{\partial T_i(x)}{\partial x_k} = \frac{a_{ik} \rho''_i(x_k - T(x))}{\sum_{j \in N} a_{ij} \rho''_i(x_j - T(x))}.$$

Since  $x_k$  and  $T(x)$  lie in  $I = [a, b]$ , we have that  $|x_k - T| \in [0, (b - a)]$ . Denote as

$$\underline{\rho}_i'' = \min_{z \in [0, (b-a)]} \rho''_i(z) \text{ and } \bar{\rho}_i'' = \rho''_i(z).$$

Since  $\rho_i \in C^2$  and  $\rho''_i > 0$  the maximum and minimum are attained and  $0 < \underline{\rho}_i'' \leq \bar{\rho}_i''$ .

Then

$$\frac{\partial T_i(x)}{\partial x_k} = \frac{a_{ik} \rho''_i(x_k - T(x))}{\sum_{j \in N} a_{ij} \rho''_i(x_j - T(x))} \geq \frac{a_{ik} \underline{\rho}_i''}{|\{j : a_{ij} = 1\}| \bar{\rho}_i''}.$$

Therefore, let  $a_{ij} = 1$ , and  $x \geq y$  Let  $\varepsilon_{ij} = \frac{\underline{\rho}_i''}{|\{l: a_{il}=1\}| \bar{\rho}_i''}$ . Then if we define  $z$  as

$$z_k = \begin{cases} x_k & k = j \\ y_k & \text{otherwise} \end{cases}$$

monotonicity implies that  $T_i(x) \geq T_i(z)$ . Moreover, since  $a_{ij} = 1$ ,

$$\frac{\partial T_i(x)}{\partial x_k} \geq \frac{\underline{\rho}_i''}{|\{j : a_{ij} = 1\}| \bar{\rho}_i''}$$

and therefore  $T_i(x) - T_i(y) \geq T_i(z) - T_i(y) \geq (x_k - y_k) \min_{x' \in I} \frac{\partial T_i(x')}{\partial x'_k} = (x_k - y_k) \varepsilon_{ij}$ .

But then

$$\frac{\partial T_i(x)}{\partial x_k} = \frac{\rho''_i(x_k - T(x))}{\sum_{j \in N_i \setminus \{k\}} \rho''_i(x_j - T(x))} \leq \frac{\bar{c}}{|N_i|} \leq \bar{c} \frac{n}{\min |N_k|} \frac{e}{n}.$$

■

## D Appendix: Vox populi, vox dei

**Proof of Proposition 21.** We proceed by proving two intermediate steps. First, we prove that  $\bar{T}_i(n)$  is an unbiased estimator for all  $n \in \mathbb{N}$  and for all  $i \in N$ . Second, we show that (51) yields that  $\bar{T}_i(n)$  is not extremely sensitive to changes coming from a single observation. Finally, by applying McDiarmid's inequality, we obtain (52). Before starting, we make a few observations. By Corollary 5 and by points 4 and 5 of Lemma

8, we have that  $\bar{T}(n)$  is a well defined odd robust opinion aggregator for all  $n \in \mathbb{N}$ . Since  $\bar{T}$  is a consensus operator, it follows that  $\bar{T}_i(n) = \bar{T}_j(n)$  for all  $i, j \in N$  and  $n \in \mathbb{N}$ . Since  $\{X_j\}_{j \in \mathbb{N}}$  are uniformly bounded and measurable and  $\bar{T}_i(n)$  is continuous for all  $i \in N$  and  $n \in \mathbb{N}$ , it follows that  $\bar{T}_i(X_1, \dots, X_n)$  is integrable for all  $i \in N$  and for all  $n \in \mathbb{N}$ .

*Step 1. For each  $i \in N$  and  $n \in \mathbb{N}$*

$$\mathbb{E}(\bar{T}_i(n)(X_1, \dots, X_n)) = \int_{\Omega} \bar{T}_i(n)(X_1(\omega), \dots, X_n(\omega)) dP = \mu. \quad (90)$$

*Proof of the Step.* For each  $j \in \mathbb{N}$  define  $\varepsilon_j = X_j - \mu$ . Since  $X_j$  is symmetric around  $\mu$  for all  $j \in \mathbb{N}$ , we have that  $\{\varepsilon_j\}_{j \in \mathbb{N}}$  is a collection of uniformly bounded and symmetric around 0 random variables. By [1, Theorem 13.46] and since  $\bar{T}(n)$  is odd for all  $n \in \mathbb{N}$ , this implies that for each  $n \in \mathbb{N}$

$$\begin{aligned} \int_{\Omega} \bar{T}_i(n)(\varepsilon_1(\omega), \dots, \varepsilon_n(\omega)) dP &= \int_{\Omega} \bar{T}_i(n)(-\varepsilon_1(\omega), \dots, -\varepsilon_n(\omega)) dP \\ &= - \int_{\Omega} \bar{T}_i(n)(\varepsilon_1(\omega), \dots, \varepsilon_n(\omega)) dP. \end{aligned}$$

It follows that for each  $n \in \mathbb{N}$

$$2 \int_{\Omega} \bar{T}_i(n)(\varepsilon_1(\omega), \dots, \varepsilon_n(\omega)) dP = 0.$$

Since  $\bar{T}$  is translation invariant, we can conclude that for each  $n \in \mathbb{N}$

$$\begin{aligned} \mathbb{E}(\bar{T}_i(n)(X_1, \dots, X_n)) &= \int_{\Omega} \bar{T}_i(n)(X_1(\omega), \dots, X_n(\omega)) dP \\ &= \int_{\Omega} \bar{T}_i(n)(\mu + \varepsilon_1(\omega), \dots, \mu + \varepsilon_n(\omega)) dP \\ &= \mu + \int_{\Omega} \bar{T}_i(n)(\varepsilon_1(\omega), \dots, \varepsilon_n(\omega)) dP = \mu \quad \forall n \in \mathbb{N}, \end{aligned}$$

proving (90). □

*Step 2. For each  $j \in N$  and for each  $n \in \mathbb{N}$*

$$\sup_{\{(x,t) \in \hat{B} \times \mathbb{R} : x + te^i \in \hat{B}\}} |\bar{T}_j(n)(x) - \bar{T}_j(n)(x + te^i)| \leq lc(n) w_i(n) \quad \forall i \in N.$$

*Proof of the Step.* Consider  $x \in \hat{B}$  and  $t \in \mathbb{R}$  such that  $x + te^i \in \hat{B}$ . Define  $y = x + te^i$ . Since  $\bar{T}_j$  is nonexpansive, it is Clarke's differentiable. Since  $\bar{T}_j$  is monotone and translation invariant, we have that  $\partial \bar{T}_j(n)(x) \subseteq \Delta_n$  for all  $n \in \mathbb{N}$ . By Lebourg's Mean Value Theorem, we have that there exist  $\lambda \in (0, 1)$  and  $p \in \partial \bar{T}_j(n)(x_\lambda)$  where  $x_\lambda = \lambda x + (1 - \lambda)y \in \hat{I}^n$  such that

$$\bar{T}_j(n)(x) - \bar{T}_j(n)(y) = p \cdot (x - y).$$

It follows that  $|\bar{T}_j(n)(x) - \bar{T}_j(n)(y)| = |p \cdot (x - y)| = |p_i(x_i - y_i)| \leq p_i |x_i - y_i| \leq \ell p_i$ . Since  $p \in \partial \bar{T}_j(n)(x_\lambda)$  and  $\bar{T}$  is a consensus operator, we have that

$$p_i \leq \limsup_{\substack{z \rightarrow x_\lambda \\ t \downarrow 0}} \frac{\bar{T}_j(z + te^i) - \bar{T}_j(z)}{t} \leq s \left( T(n), \hat{I}^n \right)_i,$$

proving that

$$\begin{aligned} |\bar{T}_j(n)(x) - \bar{T}_j(n)(x + te^i)| &= |\bar{T}_j(n)(x) - \bar{T}_j(n)(y)| \leq \ell p_i \\ &\leq \ell s \left( T(n), \hat{I}^n \right)_i \leq \ell c(n) w_i(n). \end{aligned}$$

Since  $i$ ,  $n$ , and  $j$  were arbitrarily chosen, the statement follows.  $\square$

By McDiarmid's inequality as well as Steps 1 and 2, it follows that for each  $\varepsilon > 0$

$$\begin{aligned} &P \left( \left\{ \omega \in \Omega : \left| \bar{T}_j(n)(X_1(\omega), \dots, X_n(\omega)) - \mu \right| \geq \varepsilon \right\} \right) \\ &= P \left( \left\{ \omega \in \Omega : \left| \bar{T}_j(n)(X_1(\omega), \dots, X_n(\omega)) - \mathbb{E}(\bar{T}_j(n)(X_1, \dots, X_n)) \right| \geq \varepsilon \right\} \right) \\ &\leq 2 \exp \left( - \frac{2\varepsilon^2}{\sum_{i=1}^n (\ell c(n) w_i(n))^2} \right) = 2 \exp \left( - \frac{2\varepsilon^2}{\ell^2 c(n)^2 \sum_{i=1}^n w_i(n)^2} \right) \\ &\leq 2 \exp \left( - \frac{2\varepsilon^2}{\ell^2 c(n)^2 \max_{k \in N} w_k(n) \sum_{i=1}^n w_i(n)} \right) \\ &= 2 \exp \left( - \frac{2\varepsilon^2}{\ell^2 c(n)^2 \max_{k \in N} w_k(n)} \right) \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

proving the statement.  $\blacksquare$

**Proof of Remark 2.** Let  $j \in N$ . Given Step 1 in the proof of Proposition 21, we have that  $\mathbb{E}(\bar{T}_j(n)(X_1, \dots, X_n)) = \mu$ . Recall also McDiarmid's inequality

$$\begin{aligned} &P \left( \left\{ \omega \in \Omega : \left| \bar{T}_j(n)(X_1(\omega), \dots, X_n(\omega)) - \mu \right|^2 \geq \varepsilon \right\} \right) \\ &= P \left( \left\{ \omega \in \Omega : \left| \bar{T}_j(n)(X_1(\omega), \dots, X_n(\omega)) - \mu \right| \geq \sqrt{\varepsilon} \right\} \right) \\ &= 2 \exp \left( - \frac{2\varepsilon}{\ell^2 c^2 \max_{k \in N} w_k(n)} \right) \quad \forall \varepsilon > 0. \end{aligned}$$

Observe that

$$\begin{aligned} &\text{Var}(\bar{T}_j(n)(X_1, \dots, X_n)) \\ &= \int_0^\infty P \left( \left\{ \omega \in \Omega : (\bar{T}_j(n)(X_1, \dots, X_n) - \mu)^2 \geq \varepsilon \right\} \right) d\varepsilon \\ &= \int_0^{\ell^2} P \left( \left\{ \omega \in \Omega : (\bar{T}_j(n)(X_1, \dots, X_n) - \mu)^2 \geq \varepsilon \right\} \right) d\varepsilon \\ &\leq \int_0^{\ell^2} 2 \exp \left( - \frac{2\varepsilon}{\ell^2 c^2 \max_{k \in N} w_k(n)} \right) d\varepsilon \leq \int_0^{\ell^2} 2 \exp \left( - \frac{2\ell^2}{\ell^2 c^2 \max_{k \in N} w_k(n)} \right) d\varepsilon \\ &= 2\ell^2 \exp \left( - \frac{2}{c^2 \max_{k \in N} w_k(n)} \right), \end{aligned}$$

proving (54). ■

**Proof of Proposition 22.** Fix  $i \in N$ ,  $n \in \mathbb{N}$ , and  $x \in \mathbb{R}^n$ . For each  $h \in N$  define  $g_h : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$g_h(z) = \limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{T_h(n)(y + tz) - T_h(n)(y)}{t} \quad \forall z \in \mathbb{R}^n.$$

By [25, Proposition A.3] and since  $T_h(n)$  is Lipschitz continuous, monotone, and translation invariant for all  $h \in N$ , we have that  $g_h$  is well defined and the set

$$\partial T_h(n)(x) = \{w \in \mathbb{R}^n : w \cdot z \leq g_h(z) \quad \forall z \in \mathbb{R}^n\},$$

called Clarke's differential, is a nonempty convex and compact subset of  $\Delta_n$ . Finally, we have that

$$g_h(z) = \max_{w \in \partial T_h(n)(x)} w \cdot z \quad \forall z \in \mathbb{R}^n. \quad (91)$$

Define also  $\bar{g} : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$\bar{g}(z) = \limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{\bar{T}_1(n)(y + tz) - \bar{T}_1(n)(y)}{t} \quad \forall z \in \mathbb{R}^n.$$

Moreover, we have that

$$\partial \bar{T}_1(n)(x) = \{w \in \mathbb{R}^n : w \cdot z \leq \bar{g}(z) \quad \forall z \in \mathbb{R}^n\},$$

is also a nonempty convex and compact subset of  $\Delta_n$ . Finally, we have that

$$\bar{g}(z) = \max_{w \in \partial \bar{T}_1(n)(x)} w \cdot z \quad \forall z \in \mathbb{R}^n.$$

By Theorem 2, we have that  $\bar{T}(n) \circ T(n) = \bar{T}(n)$ . It follows that  $\bar{T}_1(n) \circ T(n) = \bar{T}_1(n)$ . We thus have that

$$\partial \bar{T}_1(n)(x) = \partial \bar{T}_1(n)(T(n)(x)). \quad (92)$$

Define  $\Pi_{h=1}^n \partial T_h(n)(x)$  the collection of all  $n \times n$  square matrices whose  $h$ -th entry is an element of  $\partial T_h(n)(x)$ . From the previous part of the proof, we have that  $\Pi_{h=1}^n \partial T_h(n)(x) \subseteq \mathcal{W}$ . Define

$$\begin{aligned} & \partial \bar{T}_1(n)(T(n)(x)) \Pi_{h=1}^n \partial T_h(n)(x) \\ &= \{\tilde{w} \in \mathbb{R}^n : \exists \alpha \in \partial \bar{T}_1(n)(x), \exists W \in \Pi_{h=1}^n \partial T_h(n)(x) \text{ s.t. } \alpha^T W = \tilde{w}^T\}. \end{aligned}$$

In words,  $\tilde{w} \in \partial \bar{T}_1(n)(T(n)(x)) \Pi_{h=1}^n \partial T_h(n)(x)$  only if it is stochastic vector which is a convex linear combination of the rows of some matrix  $W \in \Pi_{h=1}^n \partial T_h(n)(x)$ . Momentarily, this set will play a fundamental role. By assumption, we have that for each  $x \in \hat{I}^n$ , for each  $i \in N$ , and for each  $h \in N$

$$g_h(e^i) \leq c(n) w_i(n) \quad \text{and} \quad c(n)^2 \max_{k \in N} w_k(n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (93)$$

Fix  $n \in \mathbb{N}$ . Fix also  $x \in \hat{I}^n$  as well as  $i \in N$  and  $h \in N$ . By (93) and (91), we have that

$$\tilde{w}_i \leq g_h(e^i) \leq c(n) w_i(n).$$

Since  $i$  was arbitrarily chosen in  $N$ , we have that the  $h$ -th row of a matrix in  $\Pi_{h=1}^n \partial T_h(n)(x)$  is  $\leq c(n) w(n)$ . Since  $h$  was arbitrarily chosen in  $N$ , we have that for each  $W \in \Pi_{h=1}^n \partial T_h(n)(x)$

$$\tilde{w}_{ij} \leq c(n) w_i(n) \quad \forall i, j \in N.$$

It follows that for each  $\tilde{w} \in \partial \bar{T}_1(n)(T(n)(x)) \Pi_{h=1}^n \partial T_h(n)(x)$  is such that  $\tilde{w}_i \leq c(n) w_i(n)$  for all  $i \in N$ . By the chain rule for Clarke's differentials (see, e.g., [14, Theorem 2.6.6]), we have that for each  $i \in N$

$$\begin{aligned} \limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{\bar{T}_1(n)(y + te^i) - \bar{T}_1(n)(y)}{t} &= \bar{g}(e^i) = \max_{\tilde{w} \in \partial \bar{T}_1(n)(x)} \tilde{w} \cdot e^i \\ &\leq c(n) w_i(n). \end{aligned}$$

Since  $x$  was arbitrarily chosen in  $\hat{I}^n$ , we have that

$$s\left(T(n), \hat{I}^n\right)_i = \sup_{x \in \hat{B}} \left( \limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{\bar{T}_1(n)(y + te^i) - \bar{T}_1(n)(y)}{t} \right) \leq c(n) w_i(n) \quad \forall i \in N.$$

Since  $c(n)^2 \max_{k \in N} w_k(n) \rightarrow 0$  as  $n \rightarrow \infty$ , we have that (55) holds.  $\blacksquare$

**Proof of Proposition 23.** Let  $\sigma^2 > 0$  be the lower bound to the error terms imposed in Golub and Jackson [27]. By Markov's inequality applied to the random variable  $-|X_i(n) - \mu|^2 + \ell^2$ , we have  $P\left(-|X_i(n) - \mu|^2 + \ell^2 \geq -\frac{\sigma^2}{2} + \ell^2\right) \leq \frac{-\sigma^2 + \ell^2}{-\frac{\sigma^2}{2} + \ell^2}$  that is

$$P\left(\frac{\sigma^2}{2} \geq |X_i(n) - \mu|^2\right) \leq \frac{-\sigma^2 + \ell^2}{-\frac{\sigma^2}{2} + \ell^2} := K < 1.$$

Therefore, at least one between

$$P\left(X_i(n) > \frac{\sigma}{\sqrt{2}} + \mu\right) \geq \frac{1-K}{2} \quad \text{and} \quad P\left(X_i(n) < -\frac{\sigma}{\sqrt{2}} + \mu\right) > \frac{1-K}{2}$$

holds. Therefore, for at least one of the two inequalities, as  $n$  goes to infinity, the number of  $i \in \{1, \dots, n\}$  for which that inequality holds is going to infinity as well. Therefore, we can without loss of generality restrict to the subsequence of  $n$  in which this happens for the first inequality. The case for the other subsequence follows by a completely similar argument.

For every  $n \in \mathbb{N}$ , let

$$\phi_i(n)(z) = \frac{\delta}{\ell} \max_{j \in \{1, \dots, n\}} (z_j)_+ + \left(1 - \frac{\delta}{\ell}\right) \sum_{j=1}^n w_{ij}(n) z_j^2.$$

Therefore, if  $z = x - ce$  with  $x \in \hat{B}$  and  $c \in \hat{I}$ ,

$$\left| \phi_i(n)(z) - \sum_{i=1}^n w_{ij}(n) z_j^2 \right| < \delta.$$

The resulting opinion aggregator is given by

$$T_i(n)(x) = \frac{\delta}{\ell} \max_{i=\{1,\dots,n\}} x_i + \left(1 - \frac{\delta}{\ell}\right) W(n)x,$$

that is robust being the convex combination of two robust opinion aggregators. Next, notice that for every  $n, t \in \mathbb{N}$  and  $x \in \hat{B}$ , we have

$$T^t(x) \geq \frac{\delta}{\ell} \max_{i=\{1,\dots,n\}} x_i + \left(1 - \frac{\delta}{\ell}\right) (W(n))^t x.$$

We prove the result by induction on  $t$ . The claim is obvious for  $t = 1$ . Suppose it holds for  $t$ . Then, for all  $j \in I$ ,

$$\begin{aligned} T_j^{t+1}(x) &= \frac{\delta}{\ell} \max_{i=\{1,\dots,n\}} T_i^t(x) + \left(1 - \frac{\delta}{\ell}\right) W_i(n) T^t(x) \\ &\geq \frac{\delta}{\ell} \max_{i=\{1,\dots,n\}} T_i^t(x) + \left(1 - \frac{\delta}{\ell}\right) W_i(n) \left( \frac{\delta}{\ell} \max_{i=\{1,\dots,n\}} x_i + \left(1 - \frac{\delta}{\ell}\right) (W(n))^t x \right) \\ &\geq \frac{\delta}{\ell} W_i(n) \left( \frac{\delta}{\ell} \max_{i=\{1,\dots,n\}} x_i + \left(1 - \frac{\delta}{\ell}\right) (W(n))^t x \right) + \left(1 - \frac{\delta}{\ell}\right) \frac{\delta}{\ell} \max_{i=\{1,\dots,n\}} x_i + \left(1 - \frac{\delta}{\ell}\right)^2 (W(n))^{t+1} x \\ &= \frac{\delta}{\ell} \max_{i=\{1,\dots,n\}} x_i + \left(1 - \frac{\delta}{\ell}\right) (W(n))^{t+1} x. \end{aligned}$$

Taking the limit for  $t$ , and letting  $\{s(n)\}_{n \in \mathbb{N}}$  denotes the influence weights in Golub and Jackson, we get

$$\bar{T}_j(x) \geq \frac{\delta}{\ell} \max_{i=\{1,\dots,n\}} x_i + \left(1 - \frac{\delta}{\ell}\right) s(n)x.$$

Notice that, by the initial part of the proof, we know that

$$\lim_{n \rightarrow \infty} P \left( \max_{i=\{1,\dots,n\}} x_i \geq \frac{\sigma}{\sqrt{2}} + \mu \right) \rightarrow 1,$$

and since  $\{W(n)\}_{n \in \mathbb{N}}$  is wise, for all  $\gamma > 0$

$$\lim_{n \rightarrow \infty} P(s(n)x > \mu - \gamma) \rightarrow 1.$$

The last three inequalities together imply that

$$\lim_{n \rightarrow \infty} P \left( \bar{T}_j(x) \geq \frac{\sigma}{2\sqrt{2}} \frac{\delta}{\ell} + \mu \right) \rightarrow 1,$$

proving the result. ■

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## E Online Appendix

Many of the proofs would be extremely facilitated if  $I$  were to be equal to  $\mathbb{R}$  (cf. the proof of Lemma 1 and Remark 7). In our case though,  $I$  is only assumed to be a closed interval with nonempty interior. Yet, we can always extend  $T$  from  $B$  to the entire space  $\mathbb{R}^n$ . Next lemma proves this. Moreover, despite the fact that  $T$  might have many extensions, it also shows that all the extensions generate the sequence of updates and therefore the same limiting behavior.

### E.1 Extension of robust opinion aggregators

Many of the proofs would be extremely facilitated if  $I$  were to be equal to  $\mathbb{R}$  (cf. the proof of Lemma 1 and Remark 7). In our case though,  $I$  is only assumed to be a closed interval with nonempty interior. Yet, we can always extend  $T$  from  $B$  to the entire space  $\mathbb{R}^n$ . Next lemma proves this.

**Lemma 9** *Let  $T$  be an opinion aggregator. The following statements are true:*

1. *If  $T$  is robust, then it admits an extension  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  which is also robust.*
2. *If  $T$  is robust and constant affine, then it admits a unique extension  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  which is robust and constant affine.*
3. *If  $T$  is normalized and monotone, then  $\|T^t(x)\|_\infty \leq \|x\|_\infty$  for all  $x \in B$  and for all  $t \in \mathbb{N}$ .*
4. *If  $x \in B$ , then there exists  $\tilde{I} \subseteq I$  which is a compact subinterval of  $I$  with nonempty interior and  $x \in \tilde{I}^n = \tilde{B}$ . Moreover, if  $T$  is robust, the restriction  $\tilde{T} = T|_{\tilde{B}}$  is a robust opinion aggregator and  $\tilde{T}^t(x) = T^t(x)$  as well as  $\tilde{T}(x) = \bar{T}(x)$  for all  $t \in \mathbb{N}$  and for all  $x \in \tilde{B}$ .*

**Proof.** 1. Since  $T$  is robust, we have that  $T_i : B \rightarrow \mathbb{R}$  is monotone and translation invariant for all  $i \in N$ . By [11, Theorem 4],  $T_i$  is a niveloid for all  $i \in N$ . By [11, Theorem 1],  $T_i$  admits an extension  $S_i : \mathbb{R}^n \rightarrow \mathbb{R}$  which is a niveloid for all  $i \in N$ . By [11, Theorem 4],  $S_i$  is normalized and monotone for all  $i \in N$ . Define  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $S(x)_i = S_i(x)$  for all  $i \in N$ . It is immediate to see that  $S$  is monotone and translation invariant. Fix  $k' \in I$ . Since  $S$  is translation invariant and  $T$  is normalized, it follows that for all  $k \in \mathbb{R}$

$$\begin{aligned} S(ke) &= S(k'e + (k - k')e) = S(k'e) + (k - k')e \\ &= T(k'e) + (k - k')e = k'e + (k - k')e = ke, \end{aligned}$$

proving that  $S$  is normalized and, in particular,  $S$  is robust.

2. We only prove the uniqueness of the extension. The existence of an extension  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  which is constant affine follows from routine arguments.<sup>45</sup> Let  $R : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a robust and constant affine opinion aggregator such that  $R(x) = T(x)$  for all  $x \in B$ . Let  $r \in \text{int } I$ . It follows that  $re \in \text{int } B$ . Consider  $x \in \mathbb{R}^n$  and  $\alpha \in (0, 1)$  such that  $\alpha x + (1 - \alpha)re \in B$ . We have that

$$\begin{aligned} \alpha R(x) + (1 - \alpha)re &= R(\alpha x + (1 - \alpha)re) = T(\alpha x + (1 - \alpha)re) \\ &= S(\alpha x + (1 - \alpha)re) = \alpha S(x) + (1 - \alpha)re, \end{aligned}$$

proving that  $R(x) = S(x)$ . Since  $x$  was arbitrarily chosen, it follows that  $S = R$ .

3. By induction, if  $T$  is normalized and monotone, then  $T^t$  is normalized and monotone. Consider  $x \in B$  and  $t \in \mathbb{N}$ . Define  $k_\star = \min_{i \in N} x_i$  and  $k^\star = \max_{i \in N} x_i$ . Note that  $\|x\|_\infty = \max\{|k_\star|, |k^\star|\}$ ,  $k_\star, k^\star \in I$ , and  $k_\star e \leq x \leq k^\star e$ . Since  $T^t$  is normalized and monotone, we have that

$$k_\star e = T^t(k_\star e) \leq T^t(x) \leq T^t(k^\star e) = k^\star e,$$

yielding that  $|T^t(x)| \leq \max\{|k_\star|e, |k^\star|e\}$  and  $\|T^t(x)\|_\infty \leq \|x\|_\infty$ . Since  $t$  and  $x$  were arbitrarily chosen, the statement follows.

4. Let  $x \in B$ . Define  $k_\star = \min_{i \in N} x_i$  and  $k^\star = \max_{i \in N} x_i$ . We have two cases:

a.  $k_\star < k^\star$ . Clearly, we have that  $k_\star, k^\star \in I$ . Note that  $\tilde{I} = [k_\star, k^\star] \subseteq I$  is compact and with nonempty interior. Moreover,  $x \in \tilde{I}^n = \tilde{B}$ .

b.  $k_\star = k^\star$ . Since  $I$  has nonempty interior, there exists  $\varepsilon > 0$  such that either  $\tilde{I} = [k_\star, k_\star + \varepsilon] \subseteq I$  or  $\tilde{I} = [k_\star - \varepsilon, k_\star] \subseteq I$ . In all these cases,  $\tilde{I}$  is compact and with nonempty interior. Moreover,  $x \in \tilde{I}^n = \tilde{B}$ .

Consider the restriction  $\tilde{T} = T|_{\tilde{B}}$ . Note that  $T(\tilde{B}) \subseteq \tilde{B}$ , yielding that  $\tilde{T}$  is a robust opinion aggregator. By induction, we have that  $\tilde{T}^t(x) = T^t(x)$  for all  $t \in \mathbb{N}$  and for all  $x \in \tilde{B}$ . It follows that for each  $x \in \tilde{B}$

$$\bar{T}(x) = \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) = \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} \tilde{T}^t(x) = \tilde{\bar{T}}(x) \quad \forall x \in \tilde{B},$$

proving the point. ■

**Remark 7** Note that the proof that  $T$  is nonexpansive becomes immediately simpler when  $I = \mathbb{R}$ . It is indeed a consequence of the standard Blackwell's technique used in dynamic programming. We report it here for ease of reference. Indeed, in this case, we

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<sup>45</sup>The proof is available upon request.

have that  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Since  $T$  is normalized, monotone, and translation invariant, we have that

$$\begin{aligned} x - y \leq \|x - y\|_\infty e &\implies x \leq y + \|x - y\|_\infty e \\ &\implies T(x) \leq T(y + \|x - y\|_\infty e) = T(y) + \|x - y\|_\infty e \\ &\implies T(x) - T(y) \leq \|x - y\|_\infty e \quad \forall x, y \in B. \end{aligned}$$

Given the symmetric role of  $x$  and  $y$ , this implies that  $|T(x) - T(y)| \leq \|x - y\|_\infty e$  and  $\|T(x) - T(y)\|_\infty \leq \|x - y\|_\infty$ .  $\blacktriangle$

## E.2 Decomposition of robust aggregators

**Lemma 10** *Let  $T$  be a normalized and monotone (resp., robust) opinion aggregator. If there exist  $\varepsilon \in (0, 1)$  and  $W \in \mathcal{W}$  such that for each  $x, y \in B$*

$$x \geq y \implies T(x) - T(y) \geq \varepsilon(Wx - Wy), \quad (94)$$

then  $S : B \rightarrow \mathbb{R}^n$ , defined by

$$S(x) = \frac{T(x) - \varepsilon Wx}{1 - \varepsilon} \quad \forall x \in B, \quad (95)$$

is a normalized and monotone (resp., robust) opinion aggregator and

$$T(x) = \varepsilon Wx + (1 - \varepsilon)S(x) \quad \forall x \in B. \quad (96)$$

**Proof.** Define  $S : B \rightarrow \mathbb{R}^n$  as in (95). Observe that:

a. For each  $k \in I$

$$S(ke) = \frac{1}{(1 - \varepsilon)} (T(ke) - \varepsilon W(ke)) = \frac{1}{(1 - \varepsilon)} (ke - \varepsilon ke) = ke.$$

b. Since  $\varepsilon \in (0, 1)$ , note that for each  $x, y \in B$

$$T(x) - T(y) \geq \varepsilon(Wx - Wy) \iff T(x) - \varepsilon Wx \geq T(y) - \varepsilon Wy \iff S(x) \geq S(y).$$

Since  $T$  satisfies (94), we have that for each  $x, y \in B$

$$x \geq y \implies T(x) - T(y) \geq \varepsilon(Wx - Wy) \implies S(x) \geq S(y).$$

c. For each  $x \in B$  there exists  $k_*, k^* \in I$  such that  $k_*e \leq x \leq k^*e$ . Since  $k_*, k^* \in I$ ,  $k_*e, k^*e \in B$  and by points a and b we have that

$$k_*e = S(k_*e) \leq S(x) \leq S(k^*e) = k^*e,$$

proving that  $S(x) \in B$ .

- d. Assume  $T$  is also robust and, in particular, translation invariant. We have that for each  $x \in B$  and for each  $k \in \mathbb{R}$  such that  $x + ke \in B$

$$\begin{aligned}
S(x + ke) &= \frac{1}{(1 - \varepsilon)} (T(x + ke) - \varepsilon W(x + ke)) \\
&= \frac{1}{(1 - \varepsilon)} (T(x) + ke - \varepsilon(Wx + W(ke))) \\
&= \frac{1}{(1 - \varepsilon)} ((T(x) - \varepsilon Wx) + (1 - \varepsilon)ke) \\
&= \frac{1}{(1 - \varepsilon)} (T(x) - \varepsilon Wx) + ke = S(x) + ke.
\end{aligned}$$

Points a–c (resp., points a–d) prove that  $S$  is a selfmap on  $B$  which is normalized and monotone (resp., normalized, monotone, and translation invariant, that is, robust). Equation (96) trivially follows.  $\blacksquare$

### E.3 Other proofs

**Proof of Theorem 11.** (i) implies (ii). Let  $\varepsilon > 0$ . There exists  $\bar{\tau} \in \mathbb{N}$  such that (68) holds. Define  $\bar{\varepsilon} = \varepsilon/\bar{\tau}$ . By assumption, there exists  $\bar{t} \in \mathbb{N}$  such that  $\|x^{t+1} - x^t\|_\infty < \bar{\varepsilon}$  for all  $t \geq \bar{t}$ . Choose  $\tau \geq \max\{\bar{t}, \bar{\tau}\} + 1 \geq 2$ . In (68), set  $m = \tau - 1$ . We obtain that

$$\|\bar{x} - x^\tau\|_\infty \leq \left\| \bar{x} - \frac{1}{\bar{\tau}} \sum_{t=1}^{\bar{\tau}} x^{(\tau-1)+t} \right\|_\infty + \left\| \frac{1}{\bar{\tau}} \sum_{t=1}^{\bar{\tau}} x^{(\tau-1)+t} - x^\tau \right\|_\infty < \varepsilon + \left\| \frac{1}{\bar{\tau}} \sum_{t=1}^{\bar{\tau}} (x^{(\tau-1)+t} - x^\tau) \right\|_\infty. \quad (97)$$

Note that for each  $t \in \{2, \dots, \bar{\tau}\}$  we have that

$$x^{(\tau-1)+t} - x^\tau = \sum_{i=\tau}^{\tau-2+t} (x^{i+1} - x^i) \quad \text{and} \quad x^{(\tau-1)+1} - x^\tau = 0.$$

Since  $\|x^{t+1} - x^t\|_\infty < \bar{\varepsilon}$  for all  $t \geq \bar{t}$  and  $\bar{\varepsilon} = \varepsilon/\bar{\tau}$ , this implies that

$$\begin{aligned}
\left\| \frac{1}{\bar{\tau}} \sum_{t=1}^{\bar{\tau}} (x^{(\tau-1)+t} - x^\tau) \right\|_\infty &= \left\| \frac{1}{\bar{\tau}} \sum_{t=2}^{\bar{\tau}} (x^{(\tau-1)+t} - x^\tau) \right\|_\infty \leq \frac{1}{\bar{\tau}} \sum_{t=2}^{\bar{\tau}} \|x^{(\tau-1)+t} - x^\tau\|_\infty \\
&= \frac{1}{\bar{\tau}} \sum_{t=2}^{\bar{\tau}} \left\| \sum_{i=\tau}^{\tau-2+t} (x^{i+1} - x^i) \right\|_\infty \leq \frac{1}{\bar{\tau}} \sum_{t=2}^{\bar{\tau}} \sum_{i=\tau}^{\tau-2+t} \|x^{i+1} - x^i\|_\infty \\
&\leq \frac{1}{\bar{\tau}} \sum_{t=2}^{\bar{\tau}} \sum_{i=\tau}^{\tau-2+t} \bar{\varepsilon} = \frac{\bar{\varepsilon}}{\bar{\tau}} \sum_{t=2}^{\bar{\tau}} (t-1) = \frac{\bar{\varepsilon}}{\bar{\tau}} \sum_{t=1}^{\bar{\tau}-1} t \\
&= \frac{\bar{\varepsilon} \bar{\tau} (\bar{\tau} - 1)}{\bar{\tau} \cdot 2} = \frac{\varepsilon (\bar{\tau} - 1)}{\bar{\tau} \cdot 2} < \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

By (97), this implies that  $\|\bar{x} - x^\tau\|_\infty < 2\varepsilon$ . Since  $\tau$  was chosen arbitrarily, we have that  $\|\bar{x} - x^\tau\|_\infty < 2\varepsilon$  for all  $\tau \geq \max\{\bar{t}, \bar{\tau}\} + 1$ , proving that  $\lim_\tau x^\tau = \bar{x}$ .

(ii) implies (i). If  $\lim_{\tau} x^{\tau} = \bar{x}$ , then clearly we have that  $\lim_{\tau} \|x^{\tau+1} - x^{\tau}\|_{\infty} = 0$ . Let  $\varepsilon > 0$  and  $m \in \mathbb{N}$ . At the same time, there exists  $\hat{\tau} \in \mathbb{N}$  such that  $\|x^{\hat{\tau}} - \bar{x}\|_{\infty} < \frac{\varepsilon}{2}$  for all  $\tau \geq \hat{\tau}$ . Define  $M = \sum_{t=1}^{\hat{\tau}} \|x^t - \bar{x}\|_{\infty} < \infty$ . Let  $\bar{\tau} > \hat{\tau}$  be such that  $\frac{1}{\bar{\tau}}M < \frac{\varepsilon}{2}$ . Consider  $\tau \geq \hat{\tau} + \bar{\tau} + 1$ . We have two cases:

1.  $m \geq \hat{\tau}$ . Since  $m + t \geq \hat{\tau}$  for all  $t \in \mathbb{N}$ , it follows that  $\|x^{m+t} - \bar{x}\|_{\infty} < \frac{\varepsilon}{2}$ . We can conclude that

$$\left\| \frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} - \bar{x} \right\|_{\infty} = \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} (x^{m+t} - \bar{x}) \right\|_{\infty} \leq \frac{1}{\tau} \sum_{t=1}^{\tau} \|x^{m+t} - \bar{x}\|_{\infty} < \frac{\varepsilon}{2} < \varepsilon.$$

2.  $m < \hat{\tau}$ . Since  $\tau > \hat{\tau}$ , observe that

$$\begin{aligned} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} - \bar{x} \right\|_{\infty} &= \left\| \frac{\hat{\tau} - m}{\tau} \frac{1}{\hat{\tau} - m} \sum_{t=1}^{\hat{\tau}-m} x^{m+t} + \frac{\tau - (\hat{\tau} - m)}{\tau} \frac{1}{\tau - (\hat{\tau} - m)} \sum_{t=\hat{\tau}-m+1}^{\tau} x^{m+t} - \bar{x} \right\|_{\infty} \\ &\leq \frac{\hat{\tau} - m}{\tau} \left\| \frac{1}{\hat{\tau} - m} \sum_{t=1}^{\hat{\tau}-m} (x^{m+t} - \bar{x}) \right\|_{\infty} \\ &\quad + \frac{\tau - (\hat{\tau} - m)}{\tau} \left\| \frac{1}{\tau - (\hat{\tau} - m)} \sum_{t=\hat{\tau}-m+1}^{\tau} (x^{m+t} - \bar{x}) \right\|_{\infty}. \end{aligned}$$

First, since  $\tau > \bar{\tau}$ , we have that

$$\begin{aligned} \frac{\hat{\tau} - m}{\tau} \left\| \frac{1}{\hat{\tau} - m} \sum_{t=1}^{\hat{\tau}-m} (x^{m+t} - \bar{x}) \right\|_{\infty} &\leq \frac{1}{\tau} \sum_{t=1}^{\hat{\tau}-m} \|x^{m+t} - \bar{x}\|_{\infty} \leq \frac{1}{\bar{\tau}} \sum_{t=1}^{\hat{\tau}-m} \|x^{m+t} - \bar{x}\|_{\infty} \\ &\leq \frac{1}{\bar{\tau}} \sum_{t=1}^{\hat{\tau}} \|x^t - \bar{x}\|_{\infty} = \frac{1}{\bar{\tau}} M < \frac{\varepsilon}{2}. \end{aligned}$$

Second, since  $m + t \geq \hat{\tau}$  for all  $t \geq \hat{\tau} - m + 1$ , we have that

$$\begin{aligned} \left\| \frac{1}{\tau - (\hat{\tau} - m)} \sum_{t=\hat{\tau}-m+1}^{\tau} (x^{m+t} - \bar{x}) \right\|_{\infty} &\leq \frac{1}{\tau - (\hat{\tau} - m)} \sum_{t=\hat{\tau}-m+1}^{\tau} \|x^{m+t} - \bar{x}\|_{\infty} \\ &< \frac{1}{\tau - (\hat{\tau} - m)} \sum_{t=\hat{\tau}-m+1}^{\tau} \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \end{aligned}$$

We can conclude that

$$\left\| \frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} - \bar{x} \right\|_{\infty} < \frac{\varepsilon}{2} + \frac{\tau - (\hat{\tau} - m)}{\tau} \frac{\varepsilon}{2} < \varepsilon.$$

Cases 1 and 2 prove the implication. ■

**Proposition 26** *The set  $B_{\text{inj}} = \{x \in \text{int } B : x_i \neq x_j \text{ whenever } i \neq j\}$  is dense in  $\text{int } B$  and  $B$ .*

**Proof.** Consider  $x \in \text{int } B$ . Define  $\delta = \frac{\min_{x_i, x_j: x_i \neq x_j} |x_i - x_j|}{2} > 0$  if all the components of  $x$  are equal set  $\delta = 0$ . We have two cases.

1.  $\delta = 0$ . Let  $y \in \mathbb{R}^n$  be such that  $y_i = i/n$  for all  $i \in N$ . Define also  $x^t = x + \frac{1}{t}y$  for all  $t \in \mathbb{N}$ . Since  $x \in \text{int } B$ , there exists  $\bar{t} \in \mathbb{N}$  such that  $x^t \in \text{int } B$  for all  $t \geq \bar{t}$ . Since  $\delta = 0$ , we have that  $x_i = x_j$  for all  $i, j \in N$ . It follows that  $x^t \in B_{\text{inj}}$  for all  $t \geq \bar{t}$ . Since  $\lim_t x^t = x$ , the statement follows.
2.  $\delta > 0$ . Let  $y \in \mathbb{R}^n$  be such that  $y_i = \delta i/n$  for all  $i \in N$ . Since  $x \in \text{int } B$ , there exists  $\bar{t} \in \mathbb{N}$  such that  $x^t \in \text{int } B$  for all  $t \geq \bar{t}$ . By contradiction, assume that there exist  $t \geq \bar{t}$  and  $i, j \in N$  such that  $i \neq j$  and

$$x_i + \frac{1}{t}y_i = x_i^t = x_j^t = x_j + \frac{1}{t}y_j \implies x_i - x_j = \frac{1}{t}(y_j - y_i).$$

We have two cases:

- a.  $x_i = x_j$ . This implies that  $y_j = y_i$ , a contradiction.
- b.  $x_i \neq x_j$ . This implies that

$$\delta < |x_i - x_j| = \frac{1}{t}|y_j - y_i| \leq |y_j - y_i| \leq \delta \frac{|j - i|}{n} < \delta,$$

a contradiction.

It follows that  $x^t \in B_{\text{inj}}$  for all  $t \geq \bar{t}$ . Since  $\lim_t x^t = x$ , the statement follows.  $\blacksquare$

## E.4 Comparison with Molavi, Tahbaz-Salehi, and Jadbabaie

Here we show the convergence and wisdom properties of the log-linear learning rule axiomatized in [47] can be analyzed by means of a linear system. In particular, [47] considered the case of agents having a belief  $\mu \in \Delta(\Theta)$  where  $\Theta$  is a finite set of possible states of the world. Given this, there is a bijection between beliefs and the profile of likelihood ratios  $\left(x(\theta, \hat{\theta})\right)_{(\theta, \hat{\theta}) \in \Theta \times \Theta}$ , with

$$x(\theta, \hat{\theta}) = \frac{\mu(\theta)}{\mu(\hat{\theta})}.$$

Their assumption of IIA allows to study the evolution of  $x(\theta, \hat{\theta})$  independently from the value of the other likelihood ratios. Therefore, fix  $(\theta, \hat{\theta})$  and let

$$x_i^t = \frac{\mu_i^t(\theta)}{\mu_i^t(\hat{\theta})}$$

denote the likelihood ratio obtained from the belief of agent  $i$  at time  $t$ . If we identify  $x_i^t$  as the probability assigned to  $\theta$  by agent  $i$  at period  $t$  then equation (3) in [47] reads as

$$\log x_i^{t+1} = \sum_{j \in N_i} a_{ijt} \log x_j^t.$$

That is

$$x_i^{t+1} = S_t(x^t) := \exp \left( \sum_{j \in N_i} a_{ijt} \log x_j^t \right).$$

In general,  $S_t$  is not robust, since it does not satisfy translation invariance. However, by defining

$$T_t(y) := \log S_t(\exp y) = \sum_{j \in N_i} a_{ijt} y_j^t$$

and letting

$$y^0 = \ln x^0 \text{ and } y^t = T(y^{t-1})$$

we obtain by induction that

$$y^t = \ln \prod_{\tau=1}^t S^\tau(\ln x^0).$$

Since  $T_t$  is a linear aggregator, one can rely to the well known results for the DeGroot model if the weights  $a_{ijt}$  are time independent, or the extensions for time varying matrices developed by [57] and [39] to study  $\lim_t y^t$ . Then, continuity of the logarithm guarantees that  $\lim_t x_i^t = \exp^{\lim_t y^t}$ .

## E.5 Aggregation of Distributions

In this section, we explain how our approach can be easily adapted to the aggregation of opinions that cannot be summarized by a unique real number. In particular, we are interested in the class of distributions (CDF's) over a compact set (we will consider  $[0, 1]$  for simplicity). Therefore, we are going to consider:

$$\mathcal{D} = \left\{ F \in [0, 1]^{\mathbb{R}^k} : \begin{array}{l} (u \leq z) \implies (F(u) \leq F(z)) \\ \lim_{z \downarrow -\infty} F(z) = 0 \\ \lim_{z \uparrow \infty} F(z) = 1 \\ \lim_{z \downarrow z_0} F(z) = F(z_0) \end{array} \right\}.$$

In this more general framework, an opinion is a map  $\mathbf{F} \in \mathcal{D}^N$ , where, as in the previous sections,  $N$  is the set of agents. It follows that a (*single agent*) *distribution aggregator* is a map

$$\widehat{T}_i : \mathcal{D}^N \rightarrow \mathcal{D}.$$

It turns out that, with a slightly stronger continuity condition, robust opinion aggregators are selfmaps on that space, too. Recall that in our standard framework a functional  $T_i : [0, 1] \rightarrow \mathbb{R}$  is a (single agent) robust aggregator if it is normalized, monotone, and translation invariant. Given a robust aggregator  $T$  we can derive a (candidate) distribution aggregator by taking the section at  $z \in [0, 1]$  of the vector of distributions  $\mathbf{F}(z) \in [0, 1]^n$  and aggregating them with  $T$ . In particular, given an opinion aggregator  $T$  we define the functional

$$\widehat{T} : \mathcal{D}^N \rightarrow \left([0, 1]^{\mathbb{R}^k}\right)^N$$

$$\widehat{T}_i(\mathbf{F})(z) = T_i(\mathbf{F}(z)), \quad \forall z \in \mathbb{R}^k.$$

**Proposition 27** *Let  $T : B \rightarrow \mathbb{R}$  be a robust opinion aggregator, then  $\widehat{T}$  is a distribution aggregator, that is  $\widehat{T}_i(\mathcal{D}^N) \subseteq \mathcal{D}$ .*

**Proof of Proposition 27.** Let  $\mathbf{F} \in \mathcal{D}^N$ , and  $i \in N$ . We need to show that  $\widehat{T}_i(\mathbf{F}) \in \mathcal{D}$ , that is  $\widehat{T}_i(\mathbf{F})$  satisfies all the four properties in  $\mathcal{D}$ . Let  $u, z \in \mathbb{R}$  with  $u \leq z$ , it follows that  $\mathbf{F}(u) \leq \mathbf{F}(z)$  and therefore by monotonicity of  $T$  we have

$$\widehat{T}_i(\mathbf{F})(u) = T_i(\mathbf{F}(u)) \leq T_i(\mathbf{F}(z)) = \widehat{T}_i(\mathbf{F})(z).$$

Now let  $\{z_n\}$  be such that  $z_n \downarrow z_0$ , it follows that  $\{\mathbf{F}(z_n)\}$  is a sequence of vectors in  $[0, 1]^n$  such that  $\mathbf{F}(z_n) \downarrow \mathbf{F}(z_0)$ . Since  $T$  is robust, it is continuous, and therefore we have that

$$\lim_{n \rightarrow \infty} \widehat{T}_i(\mathbf{F})(z_n) = \lim_{n \rightarrow \infty} T_i(\mathbf{F}(z_n)) = T_i(\mathbf{F}(z_0)) = \widehat{T}_i(\mathbf{F})(z_0).$$

Moreover, given that for all  $n \in \mathbb{N}$ ,  $\widehat{T}_i(\mathbf{F})(z_0) \leq \widehat{T}_i(\mathbf{F})(z_n)$  we have  $\widehat{T}_i(\mathbf{F})(z_n) \downarrow \widehat{T}_i(\mathbf{F})(z_0)$ . Now consider  $\{z_n\}$  such that  $z_n \downarrow -\infty$ . It follows that  $\mathbf{F}(z_n) \downarrow 0$ . Since  $T$  is robust, it is continuous, and therefore

$$\lim_{n \rightarrow \infty} \widehat{T}_i(\mathbf{F})(z_n) = \lim_{n \rightarrow \infty} T_i(\mathbf{F}(z_n)) = T_i\left(\lim_{n \rightarrow \infty} \mathbf{F}(z_n)\right) = T_i(0) = 0.$$

Finally, let  $\{z_n\}$  be such that  $z_n \uparrow \infty$ . It follows that  $\mathbf{F}(z_n) \uparrow e$  and therefore, by the continuity of  $T$ , we have

$$\lim_{n \rightarrow \infty} \widehat{T}_i(\mathbf{F})(z_n) = \lim_{n \rightarrow \infty} T_i(\mathbf{F}(z_n)) = T_i\left(\lim_{n \rightarrow \infty} \mathbf{F}(z_n)\right) = T_i(e) = 1.$$

■