

Investors' Behavior in Cryptocurrency Market

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Abstract

In this paper, we develop non-parametric tests for Prospect and Markowitz Cumulative Stochastic Spanning in order to determine the performance of investors' specific behavioral patterns in the domains of losses and gains respectively. We take into account the shapes of the related utility functions and the associated probability weighting functions, in order to determine the the returns of optimal portfolios which are created, based on the aforementioned preferences and attitudes towards risk. Spanning occurs when enhancing the opportunity set or relaxing the investment constraints, the investment possibility set does not improve for a given class of investors. Therefore, we test empirically whether added cryptocurrencies to a portfolio consisting of traditional assets, is a better choice for every individual with preferences exhibiting certain patterns of local attitudes towards risk. Thus, we are trying to determine whether the formation of a portfolio with added cryptocurrencies, as investment options, such as Bitcoin, Ethereum etc., implies that a specific pattern-attitude in investor's behavior performs better than the one consisting merely of traditional assets. We formulate a testing procedure for Prospect and Markowitz Cumulative Stochastic Spanning for two portfolio sets based on subsampling and Linear Programming. Our results indicate that if investors employ a reverse S-shaped utility function (Markowitz type) together with an inverse probability weighting function, which declares risk aversion for losses and risk seeking for gains, they outperform the portfolio consisting only of traditional assets. Analogously, we take similar results for Prospect Cumulative Stochastic Spanning where the utility function is S-shaped and the probability weighting functions is again inverse S-shaped.

Keywords and phrases: Non parametric tests, Prospect efficiency, Markowitz efficiency, Markowitz cumulative stochastic spanning, Prospect cumulative stochastic spanning, Probability weighting functions, Linear Programming, Cryptocurrencies

JEL Classification: C12, C13, C15, C44, D81, G11, G14.

1 Introduction

Prospect Theory (henceforth PT), firstly described by Kahneman and Tversky (KT, 1979), argues that investors employ an S-shaped utility function, while according to Markowitz (1952)(henceforth MT), investors employ a reverse S-shaped utility function, when they are about to form an investment strategy (or simply take a mixed bet). This investment strategy focuses on the formation of optimal portfolios (i.e. optimal weights applied on chosen assets in order to obtain maximum returns and lower variability over time) based on the available (opportunity) set of assets. In the case of the S-shaped utility function, concavity is applied on the domain of gains and convexity on the domain of losses, while for the reversed S-shaped the opposite pattern takes place. The inflation point in both cases is located at zero, which is assumed as the current wealth level of the gambler (or investor). Concavity implies risk aversion while convexity implies risk loving. In general, risk is considered as a special case of *uncertainty* (i.e. unknown probability distribution function), gains and losses can be regarded as events while investor's wealth as the state. Moreover, in classical Economics and Finance investors are assumed as rational, i.e. they exhibit broad framing (e.g. rebalancing a portfolio less frequently) of options and long term view of their actions. However, in real life people tend to be "myopic" and narrow framed¹ (which is associated with aversion to losses) and as witnessed often in most cases, irrationality prevails. Irrational reactions are based on the fact that given hindsight², individuals believe the world is predictable and uncertainty is rejected. However, if that was the case we would have no difficulty in predicting the future.

The difference between risk and uncertainty is based on the following arguments. When a decision is about to be made it can be under risk or under uncertainty. Decision under risk is when the probabilities of occurrence of the different outcomes are assumed to be known *a priori*, which advocates *Laplace's Principle of Ignorance*³. On the other hand, decision under uncertainty is when the probabilities associated with the various outcomes are not given in advance. In the investing world, decisions are made always under uncertainty unless there is some inside information about a stock, bond etc.. An investment strategy can be regarded as a prospect. Hence, risky prospects are probability distributions over the set of real numbers while uncertain prospects are maps from a state space (containing different subsets which are called events) to the set of real numbers. In our work, we elaborate uncertainty by using random probability density functions in order to simulate (in the best possible way) reality into our experiments. These random probability density functions (e.g. *Dirichlet*) will be used to calculate the decision weights as well as the expected returns of the optimal portfolios.

For us, each investor type (2 types one for Prospect and one for Markowitz) exhibits two behavioral attitudes when forming an investment strategy. In this way, one could argue that investors are more agile towards unpredictable price "turbulence", because abrupt price swings affect in-

¹Choose something based on the way it is presented and not on its core meaning.

²Learn something out of an outcome which completely changes all prior beliefs and makes all prequel events logically given their end up, e.g. a recession.

³Prior probabilities are determined in exactly the same way.

vestors' perception about their future movement. When a surprising event occurs, the surprise itself is very brief but its complications may be long-lasting. Moreover, investment decisions are based on change of wealth and not on final wealth positions. Thus, both utility functions can be regarded as value functions. These two specific functional forms have found great support by empirical evidence (stemmed from extensive psychological experiments on individuals taking mixed bets) and became ground breakers for the classical framework in Economics and Finance where all economic agents are considered as global risk averters without having the possibility to deviate from these boundaries, except for some specific wealth levels as indicated by Friedman and Savage (1948). In their work, the utility function of an investor has two concave regions with a convex region in between. They base their claim on the fact that gamblers buy lottery tickets (risk lovers), insurance (risk averters) and in many cases buy both insurance and lottery tickets at the same time. It is apparent, that this kind of behavior is self contradicting. This contradiction is based on the fact that gamblers are at the same time risk lovers and risk haters and this is something that can not be supported by the classical theoretical framework in Economics and/or Finance. It is true that many times theory is way far from reality and this can greatly complicate (and even reject) predictions or estimations.

Hence, PT and/or MT attempts to provide an answer to this behavioral paradox. It is clear that one of the principles of investors' behavior is that there exists an explicit aversion to losses, namely *loss aversion*. Loss aversion is one of the fundamental pillars of PT, with the rest invoking the observed phenomena of choice such as, source dependence, description invariance and nonlinear preferences. In other words, loss aversion can be regarded as the extreme reluctance of an individual to accept mixed prospects (e.g. bets with gains and losses). Generally speaking, decision making under uncertainty can be viewed as choices between alternative, but random, probability distributions of returns where an individual chooses in accordance to a consistent set of preferences. These preferences are not necessarily steady, but they can change and/or adapt throughout an investment period, even throughout a daily trading session.

Based on the aforementioned reasoning, real life and more specifically real investments, are much more complex and thus a more compatible and *descriptive* theory is required. On this ground, cognitive psychological tests have revealed a more complex pattern in decision making when uncertainty prevails and probable (or sure) gains, or losses (or both as in mixed bets), are to emerge. These evidence reveal that people do not behave *as if*⁴ they are maximizing the expected outcome (i.e. their final wealth position when taking a gamble) through their subjective utility function, but they rather employ different subjective and abstract *heuristics*⁵ to make their decision process easier and sort. Many scholars argue that this is what is really happening in decision making under uncertainty and also that it is the most efficient way. Individuals pay more to attention to change of wealth rather than total wealth positions. They do not necessarily act as risk averters when it comes to gambling, neither do they compute linear objective probabilities of

⁴Normative approach as implied by Expected Utility Theory.

⁵A frugal decision creating a shortcut, when optimization is out of reach or not worth the effort, while time and information is limited.

the different outcomes as implied by classical theory. They seem to subjectively overweight small probabilities of large gains and underweight large and moderate probabilities of losses. This is demonstrated by inverse probability weighting functions (henceforth p.w.f.) which are continuous, increasing and reverse S-shaped functions : $[0, 1] \mapsto [0, 1]$, for gains and losses respectively. In other words, economic agents tend to rather frame outcomes and not precisely assess them.

Moreover, psychological tests have revealed that losses loom larger than gains (i.e. the pain of paying a certain amount of money is not equal, in absolute terms, to the joy of receiving the same amount!) and that the carriers of value are gains and losses and not final assets. This last inference is demonstrated by the S-shaped utility function with a steeper graph in the domain of losses, in the neighborhood of the inflation point. All the aforementioned, contributed to the evolution of PT to Cumulative Prospect Theory (henceforth CPT) (KT, 1992) which extends PT to uncertain and risky prospects with any number of outcomes, while preserving most of its essential features. It is obvious that the formation of a portfolio is a prospect with a finite but large number of outcomes and not just a simple, nor pure bet. In other words, CPT extends PT because it applies to any finite prospect and it can also be extended to continuous distributions. It applies to both probabilistic and uncertain prospects and therefore it can accommodate some source dependence (i.e. the willingness to bet on an uncertain event depends not only on the degree of uncertainty but also on its source). Finally, it allows for different decision weights for gains and losses. To sum up, PT had two drawbacks that were confronted by CPT. PT employs a monotonic transformation of outcome probabilities. This monotonic transformation could not be applied to prospects with any number of outcomes and second, it did not always satisfy Stochastic Dominance (henceforth SD). These drawbacks required a modification of PT and the outcome was CPT. Basically, CPT has three features: an S-shaped value function, two p.w.f. (one for losses and one for gains) and loss aversion (represented by the loss aversion coefficient λ), the whole distribution rather than the mean and variance plays an important role in comparing portfolios. Variance does not fully captures the risk profile of assets unless investor utility function is quadratic and this is something that can not supported by empirical evidence.

In CPT, which exhibits a sign and rank dependent functional and it is a descriptive theoretical framework, prospects are evaluated in terms of cumulative events and the entire cumulative probability density function (CDF) is subjectively transformed. Risk aversion/seeking is determined jointly by the value function and the so-called "capacities" (i.e. decision weights which are non additive set functions that generalize the standard notion of probability), which depend on the p.w.f. as indicated in the Appendix. The value of each outcome is multiplied by a decision weight and not by an additive probability. These decision weights which are associated with an outcome, can be regarded as the marginal contribution of the respective event. Moreover, the notion of diminishing sensitivity is introduced which states that, the impact of a change diminishes with the distance from the reference point. Again, the value function is defined over variations of wealth and risk aversion for gains is more pronounced than risk seeking for losses. Finally, we have sign dependent preferences which depend on a perceived reference point outcome which is usually the

current wealth level (state) of the investor.

The theory of SD provides a systematic framework for analyzing investor behavior under uncertainty. SD uses a distribution free assumption framework, and thus non parametric testing and inference, and it aims at comparing random variables in the sense of stochastic orderings expressing common preferences and beliefs. Stochastic orderings are binary relations defined on classes of probability distributions. They rely only on general preference and belief assumptions. It can be regarded as an alternative to Mean - Variance (M-V) dominance in Modern Portfolio Theory. The M-V criterion is consistent with Expected Utility Theory (EUT), which is a *normative* theoretical framework for elliptical distributions, when the distribution is assumed normal or when the utility function is quadratic but is not consistent with Second order Stochastic Dominance. In simple words, M-V is applied if the investor wants to optimize the trade-off between the mean and variance of portfolio returns. However, the M-V rule can assign an economically irrational weight to variance. Generally speaking, variance is not a satisfactory measure because it is symmetric and it penalizes gains and losses in the same way. Thus, it is inappropriate to describe the risk of low probability events. Behavioral Economists argue that volatility is a measure of investors' sentiment fluctuation and not risk. Traditional models in Economics and Finance assume that investors evaluate portfolios according to the EUT framework: the utility of an uncertain prospect is the sum of the utilities of the outcomes, each weighted by its probability (subjective or objective), i.e. they calculate the expected return which is the average of all possible outcomes. Moreover and according to the classical framework, variance is regarded as measure of risk, albeit it is suited only for normal distributions, or log-normal, or even quadratic preferences. The first who questioned EUT was Allais (1953), where in his famous work he demonstrated the descriptive shortcoming of EUT and its critical axiom of independence.

To sum up, we have that in the EUT concept, the objects of choice are probability distributions over wealth, the valuation rule is expected utility and risk aversion (or seeking) is determined solely by a utility function. On the other hand, in CPT the objects of choice are prospects formed in terms of gains and losses. The valuation rule is a two-part cumulative functional, the value function is S-shaped and the p.w.f. are reverse S-shaped. In other words, EUT is a normative theory while CPT is a descriptive one. In a descriptive model (gain-loss asymmetry and nonlinearity in probabilities) we have the so-called "fourfold pattern" of risk attitudes which describes risk aversion for most gains and low probability for losses together with risk seeking for most losses and low probability for gains. In general, CPT coincides with rank-dependent EUT for gains.

The advantage of SD is that it provides a broader frame, which accounts for all distributions' moments, without assuming any particular family of distributions. It is appealing for asset classes and investment strategies with asymmetric risk profiles. In our work, we deal with distributions of returns of traditional financial assets such as stocks, bonds, funds, indices and cryptocurrencies. Traditionally, SD is tested pairwise (e.g. two different income distributions). Thus, in order to compare more than two prospects the notion of Stochastic Dominance Efficiency (SDE) (Post, 2002) was introduced. In this concept, a portfolio consisting of any number of assets can be compared

with another but usually this comparison is made with the portfolio of the entire market, i.e. the Market Portfolio (MP). The MP is a theoretical bundle of investments that includes every type of asset available in the financial market, with each asset being weighted in proportion to its total presence in the market. The expected return of the MP is identical to the expected return of the market as a whole. Because the MP is completely diversified, it is subject only to systematic risk⁶. Systematic risk is unpredictable and impossible to avoid. It can be mitigated through hedging or by using the right asset allocation strategy.

More formally, SDE is testing whether a given prospect is stochastically efficient relative to all mixtures of a discrete set of alternatives. Efficiency tests are typically applied to a given broad market index with limited high-order moment risk. Consistent and feasible testing procedures for SDE at any order, of a given portfolio with respect to all possible portfolios from a choice set of assets, were formulated in the work of Scaillet and Topaloglou (2010). The First and Second SD Efficiency rules (FSDE and SSDE) avoid parameterization of investor preferences and the return's distribution, and at the same time ensure that the regularity conditions of non satiation and risk aversion are satisfied. The orders of SD are compatible with economic intuition, notions and interpretations. For example, the First Order SD (FSD) employs an increasing utility function declaring non satiation (prefer more to less) for investors and permits a preliminary screening of investment alternatives those which no rational investor would ever choose. FSD is appropriate for both risk lovers and haters since the utility function may contain concave as well as convex segments. Second Order SD (SSD) employs an increasing and concave utility function declaring non-satiation and global risk aversion. This criterion is based on a stronger assumption and therefore it permits a more sensible selection of investments. Third order SD (TSD), while economic notions go up to fourth order (FOSD), employs an increasing utility function which is globally concave and its third order derivative is positive. This implies the notion of prudence on behalf of the investor and the Fourth order SD (FOSD), with a negative fourth order derivative of the utility function, implies the notion of temperance. Arguably, SSD has a well established economic interpretation in terms of EUT and Yaari's (1987) dual theory of risk.

In the EUT framework, risk aversion as well as risk seeking are determined solely by the utility function, while this is not the case in CPT. Another novelty of PT, and also CPT, is that the carriers of value are gains and losses. Thus, the combination of the value function and the p.w.f. declares a specific risk attitude, something that is analytically presented in the work of Baucells and Heukamp (2006). They argue that specific combinations between the value function and the p.w.f. can exist, while the p.w.f. can have two different forms, one for the domain of gains and one for the domain of losses. These combinations will be presented in the appendix of this paper where the numerical implementation is fully described. More specifically, they state that for the SSD conditions to hold, we must have the requirement that the value functions and the p.w.f. ought to have conjugate curvatures. This means that the nonlinear deforming effects of the value function and the p.w.f. are compounding and not offsetting. If no restriction is placed on the p.w.f. then

⁶The risk inherent in the entire market or in a market's segment.

any SD condition reduces to FSD. To put it simpler, if the p.w.f. remains unrestricted, given the specific form of the value function, it is insufficient to predict preferences between stochastically undominated prospects.

Levy and Levy (2002), formulated the notions of Prospect Stochastic Dominance (PSD) and Markowitz Stochastic Dominance (MSD). These notions are extensions of FSD and SSD. PSD is a criterion that it can be applied to any prospect. PSD analysis is insensitive to the details of any specific value function as long as it is S-shaped. By employing PSD we do not try to figure out the precise shape of the value function but whether it belongs to a given family of functions which in this case is the one of the S-shaped. The same is true for the MSD criterion and the reverse S-shaped family of functions. In their analysis they also include the associated utility functions and provide the related decision rules. In this setting, Arvanitis and Topaloglou (2017) develop consistent tests for MSD and PSD efficiency when full diversification (a mixture of wide variety of investments within a portfolio) is allowed. Generally speaking, MSD is not the "opposite" of PSD.

Stochastic spanning (Arvanitis et al., 2017 and 2018) is a model free alternative to M-V spanning of Huberman and Kandel (1987) and it occurs if introducing new securities to a portfolio or relaxing the investment constraints, the investment possibility set is not improved, for a given class of investors. Spanning involves the comparison of two choice sets, not necessarily disjoint. Pairwise dominance and portfolio efficiency analysis are special cases that assume that one or two of the choice sets is a singleton. Here, we introduce stochastic spanning to the class of Markowitz (1952) and Cumulative Prospect Theory (Kahneman and Tversky, 1992) type of investors accompanied with the related p.w.f.. The former class describes investors as risk averters in the domain of losses and risk seekers in the domain of gains while the latter describes the opposite. We propose a theoretical measure for Markowitz and Prospect Cumulative Stochastic Spanning based on SD. To check Markowitz Cumulative Spanning in our data, we develop consistent and feasible test procedures based on subsampling and Linear Programming (LP). The inclusion of the cumulative notion is based in the fact that the associated p.w.f. employ cumulative probabilities and not individual ones. In general, spanning tests evaluate all feasible portfolios. A spanning set is a reduction of the original portfolio set without loss of investment opportunities for any investor with S-shaped, or reversed S-shaped preferences. In our results we have that the portfolio of traditional assets *does not span* the enhanced traditional portfolio with cryptocurrencies, and hence the enhanced portfolio performs better in both cases (Markowitz and Prospect). The utility functions are univariate, normalized and have a bounded domain. We use a finite set of increasing convex and concave piecewise - linear functions that are constructed as convex mixtures of the elementary ramp functions as in Russel and Seo (1989) or by the utility function supported by the theory of Tversky and Kahneman (1992) and al Nowaihi, Bradley and Dhami (2008). However, the latter is non linear and provides significant computational complexity and burden. Thus, we rely on the linear functions as suggested by Russel and Seo (1989). Finally, stochastic spanning is based upon SSD criterion and hence we can call it Second order Stochastic Spanning.

2 About cryptocurrencies

In 2008, a novel paper by an unknown author(s) came into the light and shook the grounds of the world economy. The author(s)'s name (probably) was Satoshi Nakamoto and the title: *"Bitcoin: A Peer-to-Peer Electronic Cash System"*. The idea behind that paper was quite simple: create a payment system of peer-to-peer transactions without any state or institutional intervention, by solving the "double spending problem". The "double spending problem" had remained unsolved until the appearance of Nakamoto. It is basically the risk associated with the fact that a digital currency can be spent twice (or even more), because it is easy to reproduce (i.e. copy) digital information. It is apparent that digital currencies are nothing more than tokens of digital information. On the other hand, physical currencies of fiat money do not face this problem because they are not easy to be reproduced. Hence, in the digital world one could copy numerous times the tokens of her currency and just use them for transactions, creating in this way unlimited digital money.

This problem was finally solved in Nakamoto's paper by introducing time stamps of the digital transactions through a record called *"proof-of-work"*. This record serves as an accounting book, open to the public, and can not be changed without redoing the whole proof-of-work. In order to alter the contents of the record, one should hack the system and rewrite all the transactions from the beginning. The key to the solution is that as the chain (record) of transactions becomes longer (i.e each transaction is stacked in a block on the chain and thus the definition "blockchain" for this specific technology), the energy required to alter the transactions log, and thus create the option to double-spend, grows exponentially and no modern computing power has this kind of capacity to overcome this boundary (however, some argue that quantum computers in the near future may have this capability). This is due to the number of nodes forming the network which is created within the chain. Each transaction has its own digital identity, it can not be modified and it is also publicly recorded. Another solution to the double-spending problem is that the Bitcoin market (or any other cryptocurrency's) can not be inflated. The algorithm which is creating the coins, based on demand for transactions, has an upper bound. It is programmed to produce at most 21 million coins, as far as they will be mined. This mining is nothing more than the solution of cryptographic puzzles, created by the algorithm, where the reward of solving such a puzzle is the formation of a new digital coin. Enormous "mines" exist in the rural areas of Mongolia and China, where electric power is rather cheap. Being a "miner" includes, besides electricity costs, significant costs concerning the software and the hardware. The mining of the coins is crucial for the creation of new ones and the sustainability of the economic system. Hence, the cost of creating new coins equals the cost of electric power consumed plus the analogy of the cost of the hardware and the software that is being used.

As the algorithm progresses its operation through time and more digital coins are created, the harder is to solve these cryptographic puzzles and hence greater electric power is required. Thus therefore, if someone is interested in hacking the system the cost of electric power would be ridiculously high. Albeit the structure of this system is rather simple, it is practically impossible

to be hijacked. The only way for the coins to be stolen is to steal them from one's PC directly (i.e. they are stored in the hard drive in a so called "wallet"). Hence, no one can argue that the system is not safe or unreliable.

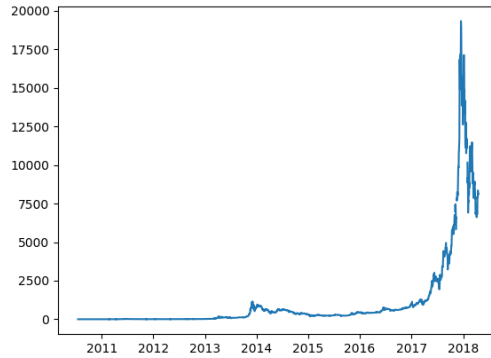


Figure 1: Bitcoin closing price in U.S. Dollars

In the years that followed, the closing price of Bitcoin started to rise from 0.05 U.S. dollars in 2009 to approximately 20,000 U.S. dollars in the beginning of 2018 (Figure 1). Today, Bitcoin has the highest market capitalization among cryptocurrencies. Moreover, hundreds of new cryptocurrencies started to appear such as : BitcoinCash, Ripple, Monero, Litecoin, Ethereum etc. The growth of their prices (Figure 2) followed the same pattern as Bitcoin's. They started really low and in a short period of time they sky rocked. In our work we employ, besides Bitcoin (BTC), Ethereum (ETH), Litecoin (LTC) and Ripple (XRP) each one for the associated reasons that create additional value. For example, ETH has the second highest market capitalization, there is an enterprise alliance that provides support to its technology and is interested in applying it in its core business. The innovation of ETH is based on the creation of "smart contracts" in order to secure and validate the transactions. A "smart contract" is a contract automatically created and executed, according to a computer algorithm, when the associated contract terms are met. LTC technology has the benefit of faster transaction confirmation and XRP is not traded in China and this fact provides greater stability to its network and price. Moreover, BTC and ETH are the basic cryptocurrencies needed to purchase tokens during Initial Coin Offerings (ICOs). These ICOs have become very popular the last couple of years because they provide funding to companies (e.g. Startups) without any intervention of the banking system.

Nowadays, more than 2000 different cryptocurrencies exist, creating an investment market that has not only witnessed remarkable gains, but severe losses too. These coins, depend their value on their circulation and the expectation of the course of the financial success, the issuer (company, organization or individual) is going to have. In other words, the "success" of the coin is directly related to the success of the company or the entrepreneurial entity that belongs to. It is apparent, that these digital coins (i.e. cryptocurrencies) are utility based coins and it would be extremely hard to replace fiat currency, at least in the short run. However, during the last years billions of

dollars have been invested in the cryptocurrency market and thus an investing behavioral paradox has emerged. The greater the market's volatility the bigger the investors' interest and under high price volatility, higher profits can be attained. The specific market is extremely volatile and high price fluctuations can occur within seconds. This phenomenon is based also on the fact that the majority of transactions (it is estimated that it is over 90%) is conducted through algorithmic (high - frequency) trading, which is the first introduction of Artificial Intelligence into Finance. Thus therefore, the classical theoretical framework fails to interpret these behaviors and our experiments reveal a behavior pattern of investors that is closer to reality. Through the Markowitz and/or Prospect Cumulative framework, an investor can choose the weights for her cryptocurrencies, within her portfolio, and perform better than the stocks/bonds and exchange traditional market. Hence, local attitudes towards risk are indeed adapting, depending on the market trend towards gains and losses. This behavior pattern can also occur during intraday transactions.

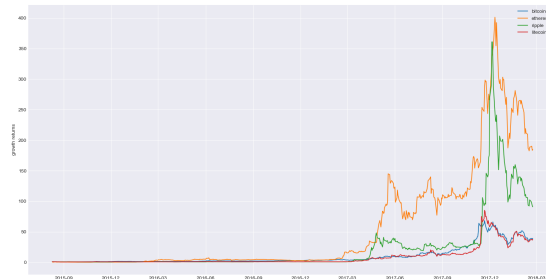


Figure 2: cryptocurrencies' growth

So far, the cryptocurrency market besides significant gains, has suffered huge losses too. Both phenomena bear resemblance (with opposite signs) to the magnitude of their extremity. These extreme gains and losses provided us with the motivation for our work. We are interested in applying the aforementioned behavioral patterns and assess the performance of the optimal constructed and related portfolios. In our first experiment, we assume that the losses domain is the down trending the market (bear market : prices are falling, encouraging selling always with respect to the expectation that they will continue to fall) while the gains domain is the up trending the market (bull market : prices are rising, encouraging buying always w.r.to the expectation that they will continue to rise) (Figure 3). These two market types are formed through market growth (Figure 3). It is well known that the majority of investors has it as a benchmark of overall economic performance the growth of the stock, bond and exchange market. It is true that the market as a whole forms the overall attitude and sentiment (which form expectations) towards investments. One interesting conclusion comes from the work of Anyfantaki et al. (2018) where they prove that the traditional market and the cryptocurrency market are segmented and can be treated separately. We also follow this reasoning because we assume that investors have as their basic investment "arena" the traditional market and that the cryptocurrencies market is simply additional. The time span is of 654 days without weekends (only business days). This is a significant difference, concerning

trading, between traditional financial assets and digital ones. The latter can be traded around the clock on an everyday basis (24/7) throughout the entire year while the former are traded only during the 5 working days of the week when the stock, bond and exchange market are open. Hence, no "weekend effect" for cryptocurrencies and all traders worldwide can watch the prices fluctuations in real time. However, it is true that the vast majority of transactions is done with algorithmic-high frequency trading and no human intervention. Thus, prices fluctuate constantly and within milliseconds of time.

In our second experiment, which elaborates a novel approach, we assume that gains and losses are realized daily. This means that every single day we separate gains and losses and we apply the associate utility functions (S-shaped and reverse S-shaped) and the corresponding p.w.f. and evaluate the performances of the optimal portfolios.

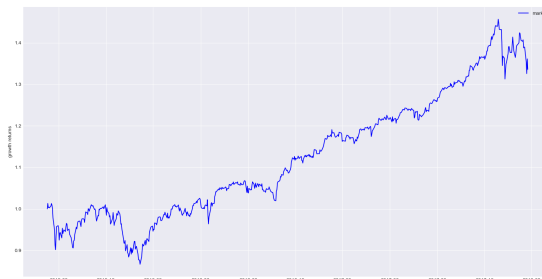


Figure 3: market growth

3 Markowitz Stochastic Dominance and Stochastic Spanning

Given a probability space (Ω, F, P) , suppose that F denotes the cdf of some probability measure on R^n . Let G be the cdf of the linear transformation $x \in R^n \rightarrow \lambda^T x$, where $\lambda \in R_+^n$ assumes values in Λ , which denotes the portfolio space. It is apparent that $\lambda e' = 1$, where $e = (1, 1, \dots, 1) \in R^n$. In order to define Markowitz Stochastic Dominance (henceforth MSD) and subsequently stochastic spanning, we consider

$$J(z_1, z_2, \lambda; F) = \int_{z_1}^{z_2} G(u, \lambda; F) du \quad (1)$$

where,

$$G(z, \lambda; F) = \int_{R^n} (\lambda^T u - z)_- dF(u) \quad (2)$$

and we also have that,

$$(x)_- = \min\{x, 0\} \quad \text{and} \quad (x)_+ = \max\{x, 0\} \quad (3)$$

Definition 1. κ weakly Markowitz-dominates λ , written as $\kappa \succeq_M \lambda$, if and only if (iff)

$$P_1(z, \lambda, \kappa, F) = J(-\infty, z, \kappa; F) - J(-\infty, z, \lambda; F) \leq 0, \quad \text{for all } z < 0 \quad (4)$$

and

$$P_2(z, \lambda, \kappa, F) = J(z, +\infty, \kappa; F) - J(z, +\infty, \lambda; F) \leq 0, \quad \text{for all } z > 0 \quad (5)$$

The definition above can be modified, due to simplicity reasons, for a bounded domain (in this case regarding gains and losses), i.e. $[a, b]$, where $a < 0 < b$.

Definition 2. K Markowitz-spans Λ , written as $K \succeq_M \Lambda$, iff

$$(\forall \lambda \in \Lambda)(\exists \kappa \in K) : \kappa \succeq_M \lambda \quad (6)$$

If K is a singleton, its element κ is termed as *Markowitz super-efficient*.

Lemma 1. Suppose that K is closed (i.e. it contains all its boundary points). Then $K \succeq_M \Lambda$ iff,

$$\max_{i=1,2} \sup_{\lambda \in \Lambda} \sup_{z \in A_i} \sup_{\kappa \in K} P_i(z, \lambda, \kappa, F) = 0 \quad (7)$$

where $A_1 = R_{--}$ and $A_2 = R_{++}$

Corollary 1. κ is *Markowitz super-efficient* iff,

$$\max_{i=1,2} \sup_{\lambda \in \Lambda} \sup_{z \in A_i} P_i(z, \lambda, \kappa, F) = 0 \quad (8)$$

4 Prospect Stochastic Dominance and Stochastic Spanning

Given a probability space (Ω, F, P) , suppose that F denotes the cdf of some probability measure on R^n . Let G be the cdf of the linear transformation $x \in R^n \rightarrow \lambda^T x$, where $\lambda \in R_+^n$ assumes values in Λ , which denotes the portfolio space. It is apparent that $\lambda e' = 1$, where $e = (1, 1, \dots, 1) \in R^n$. In order to define Prospect Stochastic Dominance (henceforth PSD) and subsequently stochastic spanning, we consider again

$$J(z_1, z_2, \lambda; F) = \int_{z_1}^{z_2} G(u, \lambda; F) du \quad (9)$$

Definition 3. κ weakly Prospect-dominates λ , written as $\kappa \succeq_P \lambda$, iff

$$P_3(z, \lambda, \kappa, F) = J(z, 0, \kappa; F) - J(z, 0, \lambda; F) \leq 0, \quad \text{for all } z \leq 0 \quad (10)$$

and

$$P_4(z, \lambda, \kappa, F) = J(0, z, \kappa; F) - J(0, z, \lambda; F) \leq 0, \quad \text{for all } z \geq 0 \quad (11)$$

The definition above can be modified, as before due to simplicity reasons, for a bounded domain (in this case regarding gains and losses), i.e. $[a, b]$, where $a < 0 < b$.

Definition 4. K Prospect-spans Λ , written as $K \succeq_P \Lambda$, iff

$$(\forall \lambda \in \Lambda)(\exists \kappa \in K) \quad : \quad \kappa \succeq_P \lambda \quad (12)$$

If K is a singleton, its element κ is termed as *Prospect super-efficient*.

Lemma 2. Suppose that K is closed. Then $K \succeq_P \Lambda$ iff,

$$\max_{i=3,4} \sup_{\lambda \in \Lambda} \sup_{z \in A_i} \sup_{\kappa \in K} P_i(z, \lambda, \kappa, F) = 0 \quad (13)$$

where $A_1 = R_-$ and $A_2 = R_+$

Corollary 2. κ is *Prospect super-efficient* iff,

$$\max_{i=3,4} \sup_{\lambda \in \Lambda} \sup_{z \in A_i} P_i(z, \lambda, \kappa, F) = 0 \quad (14)$$

In simple words, stochastic spanning (Markowitz or Prospect) occurs when the augmentation of the investment space does not enhance investment opportunities or equivalently, when the reduction of the investment space does not lead to losses of investment opportunities. SD is a pre-order rather than a partial order, thus we can view these spanning relations as SSD order preserving reductions of the portfolio opportunity set in both cases of Prospect-type and Markowitz-type. Thus, stochastic spanning occurs if the aforementioned enlargement does not change the efficient set, which includes the maximal elements of the set. The aforementioned set could also contain all the greatest elements, in contrast to the maximal ones, if these portfolios-elements follow the notion of super efficiency. A portfolio is super efficient by second order if it weakly stochastically second order dominates all other feasible portfolios of the same choice set. Obviously, stochastic super efficiency is a sufficient condition for stochastic efficiency with the converse being not true. This occurs because all super efficient portfolios must be comparable and equivalent while all efficient portfolios may be incomparable or non-equivalent. In our applications, the efficient set is empty and hence we are left with the super efficient one. The generalization of the notion of stochastic super efficiency is the notion of stochastic spanning for comparing two nested choice sets. In our numerical implementation, these two sets are the one which contains traditional assets and the other which is its enlargement with the cryptocurrencies.

Stochastic efficiency of second order places the foundations for the definitions of stochastic spanning which we implement in our work. Stochastic efficiency of second order (henceforth SSDE) is when the portfolio under examination is not strictly second order stochastically dominated by any other feasible portfolio, within the choice set. Stochastic inefficiency occurs in if there exists such a feasible portfolio.

5 Representation by Utility Functions accompanied with Probability Weighting Functions

We also provide a characterization of spanning via an appropriate family of utility functions for Markowitz spanning as well as for Prospect spanning. The family of utility functions employed are the ones that are convex in the domain of gains, and concave in the domain of losses, for Markowitz, and with exactly the opposite pattern for Prospect. These utilities accompanied with the p.w.f can characterize the behavior of a Markowitz-type and a Prospect-type investor as in Baucells and Heukamp (2006). As described in the seminal paper of Markowitz (1952), investors employ a reverse S-shaped utility function when it comes to bets whose outcomes are moderate and not too extreme. While in PT and more especially in CPT, investors employ an S-shaped utility function together with their subjective probability transformation function which here is represented by the p.w.f. In order to make the numerical implementation tractable, we found it necessary to follow Baucells and Heukamp (2006) in combination with Russell and Seo (1989). The latter, present S-shaped as well as reverse S-shaped utility function as convex mixtures of elementary ramp functions. The former, argue that the following curvature pairs are adequate to capture a behavior towards risk. In the losses domain we have convex v^P and convex w^- or concave v^M together with a concave w^- . In the gains domain we can have a concave v^P with a convex w^+ or a convex v^M together with a concave w^+ . The aforementioned combinations are denoted in Table 1 and in Table 3 we have the notations for the different parts of the value functions with the associated curvatures.

value function	concave segment	convex segment
v^P	v_{cv}^P	v_{cx}^P
v^M	v_{cv}^M	v_{cx}^M

Table 1: value function and associated curvature, where P stands for Prospect and M for Markowitz

These w' s define the p.w.f. in the losses and in the gains domain and are both reverse S-shaped probability functions with $w^-(0) = w^+(0) = 0$ and $w^-(1) = w^+(1) = 1$. Aside economic interpretation, these combinations are quite useful for the numerical implementation of the inferential procedures which we construct in what follows. In doing so, we generalize the utility characterizations of Levy and Levy (2002), where their inflation point is located at zero, they depend on change in wealth and it is not required that they are almost everywhere differentiable.

We denote the sets of Borel probability measures with W_-, W_+ , on the real line with supports R_-, R_+ respectively. First moments exist and are uniformly integrable. These sets are convex and closed with respect the topology of weak convergence and their union contains the set of degenerate measures. Following Aliprantis and Border (2006), we have that the set of degenerate measures with support inside R_{--}, R_{++} or R_-, R_+ , are the extreme points of W_-, W_+ respectively. This implies that any linear function on W_-, W_+ , is maximized at some relevant degenerate measure.

Define for Markowitz :

$$V_+^M = \left\{ v_w^M : R_{++} \rightarrow R, v_w^M(u) = \int_{R_{++}} [u1_{0 < z \leq u} + z1_{u \leq z}] dw(z), w \in W_+ \right\} \quad (15)$$

and

$$V_-^M = \left\{ v_w^M : R_{--} \rightarrow R, v_w^M(u) = \int_{R_{--}} [u1_{u \leq z} + z1_{z \leq u < 0}] dw(z), w \in W_- \right\} \quad (16)$$

It is apparent that every element of V_+^M is increasing and convex, while every element of V_-^M is increasing and concave. The union $V_+^M \cup V_-^M$, where $v_w^M(0) = 0$, is the graph of a reverse S-shaped utility function as defined by Levy and Levy (2002).

Define for Prospect :

$$V_+^P = \left\{ v_w^P : R_+ \rightarrow R, v_w^P(u) = \int_{R_+} [u1_{0 \leq u \leq z} + z1_{u \geq z}] dw(z), w \in W_+ \right\} \quad (17)$$

and

$$V_-^P = \left\{ v_w^P : R_- \rightarrow R, v_w^P(u) = \int_{R_-} [u1_{z \leq u \leq 0} + z1_{u \leq z}] dw(z), w \in W_- \right\} \quad (18)$$

It is apparent that every element of V_+^P is increasing and concave, while every element of V_-^P is increasing and convex. The union $V_+^P \cup V_-^P$, where $v_w^P(0) = 0$, is the graph of an S-shaped utility function as defined by Levy and Levy (2002).

Define the probability weighting functions (or decision weights):

Both are inverse S-shaped, which means that individuals overweight small probabilities and underweight moderate and large ones in the domain of losses and gains respectively.

$$w^+(p) = \frac{p^{0.61}}{[p^{0.61} + (1-p)^{0.61}]^{\frac{1}{0.61}}}$$

and

$$w^-(p) = \frac{p^{0.69}}{[p^{0.69} + (1-p)^{0.69}]^{\frac{1}{0.69}}}$$

Their graphs are depicted in Figure 4 and p is the cumulative probability. Clearly, the concave as well as the convex segments of the two functions can be identified. It is imperative though to define these convex and concave regions explicitly in order to employ them in the numerical implementations and be consistent with the arguments of Baucells and Heukamp (2006). Both these probability weighting functions were introduced in Kahneman and Tversky (1992) and were extracted through psychological experiments on individuals by taking pure bets. A bet can take two forms: pure or mixed. A pure bet is the one where the gambler can attain either gains or losses. A mixed bet is the one where gains and losses are both candidate outcomes. Apparently, in the traditional bond, stock or exchange market or in the more recent cryptocurrencies market,

mixed bets prevail. However, Ingersoll (2008) argues that the certain values of the exponents in the p.w. functions can induce non monotonicity. This can lead to negative decision weights and the preference for first-order stochastically dominated prospects. The problem is tackled if both exponents of w^+ and w^- are strictly above 0.279, thus in our case monotonicity is satisfied. Moreover, we can define the convex as well as the concave regions too. By starting from zero and up to the dashed green vertical line ($x = 0.68$) we have the concave segments and from the black dashed vertical line ($x = 0.24$) up to 1 we have the convex segments for both w^+ and w^- . We also have that $w^-(p) = w^+(p) \Leftrightarrow p \approx 0.23$ for $p \in (0, 1)$ and in PT we have that $w^-(p) \equiv w^+(p)$. The condition for these vertical lines is that they have to include between them an approximately linear segment. However, the curvature of both p.w.f. (inflection point) changes at the critical value of $p_{crit} \in (0.3, 0.4)$.

p.w.f.	concave segment	convex segment
w^+	w_{cv}^+	w_{cx}^+
w^-	w_{cv}^-	w_{cx}^-

Table 2: p.w.f. and associated curvature

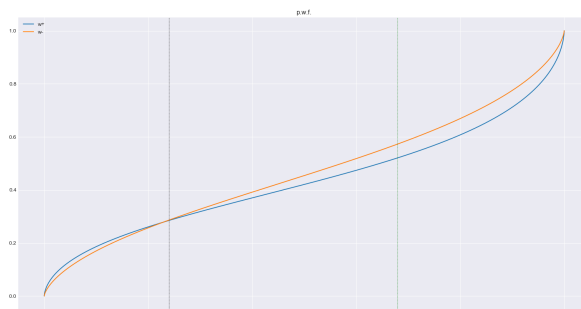


Figure 4: probability weighting functions for gains and losses with the vertical lines indicating the convex and the concave segments

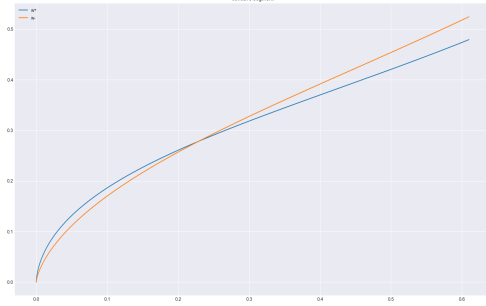
These segments are illustrated in Figure 5 and are denoted as indicated in Table 3 where concavity declares over-weighting and convexity under-weighting. The in-between almost linear segment declares indifference for moderate probabilities. For mixed prospect, the sum of w^- , w^+ can be either smaller or greater than 1. The p.w.functions form the so-called capacities that are non additive set functions that generalize the standard notion of probabilities. In general, we can argue that probability is measuring the individual degree of belief about the occurrence of a specific event. Another definition for these pwf is "capacities" for the relative events, following KT (1992).

More specifically, pwf can be distinguished into two cases: one for risk and one for uncertainty. In the case of risk we have that the pwf are continuous and increasing functions with domain and range the compact set $[0, 1]$. In the case of uncertainty we have that besides continuity and increasing monotonicity, "solvability" also holds. Thus, we say that a pwf is "solvable" if:

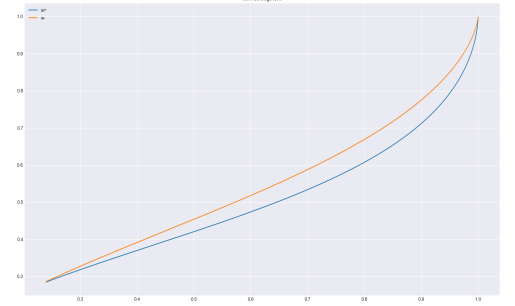
$$\forall A \subset B \quad \text{and} \quad W(A) \leq p \leq W(B)$$

there

$$\exists C \quad \text{such that} \quad W(C) = p \quad \text{and} \quad A \subset C \subset B$$



(a) concave segment



(b) convex segment

Figure 5: The concave and convex segments of w^+ and w^-

value function	p.w.f.
v_{cx}^P	w_{cx}^-
v_{cv}^P	w_{cx}^+
v_{cv}^M	w_{cv}^-
v_{cx}^M	w_{cv}^+

Table 3: combinations of the value function and the associated p.w.f. segment

In order to attain computational simplicity without altering the economic notions and through mathematical expressions, we proceed to the following transformations. We want to provide the concave and the convex segments of the p.w.f. and if we would take the segments as illustrated in Figure 5 we would restrict the associated domain for probabilities. Our goal is to provide p.w.f. with domain and range the compact set $[0, 1]$. Hence, we tackle this problem by introducing continuous, globally concave and convex functions and we derive their compositions with the p.w.f. We assume the following functions:

$$g_1 : [0, 1] \mapsto [0, 1] \quad \text{concave and weakly increasing}$$

and

$$g_2 : [0, 1] \mapsto [0, 1] \quad \text{convex and weakly increasing}$$

and

$$g_3 : [0, 1] \mapsto [0, 1] \quad \text{concave and weakly decreasing}$$

and

$$g_4 : [0, 1] \mapsto [0, 1] \quad \text{convex and weakly decreasing}$$

Thus, we obtain the following compositions by taking into concern the inflection point (p_{crit}) of the p.w.f. for gains and losses respectively:

$$h_1 = (g_1ow^+)(p), \quad p \in [0, p_{crit}] \quad : \quad \text{concave and increasing}$$

and

$$h_2 = -(g_3ow^+)(p), \quad p \in [p_{crit}, 1] \quad : \quad \text{concave and increasing}$$

then

$$w_o^+ \equiv \left\{ \begin{array}{l} h_1, \quad p \in [0, p_{crit}] \\ \text{and} \\ h_2, \quad p \in [p_{crit}, 1] \end{array} \right\} : [0, 1] \mapsto [0, 1] \quad \text{continuous, concave and increasing}$$

similarly,

$$h_3 = -(g_4ow^-)(p), \quad p \in [0, p_{crit}] \quad : \quad \text{convex and increasing}$$

and

$$h_4 = (g_2ow^-)(p), \quad p \in [p_{crit}, 1] \quad : \quad \text{convex and increasing}$$

then

$$w_o^- \equiv \left\{ \begin{array}{l} h_3, \quad p \in [0, p_{crit}] \\ \text{and} \\ h_4, \quad p \in [p_{crit}, 1] \end{array} \right\} : [0, 1] \mapsto [0, 1] \quad \text{continuous, convex and increasing}$$

5.1 Decision weights and capacities

If the prospect $f(x_i, A_i)$, which must be considered more generally as the vertices of a polyhedral choice set, is given by the probability distribution $p(A_i) = p_i$, it can be viewed as the probabilistic or risky prospect (x_i, p_i) . In this case, decision weights are defined as:

$$\pi_n^+ = w_o^+(p_n) \quad \text{and} \quad \pi_{-m}^- = w_o^-(p_{-m})$$

$$\pi_i^+ = w_o^+(p_i + \dots + p_n) - w_o^+(p_{i+1} + \dots + p_n) \quad \text{where} \quad 0 \leq i \leq n-1$$

and

$$\pi_i^- = w_o^-(p_{-m} + \dots + p_i) - w_o^-(p_{-m} + \dots + p_{i-1}) \quad \text{where} \quad 1-m \leq i \leq 0$$

we also evaluate the value function (concave-convex or convex-concave) together with the decision weights, for gains and losses, in the following way:

$$V(f) = V(f^+) + V(f^-) = \sum_{i=0}^n \pi_i^+ v(x_i) + \sum_{i=-m}^0 \pi_i^- v(x_i)$$

$V(f)$ is the overall value of a mixed prospect, π_i^+ denotes that "the outcome is at least as good as x_i " minus "the outcome is strictly better than x_i ". On the other hand, π_i^- denotes that "the outcome is at least as bad as x_i " minus "the outcome is strictly worse than x_i ".

In the case where no restrictions are placed on the pwf, then all SD conditions reduce to FSD. Also, knowledge of the value function together with an unrestricted pwf, is insufficient to predict preferences between stochastically undominated prospects. Hence, a pwf with certain constant curvatures is able to extend the known forms of SD to the CPT and Markowitz framework (Bau-cells and Heukamp, 2006).

Lemma 3. We have that

$$p(F) = \max_{i=1,2 \wedge j=+,-} \sup_{v_w^M \in V_j^M} \left[\sup_{\lambda \in \Lambda} E_\lambda [1_{u \in A_i} v_w^M(u)] - \sup_{\kappa \in K} E_\kappa [1_{u \in A_i} v_w^M(u)] \right] \quad (19)$$

where E_λ denotes the expectation w.r.t $G(u, \lambda; F)$.

If the hypotheses of Lemma 1 hold and K is convex, then $\kappa \succeq_M \lambda$ iff,

$$\sup_{v \in V^M} \left[\sup_{\lambda \in \Lambda} E_\lambda [v] - \sup_{\kappa \in K} E_\kappa [v] \right] = 0 \quad (20)$$

Lemma 4. We have that

$$p(F) = \max_{i=1,2 \wedge j=+,-} \sup_{v_w^P \in V_j^P} \left[\sup_{\lambda \in \Lambda} E_\lambda [1_{u \in A_i} v_w^P(u)] - \sup_{\kappa \in K} E_\kappa [1_{u \in A_i} v_w^P(u)] \right] \quad (21)$$

where E_λ denotes the expectation w.r.t $G(u, \lambda; F)$.

If the hypotheses of Lemma 2 hold and K is convex, then $\kappa \succeq_P \lambda$ iff,

$$\sup_{v \in V^P} \left[\sup_{\lambda \in \Lambda} E_\lambda [v] - \sup_{\kappa \in K} E_\kappa [v] \right] = 0 \quad (22)$$

6 Statistical Tests for Markowitz and Prospect Spanning

Taking into consideration all the above and by following Arvanitis et.al. (2017), we can form the test statistic of stochastic spanning. We also employ the empirical distribution function F_T , which is a consistent estimator of F , where F is the cdf of Y_0 . In empirical applications the cdf F and the temporal dependence of the underlying process are latent. The empirical cdf is associated with the random element $(Y_t)_{t=1, \dots, T}$. One problem with cdf F is that it can be unknown and another is that the optimizations may be infeasible. Hence, its empirical estimate tackles these problems and the resulted optimizations, via Linear Programming (LP), are feasible. Thus, the following random variable, which is a scaled empirical analogue of $p(F)$, plays the role of the test statistic.

$$p_T := \sqrt{T}p(F_T) = \sqrt{T} \max_{i=1,2,3,4} \sup_{\lambda \in \Lambda} \sup_{z \in A_1, A_2} \inf_{\kappa \in K} P_i(z, \lambda, \kappa, F_T) \quad (23)$$

where, the null and the alternative hypotheses of the asymptotically exact, feasible and consistent statistical test takes the following forms:

$$H_0 : p_T = 0 \quad (24)$$

and

$$H_A : p_T > 0 \quad (25)$$

The empirical joint cumulative distribution function which is constructed from our sample and is the unconstrained maximum likelihood (ML) estimate of F is :

$$F_T = T^{-1} \sum_{t=1}^T 1(x_t \leq x)$$

where,

$$1(x_t \leq x) = \begin{cases} 1 & \text{if } x_t \leq x \\ 0 & \text{if } x_t > x \end{cases}$$

We assume that the return distribution is a latent stochastic process with continuous CDF $F : R^N \rightarrow [0, 1]$ with a finite covariance matrix of full rank (N). If the null hypothesis holds, spanning occurs. If the alternative hypothesis holds, spanning does not occur. In general, we can say that the null hypothesis holds if the enlargement (or reduction) of the choice set does not change the efficient set which is subset (not necessarily a proper one) of the choice set. Moreover, it would be useful to reformulate the test statistic in terms of expected utility. We do so in order to include the two different forms of the utility functions and the probability weighting functions. Thus,

$$p_T := \sqrt{T} \sup_{\lambda \in \Lambda; v \in V^M, V^P; w \in W^-, W^+} \inf_{\kappa \in K} E_{F_T} \left[v(X^T \lambda) - v(Y^T \kappa) \right] \quad (26)$$

where,

$$Y \subset X$$

In other words, X is the augmented portfolio containing traditional assets and cryptocurrencies, while Y (henceforth traditional portfolio) is the one containing only traditional assets. Hence, we are interested in testing whether Y spans X .

In order to combine stochastic spanning with the related family of utility and probability weighting functions we have the following descriptions. Stochastic spanning occurs, in the case of S-shaped utility-value function together with inverse S-shaped probability weighting function, if no satiable, risk averse (for gains) and risk seeker (for losses) investor benefits from the enlargement of the traditional portfolio. On the other hand (reverse S-shaped value function and inverse p.w.f.), we have that stochastic spanning occurs if no satiable, risk averse (for losses) and risk seeker (for gains) investor benefits from the enlargement of the traditional portfolio.

7 Subsampling Procedure : In-sample Analysis

This consistent subsampling procedure is formulated in order to provide the critical values for the rejection areas of the test statistic. This rejection areas are obviously related with the null and the alternative hypotheses of the statistical test. We do so by employing block bootstrapping with overlapping blocks of data of daily closing prices. We choose the length of each block by the rule : $l = [T^{\frac{1}{3}}]$, where the brackets indicate the closest integer which is not greater than the number x , i.e. $[x] \leq x \leq [x] + 1 \quad \forall x \in R$ (e.g. $[2.3] = 2$), and T is the sample size. We then resample the blocks and generate the bootstrap sample. The number of blocks in our sample is $T - l + 1$. Lahiri (1999) and Andrews (2002) argue that there is little difference in the performance between the overlapping and the non overlapping method of bootstrapping. In the non overlapping case, the data are divided into T/l blocks. Afterwards, we compute the test statistic for each block - subsample. The distribution of subsample test scores can be described by the following cdf and quantile function, where $a \in [0, 1]$ is the significance level (usually at 0.05):

$$S_{T,l}(y) = (T - l + 1)^{-1} \sum_{t=1}^{T-l+1} 1(p_{l;T,t} \leq y) \quad (27)$$

and

$$q_{T,l}(1 - a) = \text{inf}_y \left\{ y : S_{T,l}(y) \geq 1 - a \right\} \quad (28)$$

In our in-sample analysis, we assume that the decision rule is to reject the null $H_0 : p_T = 0$ against the alternative $H_1 : p_T > 0$ at significance level a , if and only if $p_T > q_{T,l}(1 - a)$, or equivalently, if and only if $1 - a < S_{T,l}(p_T)$.

In our first experiment, we find that the value for the test statistic for the CPT case is 0.0726 and for the Markowitz case is 0.2034. Thus, we reject the null hypothesis, that the traditional asset class spans the augmented asset class with cryptocurrencies in favor of the alternative , because we find that $S_T^{CPT} = 0.956 > 0.95$ and $S_T^{Markowitz} = 0.99 > 0.95$.

In our second experiment (Market trend as an index), we find that the value for the test statistic for the CPT case is 0.975 and for the Markowitz case is 0.976 . Thus, we reject the null hypothesis, that the traditional asset class spans the augmented asset class with cryptocurrencies in favor of the alternative , because we find that $S_T^{CPT} = 0.99 > 0.95$ and $S_T^{Markowitz} = 0.99 > 0.95$.

However, the procedure above provides us with a test that has asymptotically correct size but empirical applications reveal that the quantile estimates may be biased and sensitive to the subsample size l in finite samples or realistic dimensions. In order to avoid these predicaments, a regression-based bias-correction method is proposed, following Arvanitis et al. (2017).

8 Backtesting : Out-of-sample Analysis

Backtesting is the process of testing a trading strategy on relevant historical data to ensure its viability before the investor risks any actual capital. An investor can simulate the trading of a strategy over an appropriate period of time and analyze the results for the levels of profitability and risk. Backtesting is a form of analysis that allows us to look backward on history and trade a strategy against historical data to see how we did. In our case, our model is tested on the entire period for experiment 1 (450 days span), where gains and losses are separated daily, and again on the whole period for experiment 2, but now gains and losses are distinguished on the basis of how the market is performing (market trend as an index). In simple words, when the market is trending upwards we assume gains and when the market is trending downwards we assume losses. The aforementioned analysis is conducted for both portfolios and for both investor types. One significant issue that has to be noted is that, while the traditional assets (e.g. S&P500 fund index) are traded weekly as long as the stock market is open (business days), cryptocurrencies can be traded around the clock, 7 days a week, 365 days a year. It is a non stopping procedure. This important difference affects prices, and subsequently investment strategies, but for the sake of simplicity we treat cryptocurrencies in the same way as traditional assets. Thus, only weekdays closing prices are incorporated and over-the-weekend prices are omitted.

8.1 Experiment 1 : the entire period

The backtesting procedure of the first experiment involves the whole sample which contains 664 business days observations (August 2016 up to March 2018) for the traditional assets and the cryptocurrencies. In order to align our tests with the theory of Prospect Stochastic Dominance and Markowitz Stochastic Dominance, we separate gains and losses daily and apply the relevant theoretical framework in the numerical implementations.

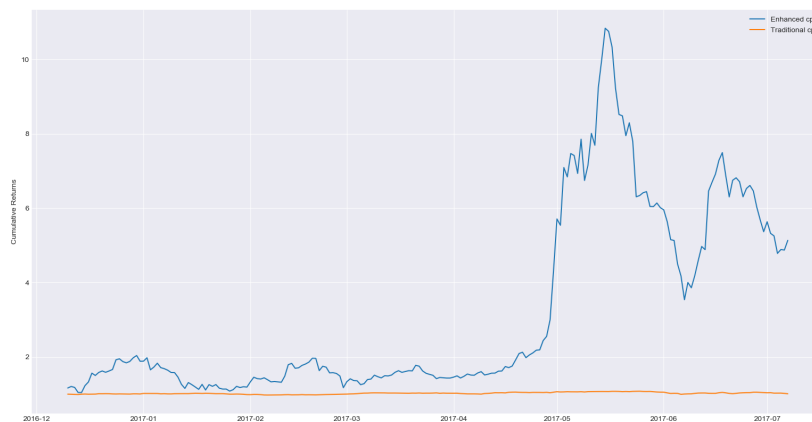


Figure 6: CPT backtesting for the entire period

As we can see in the back testing graphs (Figures 7 and 6), concerning the Markowitz type of

investors (analogously for Prospect type), the augmented portfolio, during the first days, performs better than the traditional one but not in an outstanding way. This result has to do with the fact that during that period, cryptocurrencies had not started their rally yet. When this rally started, the returns of the the portfolios started to differ dramatically. This exponential growth in the prices of cryptocurrencies created a boost in the returns of the augmented portfolio. Hence, the overall result is that the traditional portfolio does not span the augmented one, and this outcome finds support even in the mimicking of "real time" trading strategy. Our result seems apparent, due to the fact that during this period, cryptocurrencies have revealed one of the most outstanding returns in investment history. Thus, if a Markowitz type of investor was to place 1\$ in the augmented portfolio in December 2016, she would realize a growth of 700% (8 \$) in May 2017 an would end up having a growth of 350% (4.5 \$) in July 2017. Analogously, a Prospect type of investor would obtain a growth of 950% (10.5 \$) in May 2017 an would end up having a growth of 450% (5.5 \$) in July 2017. One could argue that Markowitz type of investors underperform Prospect ones.

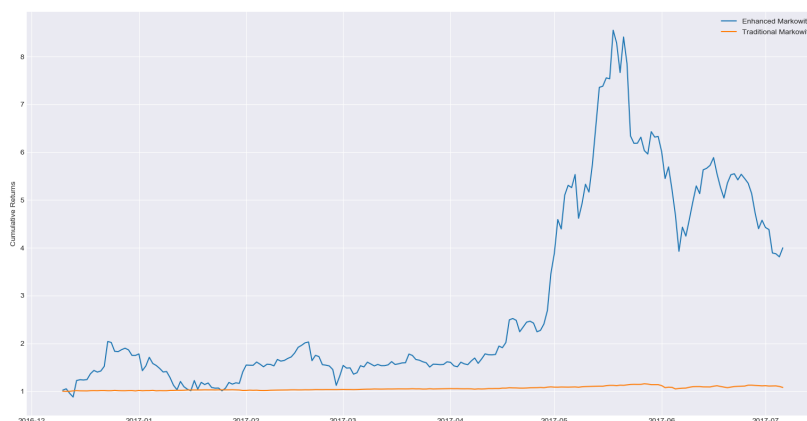


Figure 7: Markowitz backtesting for the entire period

8.2 Experiment 2 : market trend as an index

The backtesting procedure (400 days rolling window) of the second experiment involves the whole sample which contains 664 business days observations (August 2016 up to March 2018) for the traditional assets and the cryptocurrencies. In order to align our tests with the theory of Prospect Stochastic Dominance and Markowitz Stochastic Dominance, we separate gains and losses by taking the market trend as a benchmark. This means that when the market is up trending, we assume that investors are in euphoria and this is being realized as gains. On the other side, when the market is down trending, we assume that investors regard this as losses. Thus, the whole period is separated into 4 sub periods as illustrated in Figure 3. Hence, we apply the relevant theoretical framework in the numerical implementations for the two investors' types. These four subsamples are defined based on the market growth. We choose the greater peak and the lowest point that

precedes in order to define the uptrending (Bull) market and the down trending (Bear) market.

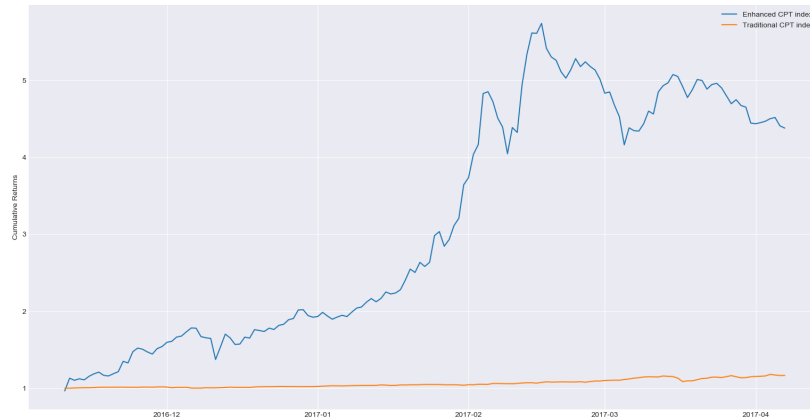


Figure 8: CPT backtesting for the entire period, market as index

As we can see in the back testing graphs (Figures 8 and 9), concerning the Markowitz type of investors (analogously for Prospect type), the augmented portfolio, during the first days, performs better than the traditional one but not in an outstanding way. This result has to do with the fact that during that period, cryptocurrencies had not started their rally yet. When this rally started, the returns of the the portfolios started to differ dramatically. This exponential growth in the prices of cryptocurrencies created a boost in the returns of the augmented portfolio. Hence, the overall result is that the traditional portfolio does not span the augmented one, and this outcome finds support even in the mimicking of "real time" trading strategy. Thus, if a Markowitz type of investor was to place 1\$ in the augmented portfolio in October 2016, she would realize a growth of 350% (4.5 \$) in April 2017 an would end up having a growth of 150% (2.5 \$) in March 2018. Analogously, a Prospect type of investor would obtain a growth of 450% (5.5 \$) in January 2018 an would end up having a growth of 150% (2.5 \$) in March 2018. One could argue that again, Markowitz type of investors underperform Prospect ones.

9 Out-of-sample Performance Assessment

In this section we compare the out - of - sample performance of the two optimal portfolios namely, Traditional and Enhanced (or as before Augmented), formed by the respective two asset sets by using both non-parametric and parametric tests.

9.1 Non-parametric Tests

There is a significant number of pairwise stochastic dominance tests presented in the literature as for example Linton et al. (2005), Barret and Donald (2003) etc. For our non parametric tests, we employ the Davidson and Duclos (2013) stochastic dominance test, because it allows for correlated

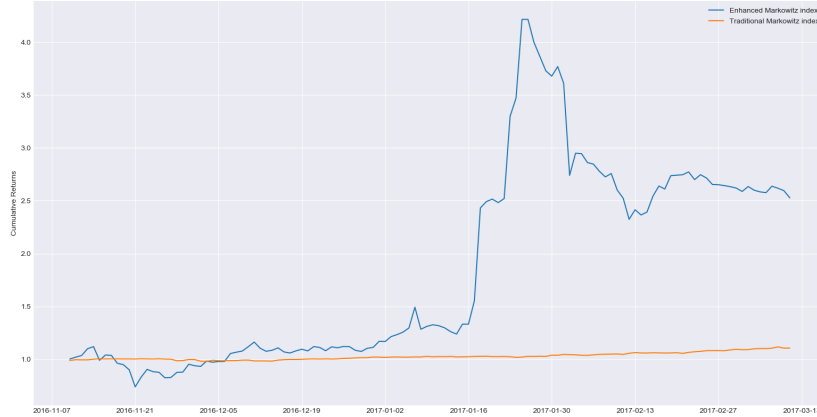


Figure 9: Markowitz backtesting for the entire period, market as index

samples and has as the null hypothesis nondominance (one portfolio does not stochastically dominates another), where the majority of the literature posits as the null the dominance. While, the rejection of dominance of one portfolio does not imply that other necessarily stochastically dominates. Hence, if we succeed in rejecting this null, we may then infer the only other possibility which is dominance, as in Davidson and Duclos (2013). In simple words, positing a null of nondominance cannot serve to infer nondominance but it can lead to the inference of dominance and they also find that the statistical testing procedures are simpler to implement than conventional procedures in which the null is dominance. Thus, we test the null of nondominance, by using second order SD criteria, of the optimal augmented portfolio over the optimal traditional one. The alternative hypothesis is that the optimal augmented portfolio stochastically dominates the optimal traditional one.

We define the empirical versions of the dominance functions, as in Davidson and Duclos (2000), in the following way:

$$\hat{D}_i^2(z) = \frac{1}{N_i} \sum_{j=1}^{N_i} (z - y_{j,i}, 0)_+ \quad ; \quad i = Tr, Aug$$

where N_i is the number of observations in the distribution sample of traditional/augmented portfolio returns, with $y_{j,i}$ being the j -th observation of the i -th sample, z the threshold of interest and the "hat" indicates the natural estimator of $D^2(z) = \int_{-\infty}^z (z - y) dF(y) = \int_R (z - y)_+ dF(y)$. Then, we can say that the augmented portfolio SSD the traditional one iff, $\hat{D}_{Tr}^2 > \hat{D}_{Aug}^2$ for all y in the joint (union) support. The null hypothesis of nondominance is not rejected unless there is dominance in the sample.

For each threshold level z , let the standardized difference of the two dominance functions form the following test statistic and in order to simulate the p-values, we use the same bootstrap methodology as in Davidson and Duclos (2013),

$$t^* = \min_z t(z) = \min_z \left\{ \frac{\hat{D}_{Tr}^2 - \hat{D}_{Aug}^2}{\left[\hat{Var}(\hat{D}_{Tr}^2) + \hat{Var}(\hat{D}_{Aug}^2) - 2\hat{Cov}(\hat{D}_{Tr}^2, \hat{D}_{Aug}^2) \right]^{1/2}} \right\}$$

where, Var and Cov are the variance and covariance of the estimated dominance functions respectively and they can be computed as follows,

$$\hat{Var}(\hat{D}_i^2(z)) = \frac{1}{N_i} \left(\frac{1}{N_i} \sum_{j=1}^{N_i} (z - y_{j,i}, 0)_+^2 - (\hat{D}_i^2(z))^2 \right) \quad ; \quad i = Tr, Aug$$

and,

$$\hat{Cov}(\hat{D}_{Tr}^2(z), \hat{D}_{Aug}^2(z)) = \frac{1}{N} \left\{ \frac{1}{N} \sum_{j=1}^N (z - y_{j,Tr}, 0)_+ \cdot (z - y_{j,Aug}, 0)_+ - \hat{D}_{Tr}^2(z) \cdot \hat{D}_{Aug}^2(z) \right\}$$

with,

$$N = N_{Tr} = N_{Aug}$$

SSD of the traditional portfolio by the augmented portfolio (for both experiments, for both investor types) implies that $t^* > 0$ always, including its smallest value. Thus, in order to test the null hypothesis that the augmented portfolio *does not SSD* the traditional one, we need only to focus on the smallest number of $t(z)$. The larger the value of t^* , the higher the likelihood of rejecting the null; and thus, the higher is the likelihood of the augmented portfolio dominating the traditional one. Finally, we have that the test statistic t^* is asymptotically normally distributed and the associated p-values are determined through bootstrapping, if the sample size is large enough, which in our case is. In order to derive meaningful test statistics and the associated p-values, the set of thresholds z includes all unique observations from the two samples (traditional and augmented) where there is at least one observations in each sample below $\min(z)$ and at least one above $\max(z)$. These unique observations lie in the joint support of the two samples. Following, Hodder et al. (2014) we can trim by 20% (10% tail cut-off for the largest and smallest returns) the joint support, in order to obtain more powerful tests. Hodder et al. argue that although the tests are more powerful but less informative regarding the tails of the two portfolios' returns distributions, an even lower cut-off level leaves their initial results unaffected. The aforementioned approach is fully described in the related appendix of their paper.

We also rely on the statistical tests developed in Scaillet and Topaloglou (2010), which can be also implemented in a non-parametric setting. The p-values of the tests are presented in Table 7. The null hypothesis in all cases is that the Augmented portfolio stochastically dominates the Traditional one by second order (SSD). The second order SD is a prerequisite in order to for the results to be aligned with the results of second order Stochastic Spanning. All resulting p-values, though block bootstrapping with overlapping blocks, are above the predefined significance level of 5%. Hence, the null hypothesis cannot be rejected and as expected, the Augmented portfolio SSD the Traditional one.

9.2 Parametric Tests

We compute a number of commonly used parametric performance measures: the Sharpe ratio, the downside Sharpe ratio (DS)(Ziemba, 2005), the upside potential (UP) and downside risk ratio (Sortino and van den Meer, 1991), the opportunity cost (Simaan, 2013), the portfolio turnover (P.T.) and a measure of the portfolio risk-adjusted returns net of transaction costs(RL). Due to the fact that cryptocurrencies exhibit asymmetric return distributions, the downside Sharpe and UP ratios are more appropriate measures than the typical Sharpe ratio.

For the DS ratio, we first need to calculate the downside risk (downside variance) which is given by the formula:

$$\sigma_{P-}^2 = \frac{\sum_{t=1}^T (\min(x_t, 0))^2}{T-1}$$

where, x_t are those returns of portfolio P at day t below 0, i.e. those days with losses. To get the total variance we use: $2\sigma_{P-}^2$, thus the DS ratio is,

$$S_P = \frac{\bar{R}_P - \bar{R}_f}{\sqrt{2\sigma_{P-}^2}}$$

where, \bar{R}_P is the average period return of portfolio P and \bar{R}_f is the average risk free rate.

The UP ratio compares the upside potential to the shortfall risk over a benchmark and is computed as follows. Let R_t be the realized daily return of portfolio P for $t = 1, \dots, T$ of the backtesting period, where T is the number of experiments performed and let p_t be respectively the return of the benchmark, which in our case is the T-bill riskless asset for the same period. Then we have,

$$UP \text{ ratio} = \frac{\frac{1}{T_1} \sum_{t=1}^{T_1} \max(R_t - p_t, 0)}{\sqrt{\frac{1}{T_2} \sum_{t=1}^{T_2} (\max(p_t - R_t, 0))^2}}, T = T_1 + T_2$$

The numerator of the above ratio is the average excess return over the benchmark and thus it reflects the upside potential. In the same sense, the denominator measures downside risk, i.e. shortfall risk over the benchmark.

Next, we compute the P.T. to get a feeling of the degree of rebalancing required to implement each one of the the four investment strategies. For any portfolio strategy P , the portfolio turnover is defined as the average of the absolute change of weights over the T rebalancing points in time and across the M available assets, i.e.

$$P.T. = \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^M (|w_{P,i,t+1} - w_{P,i,t}|)$$

where $w_{P,i,t+1}$, $w_{P,i,t}$ are the optimal weights of asset i under strategy P (traditional or augmented) at time t and $t+1$, respectively.

We also evaluate the performance of the four portfolios (two experiments) under the risk-adjusted returns measure, which is net of transaction costs, proposed by DeMiguel et al. (2009). It indicates the way that the proportional transaction cost, generated by the P.T., affects the portfolio returns. Let trc be the proportional transaction cost, and $R_{P,t+1}$ the realized return of portfolio P at time $t+1$. The change in the net of transaction cost wealth NW_P of portfolio P through time is,

$$NW_{P,t+1} = NW_{P,t}(1 + R_{P,t+1}) \left(1 - trc \times \sum_{i=1}^M (|w_{P,i,t+1} - w_{P,i,t}|) \right)$$

The portfolio return, net of transaction cost, is defined as,

$$RTC_{P,t+1} = \frac{NW_{P,t+1}}{NW_{P,t}} - 1$$

Let μ_{Tr}, μ_{Aug} be the out-of-sample mean of monthly RTC with the traditional and augmented opportunity set, respectively, and $\sigma_{Tr}, \sigma_{Aug}$ be the corresponding standard deviations. Then, the return-loss measure is,

$$R_{Loss} = \frac{\mu_{Aug}}{\sigma_{Aug}} \times \sigma_{Tr} - \mu_{Tr}$$

It evaluates the additional return needed so that the traditional portfolio performs equally well with the augmented one. We follow the literature and use 35 basis points (bps), i.e. 0.35% , for the proportional transaction cost of stocks and bonds. For cryptocurrencies, we follow Anyfantaki et al. (2018) and use the same high proportional transaction costs of 50 bps.

Finally, we use the concept of opportunity cost presented in Simaan(2013) to analyze the economic significance of the performance difference of the two optimal portfolios, in both experiments and for both investor types. Let R_{Tr}^i and R_{Aug}^i be the realized returns of the optimal traditional and augmented portfolio of investor $i = CPT, Markowitz$. Then, the opportunity cost θ is defined as the return that needs to be added to (or subtracted from) R_{Tr}^i , so that the investor is indifferent (in utility terms) between the strategies imposed by the two different investment opportunity classes.

$$E \left[U(1 + R_{Tr}^i + \theta) \right] = E \left[U(1 + R_{Aug}^i) \right] \quad ; i = CPT, Markowitz$$

A positive opportunity cost implies that an investor is better off if she includes cryptocurrencies in her portfolio, while a negative one implies that she would be worse off with the aforementioned inclusion. It is important to mention that the opportunity cost takes into account the entire probability distribution of portfolio returns and hence it is suitable to evaluate strategies even when the distribution is not normal. For the calculation of the opportunity cost we follow the literature and use the relevant *S-shaped* and *inverse S-shaped* utility functions, consistent with SSD.

S-shaped utility function for Cumulative Prospect type of investor:

$$U^P(x) = \left\{ \begin{array}{ll} x^a & \text{if } x \geq 0 \\ -c(-x)^b & \text{if } x < 0 \end{array} \right\}$$

where c is the coefficient of loss aversion (usually $c = 2.25$) and $a, b < 1$.

S-shaped utility function for Markowitz type of investor:

$$U^M(x) = \left\{ \begin{array}{ll} x^a & \text{if } x \geq 0 \\ -c(-x)^b & \text{if } x < 0 \end{array} \right\}$$

where again, c is the coefficient of loss aversion (usually $c = 2.25$) and $a, b > 1$.

We fix parameters a, b at 0.5, for the CPT case, and at 2 for Markowitz. The usage of the aforementioned utility functions is not binding, hence any other type of utility function that would satisfy the curvature required for the two cases could be used.

Tables 5 and 6 report the parametric performance measures for the traditional and the augmented portfolios, for the two investor types, in the two experiments and although they are parametric they enrich the evidence obtained from the non-parametric SD measure. The higher the value of each one of these measures, the greater the investment opportunity for cryptocurrencies. We can see that the inclusion of cryptocurrencies increases both the Sharpe ratio and the downside Sharpe ratio in both experiments. This reflects an increase in expected return per unit of risk and hence expands the investment opportunities for investors who exhibit risk aversion. Similar results also hold for the UP ratio. Furthermore, we observe that in our first experiment both traditional portfolios induce more portfolio turnover than the ones enhanced by cryptocurrencies, while in our second experiment this is true only in the case of the Markowitz type of investor. Moreover, we can see that the return-loss measure, which takes into account transaction costs, is positive in all cases for both experiments and so is the opportunity cost θ . Thus, a positive return equal to θ must be given to an investor if she does not include cryptocurrencies in her portfolio and wants to derive the same utility with an investor who does. Since the computation of the opportunity cost is done through expected utility, the whole probability distribution of the portfolios' returns (respectively) is taken into account and hence, this measure takes into account higher order moments in contrast to Sharpe's ratios. Therefore, the opportunity cost estimates provide stronger support for possible diversification benefits from the inclusion of cryptocurrencies to the investment strategies, in both experiments and for both investor types, since there exists large deviation from normality. To sum up, the aforementioned parametric performance measures and the non-parametric SD test, indicate that when the investment opportunity set is enriched with cryptocurrencies it empirically dominates the traditional one, yielding diversification benefits and providing investment opportunities.

10 Conclusions

Blockchain, and its byproduct cryptocurrencies, is one of the most important technological innovations in recent times. It succeeded in developing a rapidly growing and secure financial market,

counting already billions of invested dollars. Its high gains and losses initiated the idea of implementing some specific investors' behavioral patterns and observe their outcomes. Our findings provide strong support that when investors are not globally risk averse but they rather exhibit more complex behavioral patterns they succeed in obtaining significant returns. These returns, if it was not for cryptocurrencies and their rally in 2017/2018, could not be otherwise justified. However, it is a market that must be treated with extreme caution due to fact that it has incurred severe losses too. We provide evidence that when the investors' behavior, and subsequently her choices and thus her investment strategy is adaptable, she than can outperform by far the market and also acquire substantial diversification benefits. She can perform better if she includes in her opportunity set cryptocurrencies and then the portfolio she optimally forms is not dominated, or more accurately spanned, by the traditional one.

In our empirical experiments-applications we used actual business daily closing prices for the traditional assets as well as for the cryptocurrencies, by excluding for the latter weekend closing prices. Our period covers from August 2016 to March 2018. We constructed optimal portfolios, for the two investors' types namely Prospect and Markowitz. In our first experiment we have as the benchmark of investor behavior the trend of the overall market and in the second we treat each day separately by separating gains and losses. In both experiment the Augmented portfolio (enhanced with cryptocurrencies) not only outperforms the market portfolio but it also outperforms the Traditional one (consisting merely of bonds, stocks, equities etc.).

Investors can be either risk averse for gains and risk seeking for losses (Prospect), or vice versa (Markowitz). These two investor types are implemented in our analysis via S-shaped and reverse S-shaped value functions combined with relevant inverse S-shaped probability weighting functions. We conduct our analysis both in and out of sample by constructing and comparing optimal portfolios, for two investor types and for two different ways of how gains and losses are being understood on behalf of investors.

In the in-sample tests, we find that the augmented portfolios are not spanned by the traditional ones for both investor types. In the out-of-sample performance, using a non-parametric stochastic dominance test as well as parametric performance measures, we find that the expanded investment universe with cryptocurrencies empirically dominates the traditional one. Hence and overall, investment opportunities emerge accompanied with diversification benefits (i.e. low positive or negative correlations with the traditional assets) when cryptocurrencies are included in the aforementioned investment strategies.

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11 Appendix

Numerical Implementation and Computational Strategy

11.1 Experiment 1

This computational strategy is appropriate for the enlargement (which is regarded as large) of the K 7-simplex (traditional portfolio) to the Λ 11-simplex (augmented portfolio) but the return range $[x_{min}, x_{max}]$ is rather limited. We denote the expectation w.r.t. the empirical measure E_{F_T} . Let $R_- = \max_{i=1, \dots, n} \text{range}(X_{t_i} 1_{X_{t_i} \leq 0})_{t=1, \dots, T}$ denote the losses domain $[x_{min}, 0]$, which is partitioned into n_1 equally spaced values as $x_{min} = z_1 < \dots < z_{n_1} = 0$, where $z_n = \frac{n_1-n}{n_1-1} x_{min}$, $n = 1, \dots, n_1$; $n_1 \geq 2$. Similarly, $R_+ = \max_{i=1, \dots, n} \text{range}(X_{t_i} 1_{X_{t_i} \geq 0})_{t=1, \dots, T}$ denotes the gains domain, $[0, x_{max}]$, which is also partitioned into q_1 equally spaced values as $0 = q_1 < \dots < q_{q_1} = x_{max}$, where $q_n = \frac{n-1}{q_1-1} x_{max}$, $n = 1, \dots, q_1$; $q_1 \geq 2$. Moreover, gains and losses are separated daily and this affects the probability weighting functions (pwf) as indicated below.

Using the above, we consider the test statistic to be the following:

$$p_{T_i} = \sqrt{T} \sup_{v \in V_i} \left(\sup_{\lambda \in \Lambda} E_{F_T} \left[v(\lambda^T X) \right] - \sup_{\kappa \in K} E_{F_T} \left[v(\kappa^T Y) \right] \right); i = -, + \quad (29)$$

The term in the parenthesis is the difference between two convex optimization problems of maximizing a quasi-concave objective function over a polyhedral feasible set. The computational complexity stems from the search over all admissible value functions for Prospect and Markowitz, however this problem is tackled with the introduction of piecewise linear utility functions. Because of their linearity they can be treated either as convex or concave and together with the associated probability weighting functions they form the value functions below, for gains and losses. Moreover, the utility functions are normalized, univariate and have a bounded domain. Thus, they can be approximated with arbitrary accuracy of increasing concave and/or convex piece-wise linear functions. These linear utility functions are constructed of at most $N_1 - 1$ linear line segments with knots at N_1 possible outcomes.

$$V_- := \left\{ v \in C^0 : v(y) = \sum_{n=1}^{n_1} \pi^-(p) r_-(z_n; y) \quad , p \in W^- \right\} \quad (30)$$

and

$$V_+ := \left\{ v \in C^0 : v(y) = \sum_{n=1}^{q_1} \pi^+(p) r_+(a_n; y) \quad , p \in W^+ \right\} \quad (31)$$

with,

$$r_-(z_n; y) = \left\{ \begin{array}{ll} \min \{z_n - y, 0\} & \text{for Prospect} \\ \text{or} \\ \max \{z_n - y, 0\} & \text{for Markowitz} \end{array} \right\}$$

and

$$r_+(a_n; y) = \left\{ \begin{array}{ll} \min \{a_n - y, 0\} & \text{for Markowitz} \\ \text{or} & \\ \max \{a_n - y, 0\} & \text{for Prospect} \end{array} \right\}$$

In the formulations above, V_+ is a set of normalized, increasing and concave (convex) utility functions that are constructed as convex mixtures of elementary ramp functions $r_+(a_n; y), (a_n, y) \in R_+^2$. Similarly, V_- is a set of normalized, increasing and convex (concave) utility functions that are constructed as convex mixtures of elementary ramp functions $r_-(z_n; y), (z_n, y) \in R_-^2$. For simplicity reasons we can choose $y = 0$ because $y \in R$.

Moreover, the "capacities" are:

$$\pi_n^+ = w^+(p_n) \quad \text{and} \quad \pi_{-m}^- = w^+(p_{-m})$$

$$\pi_i^+ = w^+(p_i + \dots + p_n) - w^+(p_{i+1} + \dots + p_n) \quad \text{where} \quad 0 \leq i \leq n-1$$

and

$$\pi_i^- = w^-(p_{-m} + \dots + p_i) - w^-(p_{-m} + \dots + p_{i-1}) \quad \text{where} \quad 1-m \leq i \leq 0$$

where, the probability weighting functions (as in KT, 1992) of the cumulative probability p are:

$$w^+(p) = \frac{p^{0.61}}{[p^{0.61} + (1-p)^{0.61}]^{\frac{1}{0.61}}}, \quad p \in W^+$$

and

$$w^-(p) = \frac{p^{0.69}}{[p^{0.69} + (1-p)^{0.69}]^{\frac{1}{0.69}}}, \quad p \in W^-$$

$$W_-^t := \left\{ p_i \in Dir(a) : \sum_{n=1}^{n_1} p_i = 1 \quad \text{and} \quad \sum_{n=1}^k p_i = p \quad ; k \leq n_1 = Card([x_{min}, 0]_t) \right\} \quad (32)$$

and

$$W_+^t := \left\{ p_i \in Dir(a) : \sum_{n=1}^{q_1} p_i = 1 \quad \text{and} \quad \sum_{n=1}^k p_i = p \quad ; k \leq q_1 = Card([0, x_{max}]_t) \right\} \quad (33)$$

where, $Dir(a)$ denotes the *Dirichlet* distribution and the sets above are created for every $t = 1, \dots, T$ in order to obtain random probabilities and implement them in the associated capacities via the relevant pwf. The advantage of the Dirichlet distribution is that it samples over a probability

simplex, which in our case is of 7 dimensions and 11 dimensions respectively. Thus, the Dirichlet distribution is a categorical distribution. A categorical distribution (a generalization of the Bernoulli distribution) is a discrete probability distribution that describes the possible results of a categorical RV. A categorical distribution is very helpful when we are uncertain over what the real distribution is and the simplest way to represent that uncertainty as a probability distribution is Dirichlet.

Also let for the losses domain,

$$c_{0,p}^- = \sum_{m=i}^{n_1} (c_{1,m+1} - c_{1,m}) z_m \quad (34)$$

and

$$c_{1,p}^- = \sum_{m=i}^{n_1} p_m \quad (35)$$

and for the gains domain,

$$c_{0,p}^+ = \sum_{m=i}^{q_1} (c_{1,m+1} - c_{1,m}) a_m \quad (36)$$

and

$$c_{1,p}^+ = \sum_{m=i}^{q_1} p_m \quad (37)$$

where,

$$P^- := \{i = 1, \dots, n_1 : p_i > 0\} \cup \{n_1\} \quad \text{and} \quad (38)$$

and

$$P^+ := \{i = 1, \dots, q_1 : p_i > 0\} \cup \{q_1\} \quad (39)$$

Thus, for any given utility in V_+ (normalized, increasing and convex or similarly concave) and V_- (normalized, increasing and concave or similarly convex), $\sup E_{F_T} [v(\lambda^T Y)]$ is the optimal value of the objective function of the associated following LP problem (4 in total : 2 types of investors for 2 types of portfolios) in canonical form:

$$\text{maximize} \quad \frac{1}{T} \sum_{t=1}^T y_t \quad (40)$$

subject to

$$y_t \leq c_{1,p}^i u_t^T \lambda + c_{o,p}^i, \quad t = 1, \dots, T \quad ; p \in P^i, \quad i = +, - \quad (41)$$

$$\sum_{i=1}^M \lambda_i = 1 \quad (42)$$

$$\lambda_i \geq 0 \quad , i = 1, \dots, M \quad (43)$$

$$y_t \text{ free} \quad , t = 1, \dots, T \quad (44)$$

For the distinction between the two LPs (i.e. one for the losses domain and one for the gains domain), we apply the following rule associated with the correct selection of points for the formation of the two forms of the utilities functions for the Markowitz case combined with the relevant capacities.

For the losses:

$$V_-^M := \left\{ v : v(u) = \sum_{n=1}^{n_1} [u1_{u \leq z_n} + z_n 1_{z_n \leq u < 0}] \pi_n^- \right\} \quad (45)$$

For the gains:

$$V_+^M := \left\{ v : v(u) = \sum_{n=1}^{q_1} [u1_{u \geq a_n > 0} + z_n 1_{a_n \geq u}] \pi_n^+ \right\} \quad (46)$$

According to Russel and Seo (1989), each increasing and concave utility function can be represented by an elementary, two - piece linear utility function of the form: $v(u) = \min(u - y, 0)$. While, each increasing and convex utility function can be represented by an elementary, two - piece linear utility function of the form: $v(u) = \max(u - y, 0)$. Moreover, we are fixing n_1, q_2 for the augmented portfolio and n_1, q_2 for the traditional based on the number of assets that are in the gains and in the losses domain respectively for each t .

In the same way, for the distinction between the two LPs (i.e. one for the losses domain and one for the gains domain), we apply the following rule associated with the correct selection of points for the formation of the two forms of the utilities functions for the Prospect case, again combined with the relevant capacities.

For the losses:

$$V_-^P := \left\{ v : v(u) = \sum_{n=1}^{n_1} [z_n 1_{u \leq z_n} + u 1_{z_n \leq u < 0}] \pi_n^- \right\} \quad (47)$$

For the gains:

$$V_+^P := \left\{ v : v(u) = \sum_{n=1}^{q_1} [u 1_{0 < u \leq a_n} + z_n 1_{a_n \leq u}] \pi_n^+ \right\} \quad (48)$$

These formations are very helpful in order to separate the Markowitz case from the Prospect case. This is mandatory due to the fact that our optimizations are linear and if these distinctions are not applied, even for the probability weighting functions, one can not tell any differences in the optimization outcomes. In Figure 10 we present the two forms of the linear functions that are used in our first experiment. Moreover, we denote that the p.w.f. for each one of the above forms follow the combinations the corresponding compositions that were indicated in Table 3.

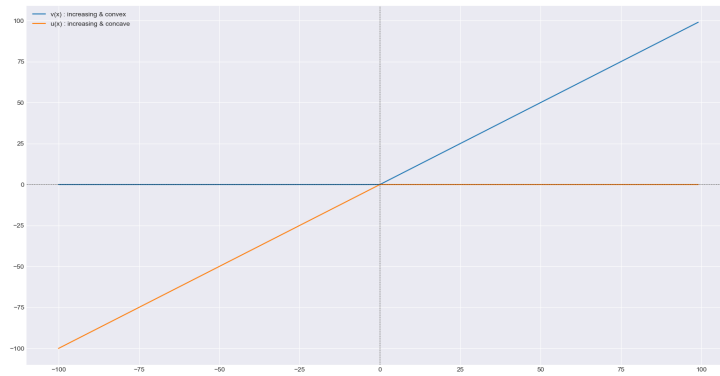


Figure 10: Linear utilities constructed from encompassing functions of max and min

The total running time of all computations for all application amounts to several minutes on a standard desktop PC with a 3.2 GHz Intel i5 processor, 16MB of RAM and using Python with the external Gurobi Optimizer solver. Our findings, that the traditional portfolio *does not span* the augmented one, both for Prospect and Markowitz, is also illustrated with scatter plots in Figures 11 and 12 (i.e. alternative hypothesis holds in both cases).

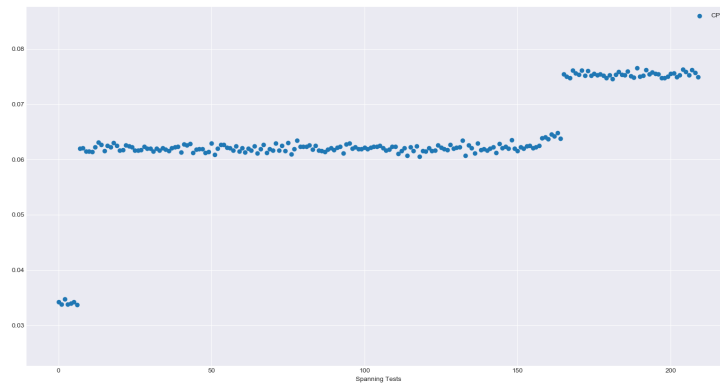


Figure 11: CPT

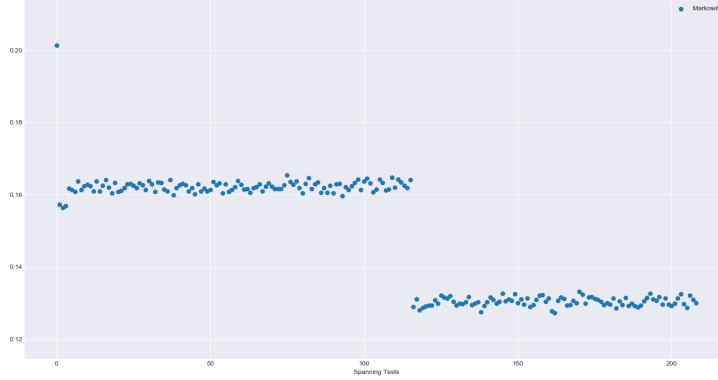


Figure 12: Markowitz

11.2 Experiment 2

In this experiment we do not separate gains and losses daily but we take the market trend as an indication (index) of gains and losses. In other words, the case of euphoria in the market (market growth is up trending) we assume it as gains and in the case of depression (market growth is down trending) we regard it as losses (Figure 3). Thus, we have 4 sub periods: 2 uptrending and 2 down trending and the analysis for experiment 2 is an adjustment of the analysis of experiment 1. Thus, the differences lie on the domains of the probability weighting functions and hence in the value and probability weighting functions to which they are their domains.

Let $BULL2 = [\min_{i=1,\dots,n}(X_{t_i}), \max_{i=1,\dots,n}(X_{t_i})]_{t=1,\dots,T}$ denote the period of the second up trending market. Hence, $BEAR2 = [\max_{i=1,\dots,n}(X_{t_i})_{t=1,\dots,T}, X_T]$ denotes the last sub period of our dataset which happens to be a down trending one. For the other 2 sub periods we partition the dataset in the following way : $C = Dataset - (BULL2 \cup BEAR2) = BULL1 \cup BEAR1$.

Thus, we partition again C into 2 sub periods: $BULL1 = [X_1, \max_{i=1,\dots,n}(X_{t_i})_{t=1,\dots,k}]$ and $BEAR1 = [\max_{i=1,\dots,n}(X_{t_i})_{t=1,\dots,k}, \min_{i=1,\dots,n}(X_{t_i})_{t=1,\dots,T}]$ with $k < T$ and we again denote the expectation w.r.t. the empirical measure E_{F_T} .

We do so in order to apply the linear utility functions and the associated probability weighting functions. Again, the linearity of the utilities leaves the dataset unaffected and the separation of the two problems depends, as before, on the p.w.f. The difference now is that the p.w.f. and the relevant capacities for the gains will be applied solely on the two smaller data sets of BULL1 and BULL2. Similarly, the p.w.f. and the relevant capacities will be applied on BEAR1 and BEAR2. Afterwards, these new data sets are concatenated and a new data set, including all the relevant information for the Prospect case and for the Markowitz case will be used in the LPs. The fact is, that regarding the p.w.f. the only difference with experiment 1 is in their domains. Thus, the associated domains for the weighting functions are the following and we also derive the relevant value functions for Prospect and Markowitz while the rest of the numerical implementation is similar with the previous experiment.

$$W_{\searrow}^t := \left\{ p_i \in Dir(a) : \sum_{n=1}^{n_1} p_i = 1 \quad \text{and} \quad \sum_{n=1}^k p_i = p \quad ; k \leq n_1 \right\} \quad (49)$$

where, $t \in m_1$ with $dim(BEAR1) = m_1 \times n_1$ and $t \in m_2$ with $dim(BEAR2) = m_2 \times n_1$

and

$$W_{\nearrow}^{t_0} := \left\{ p_i \in Dir(a) : \sum_{n=1}^{n_1} p_i = 1 \quad \text{and} \quad \sum_{n=1}^k p_i = p \quad ; k \leq n_1 \right\} \quad (50)$$

where, $t_0 \in t_1$ with $dim(BULL1) = t_1 \times n_1$ and $t \in t_2$ with $dim(BULL2) = t_2 \times n_1$

For the distinction between the two LPs (i.e. one for the losses domain and one for the gains domain), we apply the following rule associated with the correct selection of points for the formation of the two forms of the utilities functions for the Markowitz case combined with the relevant capacities.

For the down trending market:

$$V_{\searrow}^M := \left\{ v : v(u) = \sum_{n=1}^{n_1} [u1_{u \leq z_n} + z_n 1_{z_n \leq u}] \pi_n^- \right\} \quad (51)$$

For the up trending market:

$$V_{\nearrow}^M := \left\{ v : v(u) = \sum_{n=1}^{n_1} [u1_{u \geq z_n} + z_n 1_{z_n \geq u}] \pi_n^+ \right\} \quad (52)$$

In the same way, for the distinction between the two LPs (i.e. one for the losses domain and one for the gains domain), we apply the following rule associated with the correct selection of points for the formation of the two forms of the utilities functions for the Prospect case combined with the relevant capacities.

For the down trending market:

$$V_{\searrow}^P := \left\{ v : v(u) = \sum_{n=1}^{n_1} [z_n 1_{u \leq z_n} + u 1_{z_n \leq u}] \pi_n^- \right\} \quad (53)$$

For the up trending market:

$$V_{\nearrow}^P := \left\{ v : v(u) = \sum_{n=1}^{n_1} [u 1_{u \leq z_n} + z_n 1_{z_n \leq u}] \pi_n^+ \right\} \quad (54)$$

Again, the total running time of all computations for all application amounts to several minutes on a standard desktop PC with a 3.2 GHz Intel i5 processor, 16MB of RAM and using Python with the external Gurobi Optimizer solver. Our findings, that neither in this experiment the traditional portfolio *spans* the augmented one, both for Prospect and Markowitz, is also illustrated with scatter plots in Figures 13 and 14 (i.e. alternative hypothesis holds in both cases).

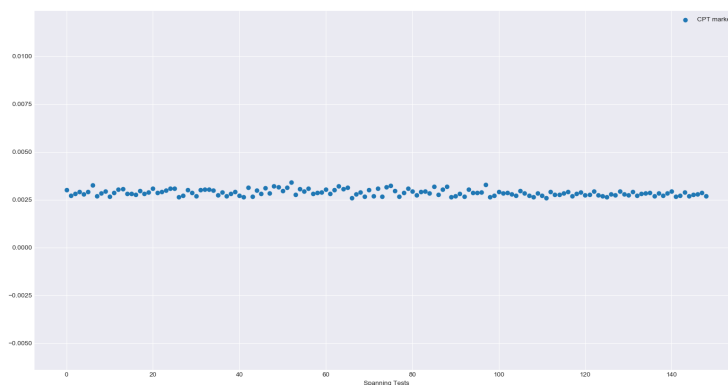


Figure 13: CPT market

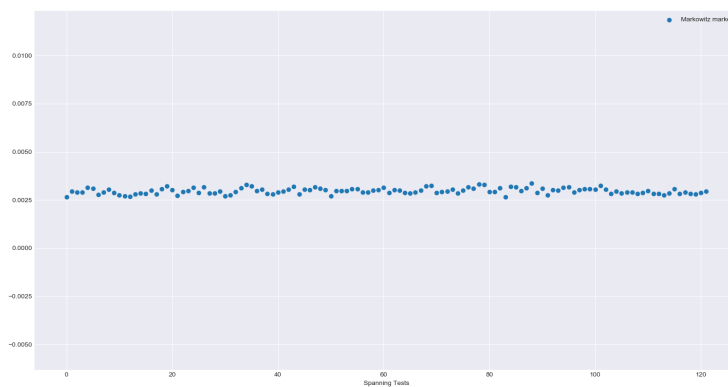


Figure 14: Markowitz market

12 Tables

Table 4: Descriptive Statistics of Daily Returns

Asset	Mean	Std	Skewness	Kurtosis	Sharpe
Market	0.00047	0.0085	-0.63	3.67	-0.49
SPY500	0.00041	0.0083	-0.62	4.33	-0.51
Bond	0.000005	0.0019	-0.28	1.2	-1.12
SMB	-0.00041	0.005	0.49	1.43	-0.8
HML	-0.00021	0.0054	0.68	1.68	-0.74
Rusell 2000	0.00042	0.0104	-0.34	1.2	-0.42
1m T-Bill	0.0002	0.00015	0.545	-1.03	-
Bitcoin	0.0066	0.0475	0.38	4.6	0.031
Ethereum	0.0128	0.1	0.88	11.4	0.078
Ripple	0.0109	0.1	4.38	34.02	0.06
Litecoin	0.008	0.077	3.1	22.4	0.037

Entries report the descriptive statistics on daily returns for the alternative asset classes used in this study. The average return, the standard deviation, the skewness, the kurtosis, as well as the Sharpe ratio are reported. The dataset covers the period from August 8, 2015 to March 29, 2018.

Table 5: Experiment 1 out-of-sample performance: Parametric portfolio measures

Performance measures	CPT Trad.	CPT Aug.	Markowitz Trad.	Markowitz Aug.
Sharpe ratio	0.027	0.17	0.065	0.108
Downside Sharpe ratio	0.016	0.18	0.038	0.095
UP ratio	0.55	1.09	0.49	0.97
Portfolio Turnover	0.5	0.44	0.54	0.11
Return Loss	0.04	-	0.13	-
Opportunity cost	0.014	-	0.0084	-

Entries report the performance measures (Sharpe ratio, Downside Sharpe ratio, UP ratio, Portfolio Turnover, Returns Loss) for the traditional and the augmented with cryptocurrencies asset classes.

Table 6: Experiment 2 out-of-sample performance: Parametric portfolio measures

Performance measures	CPT Trad.	CPT Aug.	Markowitz Trad.	Markowitz Aug.
Sharpe ratio	0.22	0.24	0.015	0.21
Downside Sharpe ratio	0.13	0.23	0.12	0.18
UP ratio	0.54	1.17	0.78	0.99
Portfolio Turnover	0.36	0.45	0.47	0.41
Return Loss	0.0042	-	0.055	-
Opportunity cost	0.013	-	0.0117	-

Entries report the performance measures (Sharpe ratio, Downside Sharpe ratio, UP ratio, Portfolio Turnover, Returns Loss) for the traditional and the augmented with cryptocurrencies asset classes.

Table 7: Scaillet & Topaloglou Non - parametric tests for SSD

Significance level at 5%	p-value
CPT market as index	0.077
Markowitz market as index	0.104
CPT	0.052
Markowitz	0.069

The null hypothesis is that the Augmented portfolio SSD the Traditional one, in every case. Thus, the null is not rejected in all cases. P-values are obtained through bootstrapping with overlapping blocks.

12.1 Composition of optimal portfolios

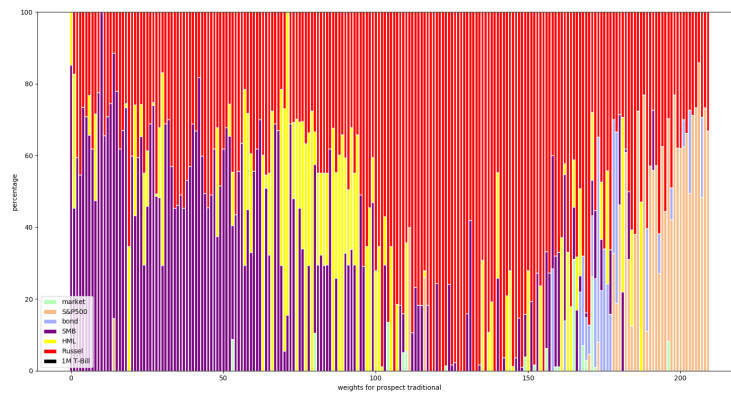


Figure 15: CPT Traditional Experiment 1

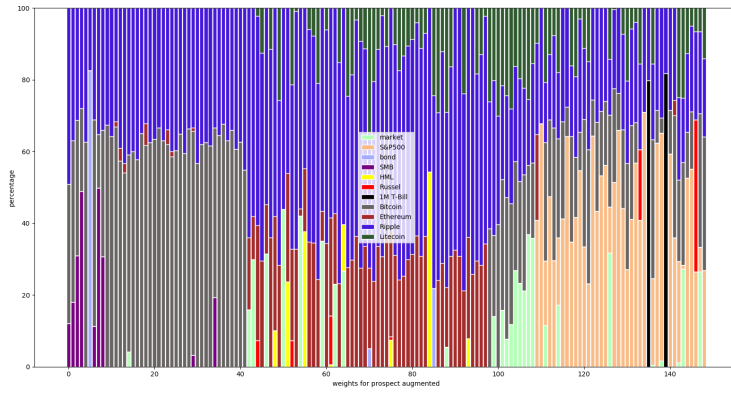


Figure 16: CPT Augmented Experiment 1

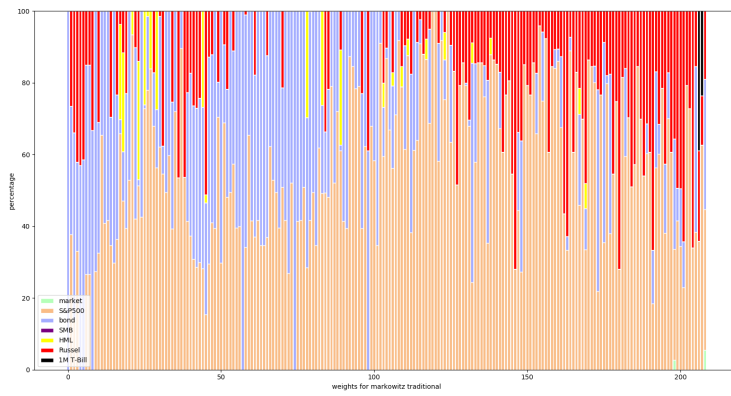


Figure 17: Markowitz Traditional Experiment 1

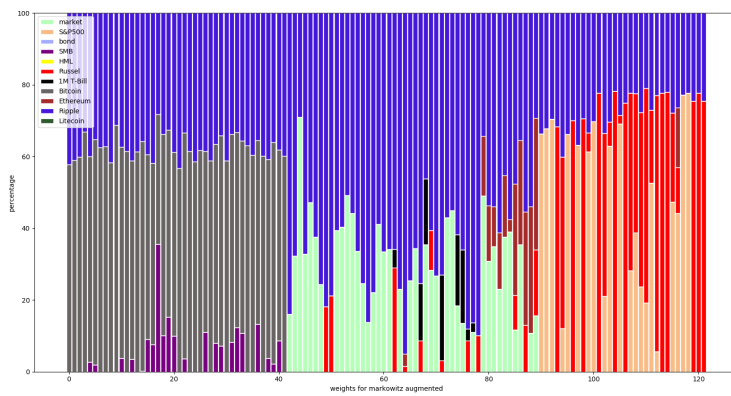


Figure 18: Markowitz Augmented Experiment 1

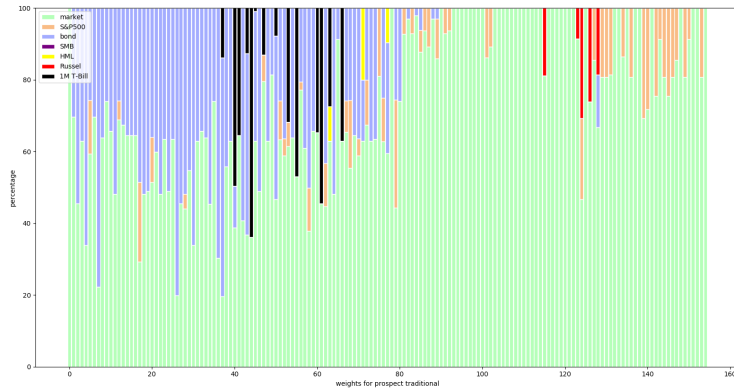


Figure 19: CPT Traditional Experiment 2

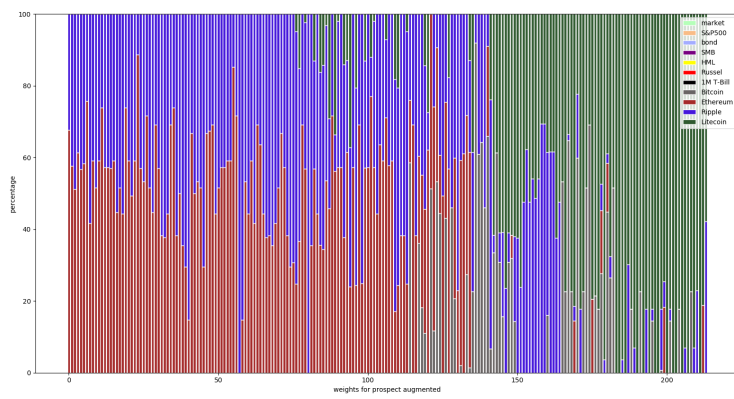


Figure 20: CPT Augmented Experiment 2



Figure 21: Markowitz Traditional Experiment 2

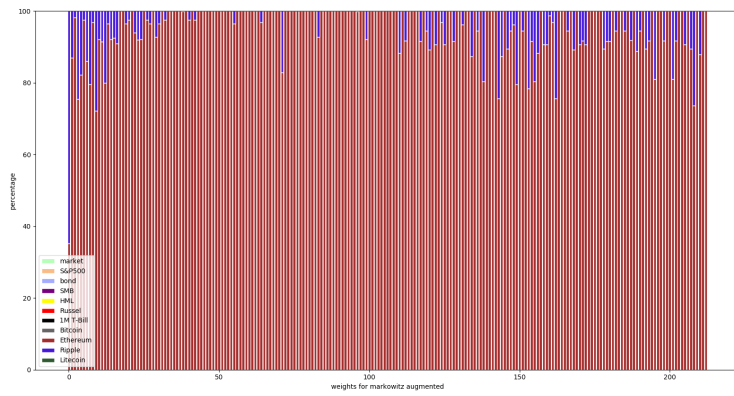


Figure 22: Markowitz Augmented Experiment 2