Explicit and Compact Representations of the One-Sided Green's Function and the Solution of Linear Difference Equations with Variable Coefficients

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Abstract

Leibniz' combinatorial formula for determinants is modified to establish a condensed and easily handled compact representation for Hessenbergians, referred to here as Leibnizian representation. Alongside, we show that the elements of a fundamental solution set associated with linear difference equations with variable coefficients of order p can be explicitly represented by p banded Hessenbergian solutions, built up solely of the variable coefficients. This yields analogous explicit representations for the elements both of the product of companion matrices and of the determinant-ratio formula of the one-sided Green's function. Combining the above results, the building elements of the aforementioned notions are endowed with compact representations of Hessenbergians formulated here by Leibnizian and nested sum representations. We show that the solution of linear difference equations with variable coefficients of order p can be explicitly expressed in terms of the first banded Hessenbergian fundamental solution, called principal determinant function, the variable coefficients, the initial conditions and the forcing terms. We also show that the one-sided Green's function coincides with the principal determinant function, when both functions are restricted to a fairly large domain. Such results make it possible to replace the principal determinant function with the one-sided Green's function in the solution formula, yielding an explicit one-sided Green's function solution representation. Accordingly, the previously stated restriction of the one-sided Green's function and its solution representation are both endowed with compact representations. The equivalence of the one-sided Green's function solution representation and the well-known single determinant solution representation is derived from first principles.

Keywords: Green's function, Linear difference equation, Variable coefficients, Compact representation, Hessenberg matrix, Hessenbergian, Fundamental set, ARMA models.
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1. Introduction

Linear difference equations with variable coefficients of order p (briefly VC-LDEs(p)) are broadly used to model discrete-time non-stationary stochastic processes such as autoregressive moving average (ARMA) models with time-dependent coefficients. This type of models, compared to those with constant coefficients, turn out to be more realistic and sensitive to abrupt and structural changes, as they are efficient approximations to non-linear ones, while linearity maintains interpretation and forecasting advantages (see [1]). Efficient explicit representations to the solution of VC-LDEs(p) having order greater than one ($p > 1$) is a long-standing research topic. There are two dominant schemes for an explicit solution representation of VC-LDEs(p), those of determinant representation (see [2, 3]) and those of compact representation (see [4, 5]). A pioneer work to link the two representation schemes was recently accomplished by Marrero and Tomeo in [6], establishing there the equivalence between the combinatorial solution representation of VC-LDEs(p) obtained by Mallik in [4] and the single determinant solution representation obtained by Kittappa in [3]. They also established in [6] a nested sum representation for Hessenbergians and, as a consequence, a compact solution representation of VC-LDEs(p).

In this context, we provide here a more condensed combinatorial representation for Hessenbergians, referred to as *Leibnizian representation* (see eq. (28) in Theorem 1). An algorithm for the symbolic computation of the Leibnizian representation of Hessenbergians is provided in Appendix C, Algorithm 1. Unlike the Leibniz combinatorial formula for kth order determinants, which consists of $k!$ singed elementary products (SEPs) and their summation index ranges over the symmetric group of permutations, the Leibnizian representation of Hessenbergians, obtained here, is a sum of 2^{k-1} distinct non-trivial SEPs, whose summation index ranges over integer intervals.

Judging from the missing cross references, the solution representations obtained in the above cited references have not been utilized in time series modelling. In contrast, the large family of time-varying coefficient models,

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such as ARMA processes with variable coefficients (TV-ARMA), which are equipped with VC-LDE(p) solution representations, employ the one-sided Green's function representation (Green's function for short) of a particular solution, because it facilitates the development of elegant and generic expressions for their fundamental properties, including the Wold-Cramér solution representation, the asymptotic properties as well as the optimal linear forecasts. This approach is originated by Miller [8] (see also [9, 10, 11]) as an alternative to the standard characteristic polynomial formulation, which has been used in the solution process of LDEs(p) and associated ARMA models with constant coefficients, but loses its strength in the presence of variable coefficients (see [9]).

Explicit representations of the Green's function strongly depend upon the availability of a fundamental set of solutions, whose elements (called linearly independent or fundamental solutions) must be explicitly expressed and computationally tractable.[1] The lack, in the general case, of a fundamental solution set associated with VC-LDEs(p), whose elements are explicitly expressed, has led to a dichotomy between the explicit representation of the properties of TV-ARMA models in terms of the Green's function and the recursive computation of this function (see for example [9]).

The two research paths in the literature concerning solution representations of VC-LDEs(p) and corresponding representations of TV-ARMA models, respectively, have been increased over time, but the overlap between them has not. The results of the present paper establish common ground developments between them and provide the mathematical framework for a unified theory of TV-ARMA models including processes with deterministic or stochastic variable coefficients (see [13]). An application of this theory is the modelling of stock volatilities during financial crises, presented in [14].

In the current study, we introduce a fundamental set of solutions associated with VC-LDEs(p), whose terms are banded Hessenbergians initiated by p distinct unit vectors of the same magnitude p, respectively (see Proposition 2 and Theorem 2). The nonzero entries of the associated Hessenberg matrices are the variable coefficients of a VC-LDE(p) evaluated at consecutive point instances. The banded Hessenbergian form of these p fundamental solutions is originated from their simultaneous construction, as a result of the infinite Gaussian elimination algorithm (see in Subsection 3.1). Banded Hessenbergians are computationally tractable due to the linear time complexity needed for their evaluation (see the discussion below Corollary 2). The first fundamental solution gives rise to the *principal determinant function*, denoted by $\xi_{t,r}$ (see Definition 2). In Proposition 4 we show that the other elements of the aforementioned fundamental solution set, and therefore the general homogeneous solution of VC-LDEs(p), can be expressed in terms of the function $\xi_{t,r}$. A particular solution is also expressed as a linear combination of $\xi_{t,r}$ times the forcing terms of the VC-LDE(p) (see eq. (**??**) in Proposition 5) and therefore as a Hessenbergian, but not, in the general case, as a banded one.

Two of the main results of this paper concern explicit and compact representations of the Green's function. The first, recovers the defining formula of the Green's function as a ratio of two determinants, but now their elements are banded Hessenbergians built up solely of the variable coefficients (see Theorem 4). The second, in Theorem 5, shows that the domain restriction of the Green's function, $H(t,r)$, involved in the solution of VC-LDEs, coincides with the same domain restriction of $\xi_{t,r}$. This result allows us to exchange the roles between $\xi_{t,r}$ and $H(t,r)$ in the solution formulas of VC-LDEs(p) (see Corollary 2). As a consequence, the main notions associated with VC-LDEs are solely expressed in terms of $\xi_{t,r}$, including the fundamental set of solutions (see Proposition 4), the product of companion matrices (see Theorem 3), the Green's function (see eq. 58) and the general homogeneous and nonhomogeneous solutions (see eqs. (60 and 64), respectively). To the extend of our Knowledge there are no fully explicit representations of the homogeneous and non-homogeneous solutions of VC-LDEs(p) (see eqs. (60) and (65) respectively), exclusively in terms of the Green's function, the variable coefficients, the initial conditions and the forcing terms. Besides, by replacing the Green's function with the principal determinant function in the results of the previously cited works on time-varying coefficient models, the above noticed dichotomy is reconciled (see [13]). In Proposition 6 we show from first principles the equivalence between the Green's function solution representation and the single determinant solution representation obtained in [3]. This equivalence result capitalizes on an alternative building process yielding the same fundamental solution set, but originated by Cramer's rule rather than the infinite Gaussian elimination algorithm (see the discussion below Proposition 6).

The paper concludes with the compact representations of the Green's function restriction, involved in the general solution of VC-LDEs(p), and of the solution itself (see Section 6), thanks to the Leibnizian and nested sum representations of Hessenbergians. In Algorithm 2 of Appendix C the Leibnizian compact representation of the Green's function and the solution of VC-LDEs(p) are verified by a symbolic computation.

2. Leibnizian Representation of Hessenbergians

In all that follows the set $\mathbb{Z}$ (resp. $\mathbb{Z}_a$) stands for the set of integers (resp. the set of integers greater than or equal to $a \in \mathbb{Z}$) and $\mathbb{C}$ for the algebraic field of complex numbers. The group of permutations on $\{1, 2, \ldots, k\}$

[1] By a computationally tractable expression, we mean an explicit form associated with an algorithm, which evaluates this form in polynomial running time.

is denoted by $\mathbb{S}_k$ and the signature $sgn(\ell)$ of $\ell \in \mathbb{S}_k$ is assigned to -1 if ℓ is an odd permutation of $\mathbb{S}_k$ and $+1$ if ℓ is an even one. The building blocks of the well known Leibnizian determinant expansion for the kth order square matrix $\mathbf{A} = [a_{i,j}]_{1\le i,j\le k}$ over $\mathbb{C}$, that is

$$\det(\mathbf{A}) = \sum_{\ell\in\mathbb{S}_k} sgn(\ell) \prod_{i=1}^{k} a_{i,\ell_i}, \tag{1}$$

are the singed elementary products, which can be formally defined as follows: Let $\ell \in \mathbb{S}_k$. A signed elementary product (SEP) of a square matrix $\mathbf{A} = [a_{i,j}]_{1\le i,j\le k}$ over $\mathbb{C}$ is an ordered pair $(\ell, sgn(\ell)\prod_{i=1}^{k} a_{i,\ell_i})$ in $\mathbb{S}_k \times \mathbb{C}$, where the second component of the ordered pair is the numerical value of the SEP in $\mathbb{C}$. We infer that two SEPs of $\mathbf{A}$, say $(\ell, sgn(\ell)\prod_{i=1}^{k} a_{i,\ell_i})$ and $(l, sgn(l)\prod_{i=1}^{k} a_{i,l_i})$, are equal if and only if $\ell = l$. In all that follows we shall use the standard notation of SEPs: $sgn(\ell)a_{1,\ell_1}a_{2,\ell_2}...a_{k,\ell_k},\ \ \ell \in \mathbb{S}_k$. The set of SEPs associated with $\mathbf{A}$ will be denoted here as $\mathcal{S}_{\mathbf{A}}$. As a consequence of the above discussion, every SEP in $\mathcal{S}_{\mathbf{A}}$ is associated with a permutation ℓ up to the bijection:

$$\mathbb{S}_k \ni \ell \mapsto sgn(\ell)a_{1,\ell_1}\dots a_{k,\ell_k} \in \mathcal{S}_{\mathbf{A}}. \tag{2}$$

It follows from (2) that the number of distinct SEPs in $\mathcal{S}_{\mathbf{A}}$ is $k!$, since $\mathrm{card}(\mathbb{S}_k) = k!$.

The kth order lower Hessenberg matrix $\mathbf{H}_k = [h_{i,j}]_{1\le i,j\le k}$ and the infinite order lower Hessenberg matrix $\mathbf{H} = [h_{i,j}]_{i,j\ge k}$ over $\mathbb{C}$ are both defined by the condition $h_{i,j} = 0$, whenever $j - i > 1$, as displayed below:

$$\mathbf{H}_k = \begin{bmatrix} h_{1,1} & h_{1,2} & 0 & \dots & 0 & 0 \\ h_{2,1} & h_{2,2} & h_{2,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{k-1,1} & h_{k-1,2} & c_{k-1,3} & \dots & h_{k-1,k-1} & h_{k-1,k} \\ h_{k,1} & h_{k,2} & h_{k,3} & \dots & h_{k,k-1} & h_{k,k} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} h_{1,1} & h_{1,2} & 0 & 0 & \dots \\ h_{2,1} & h_{2,2} & h_{2,3} & 0 & \dots \\ h_{3,1} & h_{3,2} & h_{3,3} & h_{3,4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{3}$$

From here onwards a matrix $\mathbf{H}_k$ is considered as a term of the infinite chain of lower Hessenberg matrices $\mathbf{H}_1 \sqsubset \mathbf{H}_2 \sqsubset \dots \sqsubset \mathbf{H}_k \sqsubset \dots \sqsubset \mathbf{H}$, where $\mathbf{H}_k \sqsubset \mathbf{H}$ means that $\mathbf{H}_k$ is a top submatrix of $\mathbf{H}$ consisting of the first k rows and columns of $\mathbf{H}$. The determinant of $\mathbf{H}_k$ for $k \ge 1$ satisfies the well known recurrence

$$\det(\mathbf{H}_k) = h_{k,k}\det(\mathbf{H}_{k-1}) + \sum_{i=1}^{k-1}(-1)^{k-i}h_{k,i}\prod_{j=i}^{k-1} h_{j,j+1}\det(\mathbf{H}_{i-1}), \tag{4}$$

where $\det(\mathbf{H}_0) = 1$ and $\det(\mathbf{H}_1) = h_{1,1}$ (for a proof of the recurrence formula in eq. (4) see [15]).

The zero valued entries of $\mathbf{H}_k$ positioned above the superdiagonal, that is the entries h_{ij} whose indices satisfy $j-i > 1$, will be called *trivial*, while the remaining entries of $\mathbf{H}_k$, including the entries of the superdiagonal, will be called *non-trivial*. A SEP of $\det(\mathbf{H}_k)$ will be called *trivial* if it contains at least one trivial entry. Otherwise it is called *non-trivial*. Throughout this paper the set of distinct non-trivial SEPs associated with $\det(\mathbf{H}_k)$ is denoted by $\mathcal{E}_k$. If $i, j \in \mathbb{Z}$, we adopt the integer interval notation: $[\![i, j]\!] \stackrel{\text{def}}{=} [i, j] \cap \mathbb{Z}$ and $\mathbb{I}_{k-1} \stackrel{\text{def}}{=} [\![0, 2^{k-1} - 1]\!]$.

2.1. Non-trivial SEPs and their String Structure

The non-trivial entries $h_{i,j},\ j \le i$, positioned below and including the main diagonal of $\mathbf{H}$, will be called *standard factors*, while the sign-opposite entries of the super-diagonal, i.e. the entries $-h_{i,i+1}$, will be called *non-standard factors*. By assigning $c_{i,j} = h_{i,j}$, whenever $j \ne i+1$, and $c_{i,i+1} = -h_{i,i+1}$ the matrices in eqs. (3) take the form:

$$\mathbf{H}_k = \begin{bmatrix} c_{1,1} & -c_{1,2} & 0 & \dots & 0 & 0 \\ c_{2,1} & c_{2,2} & -c_{2,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{k-1,1} & c_{k-1,2} & c_{k-1,3} & \dots & c_{k-1,k-1} & -c_{k-1,k} \\ c_{k,1} & c_{k,2} & c_{k,3} & \dots & c_{k,k-1} & c_{k,k} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} c_{1,1} & -c_{1,2} & 0 & 0 & \dots \\ c_{2,1} & c_{2,2} & -c_{2,3} & 0 & \dots \\ c_{3,1} & c_{3,2} & c_{3,3} & -c_{3,4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{5}$$

Writing the Hessenbergian recurrence in eq. (4) in terms of the entries of $\mathbf{H}_k$ in eq. (5), after some algebraic manipulations (see for details Proposition A1 (i) in Appendix can be equivalently rewritten as

$$\det(\mathbf{H}_k) = c_{k,k}\det(\mathbf{H}_{k-1}) + \sum_{j=1}^{k-1}\prod_{i=j}^{k-1} c_{k,j}c_{i,i+1}\det(\mathbf{H}_{j-1}). \tag{6}$$

The recurrence in eq. (6) consists exclusively of positively singed non-trivial SEPs, which gives a comparative advantage to the Hessenberg matrix forms in eqs. (5) over those in eqs. (3), respectively. The recurrence of

$\det(\mathbf{H}_k)$ in eq. (6) can be written in an expanded form as:

$$\det(\mathbf{H}_k) = c_{1,2}c_{2,3}\dots c_{k-1,k}c_{k,1}\det(\mathbf{H}_0) + c_{2,3}c_{3,4}\dots c_{k-1,k}c_{k,2}\det(\mathbf{H}_1) + c_{3,4}c_{3,5}\dots c_{k-1,k}c_{k,3}\det(\mathbf{H}_2) + ... + c_{k-1,k}c_{k,k-1}\det(\mathbf{H}_{k-2}) + c_{k,k}\det(\mathbf{H}_{k-1}). \quad (7)$$

We show in Proposition A1 (ii) of Appendix A, that the total number of distinct non-trivial SEPs involved in the recurrence (7) and therefore in the recurrences in eqs. (4) and (6), is $\text{card}(\mathcal{E}_k) = 2^{k-1}$. This number is considerably less than the number $k!$ of distinct SEPs involved in the Leibniz' determinant expansion in eq. (1).

In what follows, we introduce the notion of *strings* associated with an infinite order Hessenberg matrix $\mathbf{H}$ in eq. (3) or (5). Informally speaking, strings are pieces of non-trivial SEPs and pieces of strings are also strings. A formal Definition is given below:

Definition 1 (Strings). *A finite product of consecutive factors associated with* $\mathbf{H}$ *is said to be a string, denoted by* $C[j,m;\ell] = c_{j,\ell_j}c_{j+1,\ell_{j+1}}\dots c_{m,\ell_m}$*, if there is some* $k \in \mathbb{Z}_1$ *and a non-trivial SEP, say* $C \in \mathcal{E}_k$*, such that* C *includes* $C[j,m;\ell]$*, that is:*

$$C = c_{1,\ell_1}\dots \underbrace{c_{j,\ell_j}\dots c_{m,\ell_m}}_{C[j,m;\ell]} \dots c_{k,\ell_k}.$$

An initial string determined by m *and* ℓ *is defined by* $C[1,m;\ell]$ *and is shortly denoted as* $C[m;\ell]$.

Formally the string $C[j,m;\ell]$ is uniquely determined by $(j,m,\ell) \in \mathbb{Z}_1^2 \times \mathbb{S}_k$ for any $k \geq m$. Let $C[j,m;\ell]$ be a string. If $j = m$, then $C[m,m;\ell] = c_{m,\ell_m}$. If $i \geq j$ and $n \leq m$, then Definition 1 implies that $C[i,n;\ell]$ is also a string, since it is included in the same SEP as $C[j,m;\ell]$. Accordingly, $C[i,n;\ell]$ might be said to be a *substring* of $C[j,m;\ell]$. Adopting the convention $c_{0,0} = h_{0,0} = 1$, the class of all initial strings determined by $i \geq 0$ is denoted by $\mathfrak{C}[i]$. The first three classes of initial strings are: $\mathfrak{C}[0] = \{c_{0,0}\}$, $\mathfrak{C}[1] = \{c_{1,1}, c_{1,2}\}$ and $\mathfrak{C}[2] = \{c_{1,1}c_{2,2}, c_{1,1}c_{2,3}, c_{1,2}c_{2,1}, c_{1,2}c_{2,3}\}$. The major difference between initial strings and non-trivial SEPs is demonstrated by the following example: The initial string $c_{1,1}c_{2,3}$ is included in the non-trivial SEP $c_{1,1}c_{2,3}c_{3,2}$, but the string under discussion is not a SEP. On the other hand, every non-trivial SEP is an initial string, since every SEP is included in itself. As a consequence, every string is included in an initial string.

A non-trivial element (or factor) $c_{i,m}$ of $\mathbf{H}$ is said to be an *immediate successor* (IS) of the initial string $C[i-1;\ell] = c_{1,\ell_1}\dots c_{i-1,\ell_{i-1}}$, whenever $c_{1,\ell_1}\dots c_{i-1,\ell_{i-1}}c_{i,m}$ is an initial string too. For instance, the immediate successors of $c_{1,1}$ are $c_{2,2}$ and $c_{2,3}$. Some elementary properties of strings, as arrays of standard and non-standard factors, are summarized below:

Proposition 1 (Properties of Strings).
1) Every non-trivial entry $c_{i,j}$ *of* $\mathbf{H}$ *in eq.* (5) *is an IS of some initial string.*
2) Every initial string of $\mathbf{H}$ *has two ISs. One of these is non-standard (therefore the other must be standard).*
3) Let $c_{i-1,\ell_{i-1}}$ *be any standard factor of an initial string,* $C[k;\ell]$ *for some* $k \geq i-1$*. Then the only possible ISs of the initial string* $C[i-1;\ell]$ *are* $c_{i,i}$ *and* $c_{i,i+1}$*.*
4) Let c_{i,ℓ_i} *be the standard IS of the initial string* $C[i-1;\ell]$*. If the number of consecutive non-standard predecessors of* c_{i,ℓ_i} *is* m*, then* $\ell_i = i - m$*.*

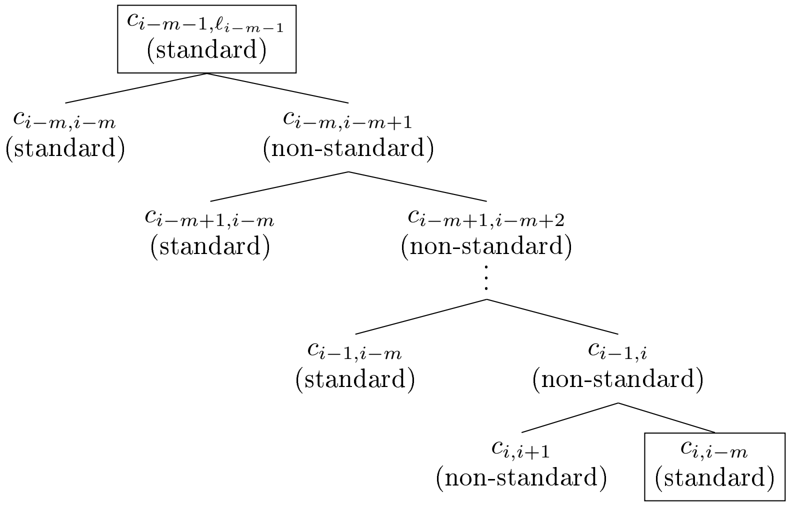
Proof. 1) It follows directly from the recurrence in eq. (7). 2) Trivially, the ISs of $c_{1,1}$ are $c_{2,2}$ and $c_{2,3}$ (standard and non-standard, respectively). An initial string, say $C[i-1;\ell] = c_{1,\ell_1}\dots c_{i-1,\ell_{i-1}}$ for $i \geq 1$ of $\mathbf{H}$ has only two ISs, since there are $(i+1)$ candidates, that is the number of the non-trivial elements of the ith row of $\mathbf{H}$, minus the number $(i-1)$ of the preceding factors, whose column indices have already occurred in the string. Since the column index $(i+1)$ has not previously used by preceding factors, we conclude that one of these ISs is the non-standard factor $c_{i,i+1}$, whence the other must be standard. 3) It suffices to show that the standard IS of $C[i-1;\ell]$ is $c_{i,i}$, provided that $c_{i-1,\ell_{i-1}}$ is standard, that is to show that $c_{m,i}$ for any $m = 1, 2..., i-1$ is not a factor of this string. If $i - m > 1$ (or $i - m \geq 2$), then $c_{m,i}$ is a trivial entry, because $c_{i-2,i}, c_{i-3,i}, ..., c_{1,i}$ are all trivial entries, and therefore are not factors of the string. Moreover, the non-standard factor $c_{i-1,i}$ is not a factor of the string, since, by hypothesis, $c_{i-1,\ell_{i-1}}$ is standard. Thus, the only available (not previously occurred) standard IS of $c_{i-1,\ell_{i-1}}$ is $c_{i,i}$, as required. 4) Property 4 is a generalization of Property 3 and it is shown in Proposition A2 of the Appendix A (see also the tree diagram below). □

As a consequence of Property 2, the class $\mathfrak{C}[k]$ consists of 2^k initial strings, since $\text{card}(\mathfrak{C}[k]) = 2\text{card}(\mathcal{E}_k)$. We remark that Property 3 follows directly from Property 4, since the number of the non-standard factors between two successive standard ones in any initial string is $m = 0$. Property 4 can be used to identify the ISs, say c_{i,ℓ_i}, of any initial string, by using the following method:

Method. Let $C[i-1,\ell] = c_{1,\ell_1}c_{2,\ell_2}...c_{i-1,\ell_{i-1}}$ be an initial string. If c_{i,ℓ_i} is the non-standard IS of $C[i-1,\ell]$, then $\ell_i = i+1$ and $c_{i,\ell_i} = c_{i,i+1}$. If c_{i,ℓ_i} is the standard the IS of $C[i-1,\ell]$, then, in order to identify it, we need to count all consecutive non-standard predecessors of c_{i,ℓ_i}. Call this number m (m could be: $0, 1, ..., i-1$). As $c_{i-m-1,\ell_{i-m-1}}$ is a standard factor, Property 4 entails that $\ell_i = i - m$ and the standard IS of C is: $c_{i,\ell_i} = c_{i,i-m}$.

Examples: 1) It follows from Property 3 that the ISs of the string $c_{1,1}c_{2,2}$ are $c_{3,3}$ (standard) and $c_{3,4}$ (non-standard). These are also the ISs of the string $c_{1,2}c_{2,1}$. 2) The ISs of the string $c_{1,1}c_{2,3}$ are $c_{3,4}$ (non-standard) and $c_{3,2}$ (standard). To see how the latter follows from Property 4, denote the standard IS of the string $c_{1,1}c_{2,3}$ as c_{3,ℓ_3}. Since $c_{1,1}$ is standard and $c_{2,3}$ is non-standard, the latter is located within two standard factors, whence $m = 1$ and $c_{3,\ell_3} = c_{3,3-1} = c_{3,2}$.

Properties 2, 3 and 4 can be visualised by a tree diagram:



2.2. Building Functions

The indexing function $\sigma_{k,i}$, yielding the compact representation of Hessenbergians (see eq. 28), is defined as a composition of two building functions: The outer building function $\mathfrak{z}_{k,i}$ and the inner building function τ_k. These functions are established in terms of integer functions in the next two paragraphs of the present Subsection.

Outer Building Function

In what follows, $\mathfrak{R}_k$ will stand for the set of k-arrays $\mathbf{r} = (r_1, r_2, \ldots, r_k)$ with components either $r_i = 0$ or 1 for $1 \le i \le k-1$ and $r_k = 1$. Trivially $\mathfrak{R}_1 = (1)$ and the number of the elements in $\mathfrak{R}_k$ is $\text{card}(\mathfrak{R}_k) = 2^{k-1}$. The outer building function, constructed below, considerably reduces the number of SEPs in the Leibniz determinant formula of Hessenbergians, built up solely of the non-trivial entries $c_{i,\mathfrak{z}_{k,i}(\mathbf{r})}$ of $\mathbf{H}_k$ for $\mathbf{r} \in \mathfrak{R}_k$ (see eq. (18)).

We introduce the function f_k, which maps every $C = c_{1,\ell_1}c_{2,\ell_2}\ldots c_{k,\ell_k} \in \mathcal{E}_k$, to $\mathbf{r} = (r_1, r_2, \ldots, r_{k-1}, 1) \in \mathfrak{R}_k$, according to the rule: $r_i = 0$, whenever c_{i,ℓ_i} is non-standard and $r_i = 1$, whenever c_{i,ℓ_i} is standard. Since the last factor of a non-trivial SEP is always standard, the last component of the array $\mathbf{r}$ has been assigned to 1, that is $r_k = 1$. As shown in Proposition A3 of Appendix A, the above rule induces the bijective mapping:

$$f_k : \mathcal{E}_k \ni C \mapsto f_k(C) \in \mathfrak{R}_k \tag{8}$$

For example a non-trivial SEP, say $C = c_{1,1}c_{2,3}c_{3,2}c_{4,5}c_{5,6}c_{6,4}c_{7,7}$ is mapped uniquely to the array $f_7(C) = (1,0,1,0,0,1,1)$ and vice versa. In order to verify that $f_7^{-1}(1,0,1,0,0,1,1) = C$, we first map the 0s, which are positioned at $i = 2,4,5$, to the non-standard factors having the same positions in C, that is: $c_{2,3}, c_{4,5}, c_{5,6}$. By virtue of Property 4 in Proposition 1, we map the 1s to standard factors as follows: The first 1 is mapped to $c_{1,1}$, as $m = 0$. The second 1 is mapped to $c_{3,3-1}$, since there is only one 0 between the first and third 1s, that is $m = 1$. Similarly, the third 1 must be mapped to $c_{6,6-2}$, since $m = 2$. The last 1 is mapped to $c_{7,7}$, since $m = 0$ and the assertion is verified.

As a consequence we have:

$$\det(\mathbf{H}_k) = \sum_{\mathbf{r} \in \mathfrak{R}_k} f_k^{-1}(\mathbf{r}). \tag{9}$$

More generally speaking, arrays in $\mathfrak{R}_k$ represent non-trivial SEPs in $\mathcal{E}_k$, preserving their string structure, as arrays of standard and non-standard factors.

If $f_k^{-1}(\mathbf{r}) = c_{1,\ell_1}c_{2,\ell_2}\ldots c_{i,\ell_i}\ldots c_{k,\ell_k} \in \mathcal{E}_k$, we call $f_{k,i}^{-1}(\mathbf{r}) = c_{i,\ell_i}$, that is the ith factor of the SEP, whence

$$f_k^{-1}(\mathbf{r}) = \prod_{i=1}^{k} f_{k,i}^{-1}(\mathbf{r}). \tag{10}$$

In what follows we assign $r_0 = 1$. It turns out that given $k, i \in \mathbb{Z}$ such that $1 \le i \le k$, then for any $\mathbf{r} \in \mathfrak{R}_k$, we have:

$$f_{k,i}^{-1}(\mathbf{r}) = \begin{cases} c_{i,i+1}, & \text{if } r_i = 0 \text{ and } i \ne k \\ c_{i,i-m}, & \text{if } r_{i-m-1} = r_i = 1 \text{ and } r_j = 0, \text{ whenever } j \in \mathbb{Z} : \ i-m \le j \le i-1. \end{cases}$$

The above piecewise expression of $f_{k,i}^{-1}(\mathbf{r})$ can be written in a single form as:

$$f_{k,i}^{-1}(r_1, r_2, ..., \ r_{i-m-2}, \ r_{i-m-1}, \ \underbrace{0, \ 0, ..., \ 0,}_{m} \ r_i, \ ..., \ r_{k-1}, 1) \ = \ c_{i,i-m}, \qquad m = -1, 0..., i-1. \tag{11}$$

As m is the number of successive 0s between $r_{i-m-1} = 1$ and $r_i = 1$, the formula in eq. (11) gives: $f_{k,i}^{-1}(\mathbf{r}) = c_{i,i-m}$. If $r_i = 0$, then m is assigned to $m = -1$, that is the formula in eq. (11) ignores all the predecessors of r_i and gives: $f_{k,i}^{-1}(\mathbf{r}) = c_{i,i-(-1)} = c_{i,i+1}$. Next, we consider two special cases: *i*) Let $r_{i-1} = r_i = 1$. Then $m = 0$ and $f_{k,i}^{-1}(\mathbf{r}) = c_{i,i}$, which is in accord with the fact that $\{j \in \mathbb{Z} : i \leq j \leq i-1\} = \emptyset$ and $\mathrm{card}(\emptyset) = 0$. *ii*) Let $r_i = 1$ and $m = i-1$. Then, on account of $r_{i-m-1} = r_{i-(i-1)-1} = r_0 = 1$, eq. (11) yields:

$$f_{k,i}^{-1}(\underbrace{0, 0, \ldots, 0,}_{i-1} 1, r_{i+1}, \ldots, r_{k-1}, 1) = c_{i,i-(i-1)} = c_{i,1}.$$

In view of eq. (10), the expression for Hessenbergians in eq. (9) can be rewritten as:

$$\det(\mathbf{H}_k) = \sum_{\mathbf{r} \in \mathfrak{R}_k} \prod_{i=1}^{k} f_{k,i}^{-1}(\mathbf{r}), \tag{12}$$

which consists of $\mathrm{card}(\mathfrak{R}_k) = \mathrm{card}(\mathcal{E}_k) = 2^{k-1}$ distinct non-trivial SEPs. An equivalent expression to eq. (12), but directly in terms of the entries $c_{i,j}$ of $\mathbf{H}_k$ in eq. (5), is obtained by introducing the function

$$\zeta_{k,i}(\mathbf{r}) \overset{\text{def}}{=} \begin{cases} -1 & \text{if } r_i = 0 \text{ and } i < k \\ m & \text{if } r_{i-m-1} = r_i = 1 \text{ and } r_j = 0 \text{ for all } j \in [\![i-m, i-1]\!], \end{cases} \tag{13}$$

which returns the number of consecutive 0s preceding the component r_i $(i \leq k)$ in an array $\mathbf{r} \in \mathfrak{R}_k$. In Proposition A4 of the Appendix, we show a single formula, equivalent to the piecewise expression of $\zeta_{k,i}$ in eq. (13), which is additionally expressed in terms of elementary functions, as displayed below

$$\zeta_{k,i}(\mathbf{r}) = r_i(i - \max_{0 \leq j < i}\{j \cdot r_j\}) - 1, \quad i \leq k, \tag{14}$$

noting that: $\max\{a_1, a_2\} = \dfrac{a_1 + a_2 + \sqrt{(a_1 - a_2)^2}}{2}$ and $\max\limits_{1 \leq j < i}\{a_j\} = \max\{\max\{a_1, a_2\}, a_3, \ldots, a_{i-1}\}$.
Letting $i = 1$ in eq. (14), it follows that $j = 0$. Recalling the convention $r_0 = 1$, it follows from $\{0 \cdot r_0\} = \{0\}$ that $\max\{0\} = 0$, whence

$$\zeta_{k,1}(\mathbf{r}) = r_1(1 - \max\{0\}) - 1 = r_1(1-0) - 1 = \begin{cases} -1 & \text{if } r_1 = 0 \\ 0 & \text{if } r_1 = 1 \end{cases} \quad \text{for all } \mathbf{r} \in \mathfrak{R}_k, \tag{15}$$

which is in accord with the Definition in eq. (13). Next, we define the function:

$$\mathfrak{z}_{k,i}(\mathbf{r}) \overset{\text{def}}{=} i - \zeta_{k,i}(\mathbf{r}). \tag{16}$$

It follows from eq. (11) that $f_{k,i}^{-1}(\mathbf{r}) = c_{i,i-\zeta_{k,i}(\mathbf{r})}$, whence:

$$f_{k,i}^{-1}(\mathbf{r}) = c_{i,\mathfrak{z}_{k,i}(\mathbf{r})}. \tag{17}$$

As a demonstrative example, let $\mathbf{r} = (1, 0, \ldots, 0, 1, 1, 0, 0, 1) \in \mathfrak{R}_k$. Applying the Definition in eq. (13), along with eqs. (16) and (17), we get: $\zeta_{k,1}(\mathbf{r}) = 0$, $\mathfrak{z}_{k,1}(\mathbf{r}) = 1 - 0 = 1$ and $f_{k,1}^{-1}(\mathbf{r}) = c_{1,\,\mathfrak{z}_{k,1}(\mathbf{r})} = c_{1,1}$. As $r_2 = r_3 = ... = r_{k-5} = r_{k-2} = r_{k-1} = 0$, we have: $\zeta_{k,2}(\mathbf{r}) = \zeta_{k,3}(\mathbf{r}) = ... = \zeta_{k,k-5}(\mathbf{r}) = \zeta_{k,k-2}(\mathbf{r}) = \zeta_{k,k-1}(\mathbf{r}) = -1$. Thus, $\mathfrak{z}_{k,2}(\mathbf{r}) = 2 - (-1) = 3, ..., \mathfrak{z}_{k,k-5}(\mathbf{r}) = k - 5 - (-1) = k - 4$, $\mathfrak{z}_{k,k-2}(\mathbf{r}) = k - 2 - (-1) = k - 1$, $\mathfrak{z}_{k,k-1}(\mathbf{r}) = k - 1 - (-1) = k$ and $f_{k,2}^{-1}(\mathbf{r}) = c_{2,\,\mathfrak{z}_{k,2}(\mathbf{r})} = c_{2,3}, ..., f_{k,k-5}^{-1}(\mathbf{r}) = c_{k-5,\mathfrak{z}_{k,k-5}(\mathbf{r})} = c_{k-5,k-4}$, $f_{k,k-2}^{-1}(\mathbf{r}) = c_{k-2,\mathfrak{z}_{k,k-2}(\mathbf{r})} = c_{k-2,k-1}$, $f_{k,k-1}^{-1}(\mathbf{r}) = c_{k-1,\,\mathfrak{z}_{k,k-1}(\mathbf{r})} = c_{k-1,k}$. As $r_{k-3} = r_{k-4} = 1$, we conclude that the number of preceding consecutive 0s of r_{k-3} is zero, whence: $\zeta_{k,k-3}(\mathbf{r}) = 0$, $\mathfrak{z}_{k,k-3}(\mathbf{r}) = k-3-0 = k-3$, and $f_{k,k-3}^{-1}(\mathbf{r}) = c_{k-3,\mathfrak{z}_{k,k-3}}(\mathbf{r}) = c_{k-3,k-3}$. Moreover, as the number of preceding consecutive 0s of r_{k-4} is $k-6$, we conclude that: $\zeta_{k,k-4}(\mathbf{r}) = k-6$, $\mathfrak{z}_{k,k-4}(\mathbf{r}) = k - 4 - (k-6) = 2$, and $f_{k,k-4}^{-1}(\mathbf{r}) = c_{k-4,\mathfrak{z}_{k,k-4}}(\mathbf{r}) = c_{k-4,2}$. Finally, as the number of preceding consecutive 0s of $r_k = 1$ is 2 we have: $\zeta_{k,k}(\mathbf{r}) = 2$, $\mathfrak{z}_{k,k}(\mathbf{r}) = k - 2$ and $f_{k,k}^{-1}(\mathbf{r}) = c_{k,\,\mathfrak{z}_{k,k}(\mathbf{r})} = c_{k,k-2}$. Hence,

$$f_k^{-1}(1, 0, \ldots, 0, 1, 1, 0, 0, 1) = c_{1,1}c_{2,3}...c_{k-5,k-4}c_{k-4,2}c_{k-3,k-3}c_{k-2,k-1}c_{k-1,k}c_{k,k-2}.$$

Taking into account eq. (17) the formula in eq. (12) can be expressed as

$$\det(\mathbf{H}_k) = \sum_{\mathbf{r} \in \mathfrak{R}_k} \prod_{i=1}^{k} c_{i,\,\mathfrak{z}_{k,i}(\mathbf{r})}, \tag{18}$$

that is an expression of $\det(\mathbf{H}_k)$ solely in terms of non-trivial entries of $\mathbf{H}_k$ in (5).

Inner Building Function

The inner building function τ_k is employed to convert integers from $\mathbb{I}_{k-1}$ to arrays in $\mathfrak{R}_k$. This makes it possible to replace the indexing set $\mathfrak{R}_k$ in eq. (18) with the integer indexing set $\mathbb{I}_{k-1} = [\![0, 2^{k-1} - 1]\!]$.

In what follows we shall make use of the conventional notation

$$\mathbf{1}_k = \underbrace{11...1}_{k} \quad (k \text{ number of } 1s),$$

that is, $\mathbf{1}_k$ represents a binary integer consisting of k consecutive 1s. Let $\mathcal{B}_k$ stand for the set of binary integers from 0 up to and including $\mathbf{1}_k$, that is $\mathcal{B}_k = \{0, 1, 10, ..., \mathbf{1}_k\}$. The binary representation of the decimal integer 2^k is the binary integer 10^k, and write for it $[2^k]_2 = 10^k \in \mathcal{B}_k$. Thus, $[2^k - 1]_2 = \mathbf{1}_k$, whence $\mathcal{B}_k$ consists of 2^k binary integers.

Let $b \in \mathcal{B}_k$ for $b \neq 0$. We can write $b = 1\, r_{i+1}...r_n...r_k$, where r_n is 0 or 1 for $i+1 \leq n \leq k$. By adding $(i-1)$ zero digits to the left side of $1\, r_{i+1}...r_k$, the binary number b is represented as:

$$\begin{array}{lccc} 1\, r_{i+1}...r_k \equiv & 0\ 0\ ...\ 0 & 1\ r_{i+1}... & r_k \\ & \uparrow & \uparrow & \uparrow \\ \text{Binary Figures}: & 2^{k-1} & 2^{k-i} & 2^0\ (\text{units}) \end{array} \qquad \text{and} \qquad \begin{array}{lcc} \mathbf{0}_k \equiv & 0\ 0\ ... & 0 \\ & \uparrow & \uparrow \\ & 2^{k-1} & 2^0 \end{array}$$

In the rest of this paper $\mathcal{2}^k$ will stand for the set of functions from $\{1, 2, ..., k\}$ to $\{0, 1\}$, that is the set of all k-arrays of 0s and 1s: $\mathcal{2}^k = \{(r_1, r_2, ..., r_{k-1}, r_k) : r_i = 0 \text{ or } 1\}$. The set $\mathcal{2}^k$ can be naturally identified with the segment of the non-negative binary integers up to and including the integer $\mathbf{1}_k$, that is the sets $\mathcal{2}^k$ and $\mathcal{B}_k$ are identified up to a bijection. For example, let $1011 \in \mathcal{B}_5$. As $k = 5$, writing $1011 = 1r_{i+1}...r_5$, it follows that $r_5 = 1$ and $r_{i+1} = 0$, whence $i + 1 = 3$ or $i - 1 = 1$. Thus adding one zero to the left side of 1011, the binary 1011 is identified with the array $(0, 1, 0, 1, 1) \in \mathcal{2}^5$.

The set $\mathfrak{R}_k$ can be defined as the subset of $\mathcal{2}^k$ consisting of the elements of $\mathcal{2}^k$ whose last component is $r_k = 1$, that is: $\mathfrak{R}_k = \{\mathbf{r} \in \mathcal{2}^k : r_k = 1\}$. Evidently $\text{card}(\mathcal{2}^k) = 2^k$ and $\text{card}(\mathfrak{R}_k) = 2^{k-1}$. Taking into account that $\text{card}(\mathcal{B}_{k-1}) = \text{card}(\mathfrak{R}_k) = 2^{k-1}$, we define the bijection $\rho_k : \mathcal{B}_{k-1} \mapsto \mathfrak{R}_k$:

$$\rho_k(\underbrace{00...01r_{i+1}...r_{k-1}}_{k-1}) = (\underbrace{0, 0, ..., 0, 1, r_{i+1}, ..., r_{k-1}, 1}_{k}) \quad \text{and} \quad \rho_k(\underbrace{00...0}_{k-1}) = (\underbrace{0, 0, ..., 0, 1}_{k}) \tag{19}$$

By identifying $\mathcal{B}_{k-1}$ with $\mathfrak{R}_k$ via ρ_k, the function f_k^{-1}, defined by (8), associates every binary integer $\mathbf{r}$ in $\mathcal{B}_{k-1}$ with the SEP $f_k^{-1}(\mathbf{r})$ in one-to-one fashion. In what follows we shall use the identification $\mathcal{B}_{k-1} \equiv \mathfrak{R}_k$.

Let $x \in \mathbb{Z}_0$ and $y \in \mathbb{Z}_1$. The largest integer not greater than the rational number x/y can be denoted with the aid of the floor function as $\lfloor x/y \rfloor$. Also $\lceil x/y \rceil$ coincides with the smallest integer not less than x/y. Formally $\lfloor x/y \rfloor$ is the quotient of the Euclidean division of x by y. If $n, k \in \mathbb{Z}_0$, the integer $(n \bmod k)$ denotes the remainder of the division of n by k. Two integers x, y are congruent modulo k if $x - y = nk$ for some $n \in \mathbb{Z}$, that is x, y have the same remainder when divided by k and we write: $x \equiv y \bmod k$. We therefore conclude that: $n \bmod 2 = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$. The binary equivalent of $n \in \mathbb{I}_{k-1}$, that is $[n]_2 = r_1 r_2 ... r_{k-1} \in \mathcal{B}_{k-1}$, can be expressed as:

$$[n]_2 = \underbrace{\lfloor n : 2^{k-2} \rfloor \bmod 2}_{r_1} \ \underbrace{\lfloor n : 2^{k-3} \rfloor \bmod 2}_{r_2} \ \ ... \ \ \underbrace{\lfloor n : 2^0 \rfloor \bmod 2}_{r_{k-1}} \tag{20}$$

which induces the bijective transformation:

$$\beta_k : \mathbb{I}_{k-1} \ni n \mapsto \beta_k(n) = [n]_2 \in \mathcal{B}_{k-1}. \tag{21}$$

The composite $\tau_k \stackrel{\text{def}}{=} \rho_k \circ \beta_k$ determines a bijection, which converts non-negative integers in $\mathbb{I}_{k-1}$ into arrays in $\mathfrak{R}_k$, that is

$$\tau_k(n) = (\lfloor n : 2^{k-2} \rfloor \bmod 2, \lfloor n : 2^{k-3} \rfloor \bmod 2, \ldots, \lfloor n : 2^0 \rfloor \bmod 2, 1), \quad n \in \mathbb{I}_{k-1}. \tag{22}$$

Some illustrative examples are given below:

If $k = 1$, then $\tau_1 : \mathbb{I}_0 \mapsto \mathfrak{R}_1$ is defined by: $\tau_1(0) = (1)$.

If $k = 2$, then:

$$\begin{aligned} \tau_2(0) &= (\lfloor 0 : 2^{2-2} \rfloor \bmod 2, 1) = (0 \bmod 2, 1) = (0, 1) \\ \tau_2(1) &= (\lfloor 1 : 2^{2-2} \rfloor \bmod 2, 1) = (1 \bmod 2, 1) = (1, 1). \end{aligned}$$

If $k = 3$, then:

$$\begin{array}{lccc} \tau_3(0) = (& \lfloor 0 : 2^{3-2} \rfloor \bmod 2, & \lfloor 0 : 2^{3-3} \rfloor \bmod 2, & 1) \\ \quad\ \ = (& 0 \bmod 2, & 0 \bmod 2, & 1) \\ \quad\ \ = (& 0, & 0, & 1) \end{array} \Bigg\| \begin{array}{lccc} \tau_3(1) = (& \lfloor 1 : 2^{3-2} \rfloor \bmod 2, & \lfloor 1 : 2^{3-3} \rfloor \bmod 2, & 1) \\ \quad\ \ = (& 0 \bmod 2, & 1 \bmod 2, & 1) \\ \quad\ \ = (& 0, & 1, & 1) \end{array}$$

$$
\begin{array}{llll}
\tau_3(2) = (& \lfloor 2:2^{3-2}\rfloor \bmod 2, & \lfloor 2:2^{3-3}\rfloor \bmod 2, & 1) \\
= (& 1 \bmod 2, & 2 \bmod 2, & 1) \\
= (& 1, & 0, & 1)
\end{array}
\quad \Bigg\| \quad
\begin{array}{llll}
\tau_3(3) = (& \lfloor 3:2^{3-2}\rfloor \bmod 2, & \lfloor 3:2^{3-3}\rfloor \bmod 2, & 1) \\
= (& 1 \bmod 2, & 3 \bmod 2, & 1) \\
= (& 1, & 1, & 1)
\end{array}
$$

If $x \in \mathbb{Z}$ and $y \in \mathbb{Z}_1$, then, applying the well known identity

$$x \mod y = x - y\lfloor \frac{x}{y} \rfloor,$$

for $x = \lfloor n : 2^{k-i} \rfloor$ $(2 \leq i \leq k)$ and $y = 2$, to eq. (22), the latter can be expressed in terms of elementary functions as:

$$\tau_k(n) = (\lfloor n : 2^{k-2} \rfloor - 2\lfloor \tfrac{\lfloor n:2^{k-2} \rfloor}{2} \rfloor, \lfloor n : 2^{k-3} \rfloor - 2\lfloor \tfrac{\lfloor n:2^{k-3} \rfloor}{2} \rfloor, \ldots, \lfloor n : 2^0 \rfloor - 2\lfloor \tfrac{\lfloor n:2^0 \rfloor}{2} \rfloor, 1). \tag{23}$$

2.3. Leibnizian Representation of Hessenbergians

The main result of our previous analysis is stated and proved by Theorem 1 in the present Subsection.

Taking into account that $\tau_k(n)$, $n \in \mathbb{I}_{k-1}$ in eq. (23) defines a bijection, we can substitute $\tau_k(n)$ for $\mathbf{r}$ in eq. (9) to get:

$$\det(\mathbf{H}_k) = \sum_{n=0}^{2^{k-1}-1} f_k^{-1}(\tau_k(n)). \tag{24}$$

For each $i \in \mathbb{Z}$ such that $1 \leq i \leq k$, we define the function

$$\sigma_{k,i}(n) \stackrel{\text{def}}{=} \mathfrak{z}_{k,i} \circ \tau_k(n) \quad \text{for } n \in \mathbb{I}_{k-1}, \tag{25}$$

that is a composite of elementary functions defined over intervals of integers. In view of eq. (25), we can rewrite eq. (17) as:

$$f_{k,i}^{-1}(\tau_k(n)) = c_{i,\mathfrak{z}_{k,i}(\tau_k(n))} = c_{i,\sigma_{k,i}(n)} \quad \text{for } n \in \mathbb{I}_{k-1}. \tag{26}$$

Substituting $\tau_k(n)$ for $\mathbf{r}$ in eq. (10), the latter takes the form

$$f_k^{-1}(\tau_k(n)) = f_{k,1}^{-1}(\tau_k(n)) f_{k,2}^{-1}(\tau_k(n)) \ldots f_{k,k}^{-1}(\tau_k(n)) \in \mathcal{E}_k \quad \text{for} \quad n \in \mathbb{I}_{k-1},$$

that also defines a bijection, since f_k^{-1}, τ_k are bijections. Applying eq. (26) to the foregoing function formula, the latter can be equivalently written as

$$(f_k^{-1} \circ \tau_k)(n) = c_{1,\sigma_{k,1}(n)} c_{2,\sigma_{k,2}(n)} \ldots c_{k,\sigma_{k,k}(n)} \in \mathcal{E}_k \quad \text{for} \quad n \in \mathbb{I}_{k-1}, \tag{27}$$

or in a mapping form as: $f_k^{-1} \circ \tau_k : \mathbb{I}_{k-1} \ni n \mapsto c_{1,\sigma_{k,1}(n)} c_{2,\sigma_{k,2}(n)} \ldots c_{k,\sigma_{k,k}(n)} \in \mathcal{E}_k$. The latter has to be compared with the bijective mapping in eq. (2).

Theorem 1. *The Leibnizian representation of the* kth *order Hessenbergian* $\det(\mathbf{H}_k)$ *in terms of non-trivial entries of* $\mathbf{H}_k$*, defined in eq.* (5)*, is:*

$$\det(\mathbf{H}_k) = \sum_{n=0}^{2^{k-1}-1} \prod_{i=1}^{k} c_{i,\sigma_{k,i}(n)}. \tag{28}$$

Proof. Taking into account that the function in eq. (27) is bijective and starting with eq. (24), the result follows from

$$
\begin{aligned}
\det(\mathbf{H}_k) \quad &= \quad \sum_{n=0}^{2^{k-1}-1} f_k^{-1}(\tau_k(n)) \\
\text{(by eq. (27))} \quad &= \quad \sum_{n=0}^{2^{k-1}-1} \prod_{i=1}^{k} c_{i,\sigma_{k,i}(n)},
\end{aligned}
$$

as claimed. □

The compact representation of Hessenbergians in eq. (28) must be compared with the corresponding nested sum representation of Hessenbergians (see [6] (or eq. (70) herein)) and Mallik's combinatorial formula in [4], as adjusted for Hessenbergians in [6] (see eq. (9) therein). The formula in eq. (28) is also an explicit and compact alternative representation to the recurrence formula (4) for the kth order Hessenbergian. An algorithm for its evaluation associated with a computer program are presented in Appendix C, Algorithm 1. The program is formulated by the Mathematica symbolic language and verifies the formula (28), by yielding an identical result to the one obtained directly by algorithms evaluating determinants, including the recurrence in eq. (4).

3. Linear Difference Equations with Variable Coefficients

Linear difference equations with variable coefficients of order $p \geq 1$ (briefly VC-LDEs(p)) are defined by recurrences of the form

$$y_t = \sum_{m=1}^{p} \phi_m(t) y_{t-m} + v_t, \tag{29}$$

assuming that s is a fixed integer, $\phi_m(t)$, $1 \leq m \leq p$ (variable coefficients) and v_t (forcing term) are complex valued functions defined for all $t \in \mathbb{Z}_{s+1}$. We further assume that $\phi_p(t) \neq 0$ for all $t \in \mathbb{Z}_{s+1}$, ensuring that eq. (29) is of p order. Taking into account that all the parameters in eq. (29) are well defined for any integer $t \geq r+1$, whenever $r \geq s$, by virtue of the existence and uniqueness of the solutions for an initial value problem (see [12], Theorem 2.7., p. 66), a solution of eq. (29) on $\mathbb{Z}_{r+1-p}$, is defined as an explicit expression of y_t for any $t \geq r+1$, written in terms of the initial values $\{y_{r+1-p}, ..., y_r\}$, the variable coefficients and the forcing terms.

3.1. A Unified Construction Process for a Fundamental Solution Set

In this Subsection we present a unified construction process for the elements of a fundamental solution set associated with VC-LDEs(p) (see the first p column sequences in eq. (31) below). This is based upon the row-finite system representation of VC-LDEs(p) (see eq. (30 below) and the Gaussian elimination algorithm applied to it, but implemented with a rightmost pivoting. In contrast to the recursive formulation of individual solution sequences constructed by eq. (29), which necessarily takes on a sequence of p prescribed values, the unified process, presented here, constructs simultaneously all the fundamental solution sequences starting from their $(p+1)$ term, without assuming any prescribed values. This is due to the row canonical form of the reduced system coefficient matrix constructed by the infinite Gaussian elimination algorithm, as discussed below.

Row-finite $\omega \times \omega$ (infinite) linear systems, in their general form, were first studied by Toeplitz [16] (1909), who extended some fundamental results, established on finite linear systems, to cover row-finite ones. Such systems are also represent linear difference equations of irregular order, that is whenever $\phi_p(t) = 0$ for some $t \geq s+1$. Their solution representation was further developed by Fulkerson [17] (1951). He devised and proved the existence of a reduced form, identified here as Fulkerson's row reduced echelon form (FRREF) for any arbitrary row-finite matrix[2]. An FRREF of a row-finite matrix, say $\mathbf{A}$, satisfies three out of four postulates of finite matrices in row reduced echelon form (RREF). It turns out that $\mathbf{A}$ and an FRREF of $\mathbf{A}$ are left associates. Left association generalizes the notion of row-equivalence of finite matrices (see [18]). The RREF of a finite matrix is uniquely associated with the matrix, and therefore it is called row canonical form of the matrix. An FRREF of a row-finite matrix, $\mathbf{A}$, is a quasi-canonical form of $\mathbf{A}$, in the sense that two FRREFs of a row-finite matrix differ only by a permutation of rows. As a consequence, the advantages gained by the row-canonical RREFs for the solution representation of finite systems, are extended to row-finite ones by their quasi-canonical FRREF. The arguments in [17] establishing the existence of an FRREF for a row-finite matrix, invoked the countable axiom of choice. In contrast to the non-constructive nature of this axiom, a modified version of the Gauss-Jordan elimination algorithm has been recently introduced by Paraskevopoulos in [19], which constructs the FRREF of an arbitrary row-finite matrix and called infinite Gauss-Jordan elimination (IGJE) algorithm. In a companion paper (see [18]), he further developed the IGJE algorithm capitalizing on the type and the form of the general solution of row-finite linear systems. If the dimension of the column-null space of the coefficient matrix is infinite, then the IGJE algorithm yields a Schauder basis of the column-null space, relative to the Fréchet metric, otherwise it yields a finite basis of vector spaces. The latter type of basis coincides with a fundamental solution set associated with the row-finite system representation of VC-LDEs(p).

The solution sequence constructed by the IGJE algorithm is solely the result of a rightmost pivot elimination strategy. As a counter example, employing a VC-LDE(1), it is shown in the above cited reference that the conventional Gauss-Jordan elimination algorithm, implemented by a leftmost pivoting, fails to construct both a row-equivalent reduced matrix and the solution of the original VC-LDE(1). This was the main barrier preventing researchers from choosing the IGJE algorithm for solving VC-LDEs and more generally row-finite systems.

Eq. (29) can be equivalently represented by a row-finte linear system whose coefficient matrix is row-finite with elements the variable coefficients at corresponding instances, starting at fixed instance $r+1$:

$$\begin{bmatrix} \phi_p(r+1) & \phi_{p-1}(r+1) & \phi_{p-2}(r+1) \ldots & \phi_1(r+1) & -1 & 0 & 0 & \ldots \\ 0 & \phi_p(r+2) & \phi_{p-1}(r+2) \ldots & \phi_2(r+2) & \phi_1(r+2) & -1 & 0 & \ldots \\ 0 & 0 & \phi_p(r+3) \ldots & \phi_3(r+3) & \phi_2(r+3) & \phi_1(r+3) & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} y_{r+1-p} \\ y_{r+2-p} \\ y_{r+3-p} \\ \vdots \\ y_r \\ y_{r+1} \\ y_{r+2} \\ \vdots \end{bmatrix} = - \begin{bmatrix} v_{r+1} \\ v_{r+2} \\ \vdots \end{bmatrix}. \tag{30}$$

[2]A row-finite matrix is an $\omega \times \omega$ infinite matrix, each row of which comprises a finite number of non-zero entries.

As the coefficient matrix of eq. (30), say $\mathbf{A}$, is in lower echelon form, the infinite Gaussian elimination part (IGE) [3] of the IGJE algorithm is only needed for reducing $\mathbf{A}$ to its FRREF. In this case the reduced matrix is unique and is denoted by $\mathbf{FRREF}(\mathbf{A})$. Implementing the IGE with rightmost pivoting, the pivot elements are the (-1)s occupying the pth superdiagonal of $\mathbf{A}$. A step-by step construction of $\mathbf{FRREF}(\mathbf{A})$ is presented in Appendix B. As shown there, the elimination process leads to a recurrence analogous to that in eq. (29).

The IGE algorithm constructs the $\mathbf{FRREF}(\mathbf{A})$, which is given by

$$\mathbf{FRREF}(\mathbf{A}) = \begin{bmatrix} -\xi^{(p)}_{r+1,r} & -\xi^{(p-1)}_{r+1,r} & \dots & -\xi^{(1)}_{r+1,r} & 1 & 0 & 0 & \dots \\ -\xi^{(p)}_{r+2,r} & -\xi^{(p-1)}_{r+2,r} & \dots & -\xi^{(1)}_{r+2,r} & 0 & 1 & 0 & \dots \\ -\xi^{(p)}_{r+3,r} & -\xi^{(p-1)}_{r+3,r} & \dots & -\xi^{(1)}_{r+3,r} & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{31}$$

In the next Subsection we show that the algorithmic outcomes $\xi^{(m)}_{r+i,r}$ are expansions of banded Hessenbergians of the form: $\xi^{(m)}_{t,r} = \det(\mathbf{\Phi}^{(m)}_{t,r})$, where $\mathbf{\Phi}^{(m)}_{r+1,r} \stackrel{\text{def}}{=} [\phi_m(r+1)]$ (i.e., $\mathbf{\Phi}^{(m)}_{r+1,r}$ is assigned to a 1×1 matrix) and

$$\mathbf{\Phi}^{(m)}_{t,r} = \begin{bmatrix} \phi_m(r+1) & -1 & & & & & \\ \phi_{m+1}(r+2) & \phi_1(r+2) & \ddots & & & & \\ \vdots & \vdots & \ddots & \ddots & & & \\ \phi_p(r+p+1-m) & \phi_{p-m}(r+p+1-m) & \ddots & \ddots & & & \\ & \vdots & \ddots & \ddots & \ddots & & \\ & \phi_p(r+1+p) & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & \phi_p(t-1) & \phi_{p-1}(t-1) & \cdots & -1 \\ & & & & \phi_p(t) & \cdots & \phi_1(t) \end{bmatrix}, \quad t \geq r+2. \tag{32}$$

Here and in what follows empty spaces in a matrix have to be replaced by zeros. For each $m \in [\![1,p]\!]$, we define the mth unit vector by: $\mathbf{e}_{p+1-m} = [\delta_{p+1-m,j}]_{1\leq j\leq p}$, where $\delta_{i,j}$ is the Kronecker delta. We further define the augmented sequence $\xi^{(m)}_{.,r} \stackrel{\text{def}}{=} [\mathbf{e}_{p+1-m} +\!\!+ [\xi^{(m)}_{r+1,r}, \xi^{(m)}_{r+2,s}, ...]]'$, where $[...]'$ stands for transposition and "$+\!\!+$" for concatenation of vectors. As $\mathbf{FRREF}(\mathbf{A})$ is a row-canonical form of $\mathbf{A}$ we conclude: First, for any $m \in [\![1,p]\!]$ the augmented sequence $\xi^{(m)}_{.,r}$, is a solution of the homogeneous system $\mathbf{A}\ \mathbf{y} = \mathbf{0}$ associated with eq. (30)). This is conveniently verified by $\mathbf{A}\ \xi^{(m)}_{.,r} = \mathbf{0}$ (see also Example 1 below). The above stated result is independently proved by Proposition 2 in Subsection 3.2. The terms $\xi^{(m)}_{r+i,r}$, $i \geq 1$, of each individual $\xi^{(m)}_{.,r}$ can also be constructed recursively via eq. (29), when $v_{r+i} = 0$ for all $i \geq 1$, taking on the components of $\mathbf{e}_{p+1-m}$ as initial values. Second, the set $\varXi_r = \{\xi^{(m)}_{.,r}\}_{1\leq m\leq p}$ turns out to be a fundamental solution set associated with eq. (30)) (or eq. (29). This result is independently established by Theorem 2) in Subsection 3.2. The fundamental solution sequence $\xi^{(1)}_{.,r}$, termed principal fundamental solution, is the first element of $\varXi_r$, but its terms $\xi^{(1)}_{t,r}$ for $t \geq r+1$ occupy the pth opposite signed column of $\mathbf{FRREF}(\mathbf{A})$, that is: $\xi^{(1)}_{.,r} = [\mathbf{e}_p +\!\!+ [\xi^{(m)}_{r+1,r}, \xi^{(1)}_{r+2,r}, ...]]' = [0,0,...,1,\xi^{(1)}_{r+1,r}, \xi^{(1)}_{r+2,r}, ...]'$.

Example 1. In this Example we apply the IGE algorithm to a row-finite system representation of VC-LDE(2). The IGE algorithm applies to eq. (30) for $p = 2$ and simultaneously constructs the terms $\{\xi^{(1)}(t,r)\}_{t\geq r+1}$ and $\{\xi^{(2)}(t,r)\}_{t\geq r+1}$ of the two fundamental solutions. The opposite signed terms of these sequences occupy the first and second columns of $\mathbf{FRREF}(\mathbf{A})$. The first three opposite sign terms of the second column in eq. (31) are:

$$\begin{aligned} &\xi^{(1)}_{r+1,r} = \phi_1(r+1), \quad \xi^{(1)}_{r+2,r} = \phi_1(r+1)\phi_1(r+2) + \phi_2(r+2), \\ &\xi^{(1)}_{r+3,r} = \phi_1(r+1)[\phi_1(r+2)\phi_1(r+3) + \phi_2(r+3)] + \phi_2(r+2)\phi_1(r+3), ... \end{aligned}$$

(see the Appendix B for a verification of $\mathbf{A}\ [0,1,\xi^{(1)}_{r+1,r}, \xi^{(1)}_{r+2,r}, ...]' = \mathbf{0}$). The above sequence augmented on its left by the unit vector $\mathbf{e}_2 = [0,1]$ yields the first fundamental solution. The algorithmic outcomes are expansions of the following Hessenbergians (determinant of a lower Hessenberg matrix):

$$\xi^{(1)}_{r+1,r} = \phi_1(r+1),\ \xi^{(1)}_{r+2,r} = \begin{vmatrix} \phi_1(r+1) & -1 \\ \phi_2(r+2) & \phi_1(r+2) \end{vmatrix},\ \xi^{(1)}_{r+3,r} = \begin{vmatrix} \phi_1(r+1) & -1 & \\ \phi_2(r+2) & \phi_1(r+2) & -1 \\ & \phi_2(r+3) & \phi_1(r+3) \end{vmatrix}, ...$$

The corresponding terms of the first column are:

$$\xi^{(2)}_{r+1,r} = \phi_2(r+1), \quad \xi^{(2)}_{r+2,r} = \phi_2(r+1)\phi_1(r+2), \quad \xi^{(1)}_{r+3,r} = \phi_2(r+1)[\phi_1(r+2)\phi_1(r+3) + \phi_2(r+3)], ...$$

[3] A recursive alternative to the infinite Gaussian elimination algorithm is presented in [20].

This sequence augmented on its left by the unit vector $\mathbf{e}_1 = [1, 0]$ yields the second fundamental solution. By analogy, these are expansions of the following Hessenbergians:

$$\xi^{(2)}_{r+1,r} = \phi_2(r+1),\ \xi^{(2)}_{r+2,r} = \begin{vmatrix} \phi_2(r+1) & -1 \\ & \phi_1(r+2) \end{vmatrix},\ \xi^{(2)}_{r+3,r} = \begin{vmatrix} \phi_2(r+1) & -1 & \\ & \phi_1(r+2) & -1 \\ & \phi_2(r+3) & \phi_1(r+3) \end{vmatrix}, \dots$$

Applying the same sequence of row elementary operations, used by the IGE for the row reduction of $\mathbf{A}$ to $\mathbf{FRREF}(\mathbf{A})$, but now to the sequence $\{-\upsilon_{r+i}\}_{i\geq 1}$, a particular solution sequence is constructed (see Appendix B). The process leads to a recurrence, which equivalently constructs the particular solution. This solution sequence is also explicitly represented by a Hessenbergian function, but not a banded one, as shown in Proposition 5. The general solution is a linear combination of the fundamental solutions with coefficients arbitrary initial condition values $y_{r+1-m} = a_m$ for $1 \leq m \leq p$ (that is the general homogeneous solution, see Proposition 3) plus the aforementioned particular solution (see eq. (64)).

3.2. Fundamental Set of Solutions

A fundamental set of solutions associated with VC-LDEs(p) plays a significant role in the explicit representation of the Green's function, the companion matrix product (or the Casorati matrix) as well as the general solution of VC-LDEs(p). The existence of such solution sets is theoretically guaranteed by the fundamental Theorem of VC-LDEs(p) (see [12] p. 74). As a consequence of the superposition principle (see the previously cited reference) the homogeneous solution of VC-LDEs(p) can be expressed as a linear combination of fundamental solutions whose coefficients are expressions involving the initial condition values.

In the previous Subsection, the IGE algorithm was employed to construct simultaneously the fundamental solution sequences $\xi^{(m)}_{\cdot,r}$ for $1 \leq m \leq p$ associated with eq. (29). However, the banded Hessenbergian structure of the IGE algorithmic outcomes must be formally established. For this purpose, we define m functions $\xi^{(m)}_{t,r}$ $(1 \leq m \leq p)$ of two independent variables (t, r), showing independently that for each fixed $r \geq s$ they form a fundamental solution set.

Given some $r \geq s$, the homogeneous linear difference equation associated with eq. (29) is of the form

$$y_t = \sum_{m=1}^{p} \phi_m(t) y_{t-m}, \quad t \geq r+1, \tag{33}$$

that is eq. (29) applied with forcing terms $\upsilon_t = 0$ for all $t \geq s+1$. The linear difference operator associated with eq. (33) is

$$\Phi_t(B) = 1 - \sum_{m=1}^{p} \phi_m(t) B^m, \quad t \geq r+1, \tag{34}$$

where B is the backshift (or lag) operator. Eq. (33) can be equivalently rephrased as: $\Phi_t(B) y_t = \mathbf{0}$.

One of the objectives of this Subsection is to provide an explicit solution function (or sequence) y_t of eq. (33) on $\mathbb{Z}_{r+1}$ for a fixed $r \geq s$, solely expressed in terms of the initial values $\{y_{r+1-m}\}_{1\leq m\leq p}$ and the variable coefficients $\phi_m(t)$. Hereafter, in absence of ambiguity, we use the common notation, y_t, in place of the formal solution notation $y_{t,r}$ of the corresponding initial value problem, since r for $r \geq s$ is assumed to be fixed.

For each $m \in [\![1, p]\!]$, we define the two variable function $\xi^{(m)}_{t,r}$ for $(t, r) \in \mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ associated with eq. (33) (or (34)) as follows

$$\xi^{(m)}_{t,r} = \begin{cases} \phi_m(r+1) & \text{if } t = r+1, \\ \det(\mathbf{\Phi}^{(m)}_{t,r}) & \text{if } r+2 \leq t, \\ 1 & \text{if } t = r-m+1, \\ 0 & \text{elsewhere,} \end{cases} \quad (r \geq s \text{ and } t \geq s+1-p) \tag{35}$$

where $\mathbf{\Phi}^{(m)}_{t,r}$ is given by eq. (32). The matrices $\mathbf{\Phi}^{(m)}_{t,r}$ for $t \geq r+1$ are banded lower Hessenberg matrices of order $k = t - r$. After a large enough t $(t \geq r+p+1-m)$, $\mathbf{\Phi}^{(m)}_{t,r}$ admits a fixed total bandwidth $(p+1)$, noticing that the matrices $\mathbf{\Phi}^{(m_1)}_{t,r}$ and $\mathbf{\Phi}^{(m_2)}_{t,r}$ for $m_1 \neq m_2$ differ only in their first column (see eq. (32). The determinant $\xi^{(m)}_{t,r} = |\mathbf{\Phi}^{(m)}_{t,r})|$ for $t > r$, is a banded Hessenbergian associated with the prescribed values:

$$\xi^{(m)}_{r+1-m,r} = 1 \text{ and } \xi^{(m)}_{r-i,r} = 0, \text{ whenever } i \neq m-1. \tag{36}$$

For each $m \in [\![1, p]\!]$, eqs. (36) describe the initial condition unit vector $\mathbf{e}_{p+1-m} = [\xi^{(m)}_{r+1-p,r}, \xi^{(m)}_{r+2-p,r}, ..., \xi^{(m)}_{r,r}]$. If $m = 1$, then $[\xi^{(1)}_{r+1-p,r}, \xi^{(1)}_{r-1,r}, ..., \xi^{(1)}_{r,r}] = [0, 0, ..., 1] = \mathbf{e}_p$. Some useful values $\xi^{(m)}_{t,r}$ for $1 \leq m \leq p$ are given below:

$$\xi^{(1)}_{t,t} = \xi^{(1)}_{r,r} = \xi^{(m)}_{r-m+1,r} = 1, \xi^{(1)}_{r,r+i} = \xi^{(1)}_{t-i,t} = 0\ (i > 0), \xi^{(m)}_{t+1,t} = \phi_m(t+1), \xi^{(m)}_{r+2,r} = \begin{vmatrix} \phi_m(r+1) & -1 \\ \phi_{m+1}(r+2) & \phi_1(r+2) \end{vmatrix}.$$

In what follows we shall also use the notation $\xi_{\cdot,r}^{(m)} = \{\xi_{t,r}^{(m)}\}_{t\in\mathbb{Z}_{r+1-p}}$ for a fixed $r \geq s$. By an abuse of terminology, the functions $\xi_{t,r}^{(m)}$ defined on $\mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ in eq. (35) will be referred to as banded Hessenbergians.

Taking into account that $\mathbf{\Phi}_{t,r}^{(m)}$ in eq. (32) is a banded Hessenberg matrix, the matrix $\mathbf{H}_{t-r}$ in eq. (5) can be identified with $\mathbf{\Phi}_{t,r}^{(m)}$ via the assignment

$$c_{i,j} = \begin{cases} \phi_{i-1+m}(r+i) & \text{if } j=1 \text{ and } 1 \leq i \leq p+1-m \\ \phi_{i+1-j}(r+i) & \text{if } 2 \leq j \leq i \leq t-r \text{ and } 1 \leq i-j+1 \leq p \\ 1 & \text{if } j = i+1 \\ 0 & \text{elsewhere,} \end{cases} \tag{37}$$

provided that $m \in [\![1,p]\!]$ is fixed, each time we referred to eq. (37).

Proposition 2. *Let $m \in [\![1,p]\!]$. The sequence $\{\xi_{t,r}^{(m)}\}_{t\in\mathbb{Z}_{r+1}}$ for any arbitrary but fixed $r \geq s$, solves eq.* (33), *assuming the initial condition unit vector $[\xi_{r+1-p,r}^{(m)}, \xi_{r+2-p,r}^{(m)}, ..., \xi_{r,r}^{(m)}] = \mathbf{e}_{p+1-m}$.*

Proof. Taking into account that $c_{l,l+1} = 1$, the recurrence in eq. (7) applied for $i = k = t-r$ (the order of the matrix) takes the form:

$$|\mathbf{H}_i| = c_{i,1}|\mathbf{H}_0| + c_{i,2}|\mathbf{H}_1| + ... + c_{i,i-1}|\mathbf{H}_{i-2}| + c_{i,i}|\mathbf{H}_{i-1}|. \tag{38}$$

We examine the following cases:
i) Let $1 \leq i \leq p+1-m$. This inequality can be equivalently written as $r+1 \leq t \leq r+p+1-m$, which means that $\mathbf{\Phi}_{t,r}^{(m)}$ in eq. (32) is a full lower Hessenberg matrix. We can equivalently write the above inequality as $t = r+p+1-m-l$, whenever $0 \leq l \leq p-m$. As $i = t-r = p+1-m-l$, it follows from eq. (37) that $c_{i,1} = c_{t-r,1} = \phi_{p+1-m-l-1+m}(r+p+1-m-l) = \phi_{p-l}(t)$ and $c_{i,2} = c_{t-r,2} = \phi_{p+1-m-l-2+1}(r+p+1-m-l) = \phi_{p-m-l}(t)$. Proceeding in this way, the remaining values of $c_{i,j}$ for $3 \leq j \leq p$ are given by: $c_{t-r,3} = \phi_{t-r-2}(t) = \phi_{p-m-l}(t), ..., c_{t-r,t-r-1} = \phi_2(t), c_{t-r,t-r} = \phi_1(t)$. The left-hand side of eq. (38) we can be replaced with $|\mathbf{H}_{t-r}| = \xi_{t,r}^{(m)}$. Working on the right-hand side of eq. (38), we can replace: $|\mathbf{H}_0|$ with 1 (or $\xi_{r+1-m,r}^{(m)}$), $|\mathbf{H}_1|$ with $\phi_m(r+1) = \xi_{r+1,r}^{(m)}, ..., |\mathbf{H}_{t-r-2}|$ with $\xi_{t-2,r}^{(m)}$ and $|\mathbf{H}_{t-r-1}|$ with $\xi_{t-1,r}^{(m)}$, whence:

$$\xi_{t,r}^{(m)} = \underbrace{\phi_{p-l}(t)\xi_{r+1-m,r}^{(m)} + \phi_{p-l-1}(t)\xi_{r+2-m,r}^{(m)} + ... + \phi_{p-l+1-m}(t)\xi_{r,r}^{(m)}}_{\text{initial values}} + \phi_{p-l-m}(t)\xi_{r+1,r}^{(m)} + ... + \phi_2(t)\xi_{t-2,r}^{(m)} + \phi_1(t)\xi_{t-1,r}^{(m)},$$

where the values of $\xi_{r+1-m,r}^{(m)}$ up to and including $\xi_{r,r}^{(m)}$ are initial values defined in eq. (36), that is $\xi_{r+1-m,r}^{(m)} = 1$ and the remaining initial values are zero, whenever $m \neq 1$. If $m = 1$, then $\xi_{r,r}^{(1)} = \xi_{r+1-m,r}^{(m)} = 1$.
ii) Let $i > p+1-m$. As $t-r-1+m > p$, in view of eq. (32), the matrix $\mathbf{\Phi}_{t,r}^{(m)}$ is a lower banded Hessenberg matrix. This means that the first values of $c_{i,j}$ in eq. (38) starting from $c_{i,1}$ up to and including $c_{i,i-p}$ are zero. Applying eq. (37), we have: $c_{i,i} = \phi_1(t)$, $c_{i,i-1} = \phi_2(t), ..., c_{i,i-p+1} = \phi_p(t)$, which is in accord with eq. (37), since: $c_{t-r,1} = ... = c_{t-r,t-r-p} = 0$. Substituting the above values of $c_{i,j}$ and replacing $\det(\mathbf{H}_{t-r-j})$ with $\xi_{t-j,r}^{(m)}$ for $0 \leq j \leq p$ in the recurrence (7), the latter takes the form:

$$\xi_{t,r}^{(m)} = \phi_p(t)\xi_{t-p,r}^{(m)} + \phi_{p-1}(t)\xi_{t-p+1,r}^{(m)} + ... + \phi_2(t)\xi_{t-2,r}^{(m)} + \phi_1(t)\xi_{t-1,r}^{(m)}.$$

In both cases $\xi_{\cdot,r}^{(m)}$ satisfies eq. (33), as required. □

An alternative to the second part in the proof of Proposition 2 can be deduced from Lemma 1(ii) below.

Theorem 2. *For each fixed $r \geq s$, the set $\Xi_r = \{\xi_{\cdot,r}^{(1)}, \xi_{\cdot,r}^{(2)}, ..., \xi_{\cdot,r}^{(p)}\}$, consisting of p functions (or sequences) defined over the same domain $\mathbb{Z}_{r+1-p}$, is a fundamental set of solutions associated with eq.* (33).

Proof. Notice first that each sequence $\xi_{\cdot,r}^{(m)}$ (the mth element of Ξ_r) starts with the value $\xi_{r+1-p,r}^{(m)}$, whence the domain of the function $\xi_{\cdot,r}^{(m)}$ is $\mathbb{Z}_{r+1-p}$. Taking into account that the elements of Ξ_r are solutions of eq. (33) (see Proposition 2), it suffices to verify that the set Ξ_r is linearly independent. Equivalently, it must be shown that the Casoratian of the matrix

$$\mathbf{\Xi}_{t,r} = \begin{bmatrix} \xi_{t,r}^{(1)} & \xi_{t,r}^{(2)} & \cdots & \xi_{t,r}^{(p)} \\ \xi_{t-1,r}^{(1)} & \xi_{t-1,r}^{(2)} & \cdots & \xi_{t-1,r}^{(p)} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \xi_{t-p+1,r}^{(1)} & \xi_{t-p+1,r}^{(2)} & \cdots & \xi_{t-p+1,r}^{(p)} \end{bmatrix} \tag{39}$$

associated with the solution set Ξ_r is nonzero for all $t \geq r$. Definition (35) entails that the matrix $\mathbf{\Xi}_{r,r}$ is the identity matrix of order p, that is $\mathbf{\Xi}_{r,r} = \mathbf{I}_p$. Therefore the first Casoratian $|\mathbf{\Xi}_{r,r}|$ of the set Ξ_r is $|\mathbf{\Xi}_{r,r}| = 1 \neq 0$. It follows that $|\mathbf{\Xi}_{t,r}| \neq 0$ for all $t \geq r$ (see Lemma 1.3 in [8], applied for $a = r-p+1$) and the set Ξ_r is linearly independent. That is Ξ_r is a fundamental set of solutions of eq. (33). □

For any fixed $r \geq s$, the dimension of the homogeneous solution space of $\Phi_t(B)$ for $t \geq r+1$ in eq. (34) is p. As Ξ_r spans this space and card$(\Xi_r) = p$, it follows that the set Ξ_r is a basis of the column-null space of the coefficient matrix of eq. (30). Besides, as $|\mathbf{\Xi}_{t,r}| \neq 0$ for all $t \geq r$, Corollary 1 below, follows immediately.

Corollary 1. *The Casorati matrix $\mathbf{\Xi}_{t,r}$ in eq. (39) is invertible (or non-singular) for all $t \geq r$ and any $r \geq s$.*

Corollary (1) is also established independently, as a consequence of Theorem 3 in Subsection 3.4.

Proposition 3. *Let $r \geq s$. A solution sequence $\{y_t\}_{t\in\mathbb{Z}_{r+1}}$ of eq. (33), can be explicitly and uniquely expressed in terms of any sequence of prescribed values, say $\{y_{r+1-m}\}_{1\leq m\leq p}$, as*

$$y_t = \sum_{m=1}^{p} \xi_{t,r}^{(m)} y_{r+1-m}. \tag{40}$$

Proof. As Ξ_r (defined in Theorem 2 is a fundamental set of solutions whose elements are defined over $\mathbb{Z}_{r+1-p}$, that is $\xi_{.,r}^{(m)} = \{\xi_{t,r}^{(m)}\}_{t\in\mathbb{Z}_{r+1-p}}$, there exist unique scalars a_m for $1 \leq m \leq p$ such that:

$$y_t = \sum_{m=1}^{p} a_m \xi_{t,r}^{(m)}, \text{ for all } \ t \geq r+1-p. \tag{41}$$

Taking into account that $[\xi_{r+1-p,r}^{(m)}, \xi_{r+2-p,r}^{(m)}, ..., \xi_{r,r}^{(m)}] = \mathbf{e}_{p+1-m}$ for $1 \leq m \leq p$ (see eqs. (36)), applying eq. (41) for $t = r+1-j$ with $1 \leq j \leq p$, we have: $y_{r+1-j} = \sum_{m=1}^{p} a_m \xi_{r+1-j,r}^{(m)} = a_j$, for all $j = 1, ..., p$ as claimed. □

3.3. Principal Determinant

Using the notation of the previous Subsection, the building element for the remaining results of this paper is the first banded Hessenbergian function $\xi_{t,r}^{(1)}$, as presented below. We show that the terms $\{\xi_{t,r}^{(m)}\}_{t\geq r+1}$ of the fundamental solution $\xi_{.,r}^{(m)}$ for $1 \leq m \leq p$ can be expressed in terms of $\xi_{t,r}^{(1)}$ and the variable coefficients. This result yields an explicit representation of the general homogeneous solution of eq. (33) exclusively in terms of $\xi_{t,r}^{(1)}$, the variable coefficients and the initial conditions.

Definition 2. *The principal matrix associated with the difference operator in eq. (34) is denoted by $\mathbf{\Phi}_{t,r}$ and is defined by setting $m = 1$ in the first branch of eq. (35), that is*

$$\mathbf{\Phi}_{t,r} = \begin{bmatrix} \phi_1(r+1) & -1 & & & & & \\ \phi_2(r+2) & \phi_1(r+2) & \ddots & & & & \\ \vdots & \vdots & \ddots & \ddots & & & \\ \phi_p(r+p) & \phi_{p-1}(r+p) & \ddots & \ddots & & & \\ & \vdots & \ddots & \ddots & \ddots & & \\ & \phi_p(r+p+1) & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & \phi_p(t-1) & \phi_{p-1}(t-1) & \cdots & -1 \\ & & & & \phi_p(t) & \cdots & \phi_1(t) \end{bmatrix} \tag{42}$$

The determinant of $\mathbf{\Phi}_{t,r}$ is called principal determinant and denoted by: $\xi_{t,r} \stackrel{\text{def}}{=} \xi_{t,r}^{(1)}$. The two variable function $\xi_{t,r}$ on $\mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$, defined in eq. (35) for $m = 1$, will be referred to as principal determinant function.

In Proposition 4, we show a generic feature of the principal determinant function $\xi_{t,r}$, that is every term $\xi_{t,r}^{(m)}$ for $t \geq r+1$ of the corresponding fundamental solution can be expressed in terms of the principal determinant. In order to simplify proofs, in place of banded Hessenbergians, we shall use full Hessenbergians of order $(t-r)$, as follows: $\xi_{t,r}^{(m)} = \det(\mathbf{\Phi}_{t,r}^{(m)})$, where

$$\mathbf{\Phi}_{t,r}^{(m)} = \begin{bmatrix} \phi_m(r+1) & -1 & & & & \\ \phi_{m+1}(r+2) & \phi_1(r+2) & \ddots & & & \\ \phi_{m+2}(r+3) & \phi_2(r+3) & \ddots & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ \phi_{m+t-r-2}(t-1) & \phi_{t-r-2}(t-1) & \cdots & \phi_2(t-1) & \phi_1(t-1) & -1 \\ \phi_{m+t-r-1}(t) & \phi_{t-r-1}(t) & \cdots & \phi_3(t) & \phi_2(t) & \phi_1(t) \end{bmatrix}, \tag{43}$$

coupled with the convention: $\phi_i(t) = 0$, whenever $i > p$ and for all $t \in \mathbb{Z}_{r+1}$. Under this convention full Hessenbergians turn into banded ones and in this case the matrices in eqs. (32) and (43) coincide.

First we state and prove the following Lemma:

Lemma 1. i) *The cofactor of the coefficient* $\phi_{m+i-1}(r+i)$ *for* $1 \leq i \leq t-r$ *in the first column of* $\mathbf{\Phi}^{(m)}_{t,r}$ *in eq.* (43) *coincides with* $\xi_{t,r+i}$. ii) *The cofactor of the coefficient* $\phi_n(t)$ *for* $1 \leq n \leq t-r-1$, *in the last row of* $\xi^{(m)}_{t,r}$ *coincides with* $\xi^{(m)}_{t-n,r}$. *If* $n = m+t-r-1$, *then* $\xi^{(m)}_{t-n,r} = \xi^{(m)}_{r+1-m,r} = 1$.

Proof. i) Let us call $M^{(0)}_{1,1}$ the minor of the $(1,1)$ entry of $\mathbf{\Phi}^{(m)}_{t,r}$ (occupied by $\phi_m(r+1)$). $M^{(0)}_{1,1}$ is the determinant of the $(t-r-1)\times(t-r-1)$ submatrix of $\mathbf{\Phi}^{(m)}_{t,r}$, formed by deleting the first row and column of $\mathbf{\Phi}^{(m)}_{t,r}$. It follows directly that $M^{(0)}_{1,1} = \xi_{t,r+1}$. The latter result multiplied by $(-1)^{1+1}$ yields the cofactor of $\phi_m(r+1)$, that is: $Cof[\phi_m(r+1)] = (-1)^{1+1}\xi_{t,r+1} = \xi_{t,r+1}$. In view of eq. (43), it remains to show the assertion for any i such that $1 < i \leq m+t-r-1$. The minor of the $(i,1)$ entry of $\mathbf{\Phi}^{(m)}_{t,r}$ (occupied by $\phi_{m+i-1}(r+i)$), call it as $M^{(0)}_{i,1}$, is derived by deleting the first column and the ith row of $\mathbf{\Phi}^{(m)}_{t,r}$, that is:

$$M^{(0)}_{i,1} = \begin{vmatrix} -1 & & & & & & & & \\ \phi_1(r+2) & -1 & & & & & & & \\ \phi_2(r+3) & \phi_1(r+3) & & & & & & & \\ \vdots & \vdots & & & & & & & \\ \phi_{i-2}(r+i-1) & \phi_{i-3}(r+i-1) & \dots & -1 & 0 & & & & \\ \phi_i(r+i+1) & \phi_{i-1}(r+i+1) & \dots & \phi_2(r+i+1) & \phi_1(r+i+1) & -1 & & & \\ \vdots & \vdots & & \vdots & \vdots & & & & \\ \phi_{t-r-2}(t-1) & \phi_{t-r-3}(t-1) & \cdots & \phi_{t-1-r-i}(t-1) & \phi_{t-2-r-i}(t-1) & \phi_{t-3-r-i}(t-1) & \cdots & \phi_1(t-1) & -1 \\ \phi_{t-r-1}(t) & \phi_{t-r-2}(t) & \dots & \phi_{t-r-i}(t) & \phi_{t-1-r-i}(t) & \phi_{t-2-r-i}(t) & \dots & \phi_2(t) & \phi_1(t) \end{vmatrix}.$$

We observe that the main diagonal elements of $M^{(0)}_{i,1}$ up to and including the entry $(i-1,i-1)$ are occupied by (-1)s, while the main diagonal element next to $(i-1,i-1)$ is occupied by $\phi_1(r+i+1)$. In what follows we compute $M^{(0)}_{i,1}$ recursively. Deleting the first row and column of $M^{(0)}_{i,1}$ the determinant of the resulting submatrix is the minor of the $(1,1)$ entry of $M^{(0)}_{i,1}$, denoted by: $M^{(1)}_{i,1}$. Proceeding in this way we denote $M^{(j)}_{i,1}$ for $1 \leq j \leq i-1$, the minor of the $(1,1)$ entry of $M^{(j-1)}_{i,1}$, that is the (j,j) element of $M^{(0)}_{i,1}$, occupied by (-1). As the first row of $M^{(j-1)}_{i,1}$ is $(-1,0,...,0)$ for any $j \in [\![1,i-1]\!]$, expanding $M^{(j-1)}_{i,1}$ along the first row, we obtain the recurrence:

$$M^{(j-1)}_{i,1} = (-1)M^{(j)}_{i,1}, \qquad 1 \leq j \leq i-1. \tag{44}$$

In particular, if $j=1$ the recurrence in eq. (44) gives: $M^{(0)}_{i,1} = (-1)M^{(1)}_{i,1}$. If $j = i-1$, the minor of the $(1,1)$ entry of $M^{(i-2)}_{i,1}$, that is the $(i-1,i-1)$ element of $M^{(0)}_{i,1}$ (occupied by the last (-1) in the main diagonal of $M^{(0)}_{i,1}$), is given by:

$$M^{(i-1)}_{i,1} = \begin{vmatrix} \phi_1(r+i+1) & -1 & & & \\ \phi_2(r+i+2) & \phi_1(r+i+2) & -1 & & \\ \vdots & \vdots & \vdots & & \\ \phi_{t-2-r-i}(t-1) & \phi_{t-3-r-i}(t-1) & \cdots & \phi_1(t-1) & -1 \\ \phi_{t-1-r-i}(t) & \phi_{t-2-r-i}(t) & \cdots & \phi_2(t) & \phi_1(t) \end{vmatrix} = \xi_{t,r+i}.$$

The recurrence (44) yields: $M^{(0)}_{i,1} = (-1)M^{(1)}_{i,j} = (-1)^2M^{(2)}_{i,j} = ... = (-1)^{i-1}M^{(i-1)}_{i,j} = (-1)^{i-1}\xi_{t,r+i}$. Accordingly the cofactor of the $(i,1)$ entry of $\mathbf{\Phi}^{(m)}_{t,r}$ is $Cof[\phi_{i-1}(r+i)] = (-1)^{i+1}M^{(0)}_{i,1} = (-1)^{i+1}(-1)^{i-1}\xi_{t,r+i} = \xi_{t,r+i}$, as claimed.
ii) Working similarly as in part (i), but expanding $\mathbf{\Phi}^{(m)}_{t,r}$ along the last row (instead of the first column), the result follows. □

As a direct consequence of Lemma 1(ii), the cofactor expansion of $\xi^{(m)}_{t,r}$ along the last row gives

$$\xi^{(m)}_{t,r} = \phi_1(t)\xi^{(m)}_{t-1,r} + \phi_2(t)\xi^{(m)}_{t-2,r} + ... + \phi_p(t)\xi^{(m)}_{t-p,r} \text{ for } t \geq r+1 \tag{45}$$

which re-establishes the second part in the proof of Proposition 2. In Proposition A6 of Appendix A, the recurrence in eq. (45) is employed to show from first principles a property of Casoratians associated with VC-LDEs(p) (see eq. (2.5), p. 39, in [8]), that is, they satisfy a first order linear recurrence.

Proposition 4. *The terms* $\xi^{(m)}_{t,r}$ *for* $1 \leq m \leq p$ *and* $t \geq r+1$ *of each fundamental solution* $\xi^{(m)}_{\cdot,r}$ *can be expressed in terms of the principal determinant function as:*

$$\xi^{(m)}_{t,r} = \sum_{i=1}^{p-m+1} \phi_{m-1+i}(r+i)\xi_{t,r+i}. \tag{46}$$

Proof. Applying Lemma 1(i) to $\xi_{t,r}^{(m)}$ in eq. (32), the cofactor expansion of $\xi_{t,r}^{(m)}$ along the first column gives $\xi_{t,r}^{(m)} = \phi_m(r+1)\xi_{t,r+1} + \phi_{m+1}(r+2)\xi_{t,r+2} + ... + \phi_p(r+p+1-m)\xi_{t,r+p+1-m}$, whence

$$\xi_{t,r}^{(m)} = \sum_{i=m}^{p} \phi_i(r+i+1-m)\xi_{t,r+i+1-m} = \sum_{i=1}^{p+1-m} \phi_{m-1+i}(r+i)\xi_{t,r+i},$$

as required. □

Applying eq. (46) to eq. (40), we obtain an explicit representation of the general solution y_t of eq. (33) in terms of the principal determinant function and any sequence of prescribed values $\{y_{r+1-m}\}_{1\leq m\leq p}$ for a fixed $r \geq s$, as follows:

$$y_t = \sum_{m=1}^{p} \sum_{i=1}^{p+1-m} \phi_{m-1+i}(r+i)\xi_{t,r+i}y_{r+1-m} \quad \text{for all } t \geq r+1. \tag{47}$$

3.4. Companion Matrix Product

We show in the current Subsection that the elements of the product of companion matrices associated with the difference operator in eq. (34) can be explicitly represented by banded Hessenbergians.

Let $t \in \mathbb{Z}_{s+1}$. The companion matrix of order p is given by

$$\mathbf{\Gamma}_t = \begin{bmatrix} \phi_1(t) & \phi_2(t) & ... & \phi_{p-1}(t) & \phi_p(t) \\ 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & 1 & 0 \end{bmatrix}. \tag{48}$$

Eq. (33) for $r \geq s$ can be expressed, as a vector equation, by:

$$\mathbf{y}_t = \mathbf{\Gamma}_t \ \mathbf{y}_{t-1}, \quad t \geq r+1. \tag{49}$$

An extended Definition of the companion matrix product, including the case $t = r$, is given by

$$\mathbf{F}_{t,r} \stackrel{\text{def}}{=} \begin{cases} \mathbf{\Gamma}_t\mathbf{\Gamma}_{t-1}...\mathbf{\Gamma}_{r+1}, & \text{if } t \geq r+1 \\ \mathbf{I}_p, & \text{if } t = r. \end{cases} \tag{50}$$

$\mathbf{F}_{t,r}$ is invertible, since $\mathbf{\Gamma}_i$ is invertible for all $i \in \mathbb{Z}_{s+1}$. As $\mathbf{F}_{r,r} = \mathbf{I}_p$, we further conclude that $\mathbf{F}_{t,r}$ is invertible for all $t \in \mathbb{Z}_r$ and any $r \geq s$. Taking into account that the matrix multiplication is non-commutative , we can alternatively use the condense notation: $\mathbf{\Gamma}_t\mathbf{\Gamma}_{t-1}...\mathbf{\Gamma}_{r+1} = \prod_{i=r}^{t-1} \mathbf{\Gamma}_{t-i+r}$.

Let $\mathbf{y}_r = [y_r, y_{r-1}, ..., y_{r-p+1}]'$ be an initial condition vector associated with eq. (33). Then the unique vector solution of the corresponding initial value problem associated with eq. (33) can also be described by the vector equation:

$$\mathbf{y}_t = \mathbf{F}_{t,r} \ \mathbf{y}_r \quad \text{for } t \geq r. \tag{51}$$

This is an alternative interpretation to the solution in eq. (40). If $t = r$, then $\mathbf{y}_r = \mathbf{I}_p \ \mathbf{y}_r$, as expected.

In all that follows $\mathbf{\Xi}_{t,r}$ stands for the Casorati matrix defined in eq. (39). Before proving the main result of this Subsection in Theorem 3, we recall an elementary result from linear algebra:

Remark 1. *Let* $\mathbf{A}, \mathbf{B}$ *be* $k \times k$ *complex matrices. If* $\mathbf{A} \ \mathbf{x} = \mathbf{B} \ \mathbf{x}$ *for all* $\mathbf{x} \in \mathbb{C}^k$*, then* $\mathbf{A} = \mathbf{B}$.

Theorem 3. *The product of companion matrices* $\mathbf{F}_{t,r}$ *associated with eq.* (33) *coincides with the Casorati matrix* $\mathbf{\Xi}_{t,r}$ *for all* $t \in \mathbb{Z}_r$ *and any fixed* $r \geq s$*, given by eq.* (39).

Proof. Let $\mathbf{y}_r = [y_r, y_{r-1}, ..., y_{r-p+1}]'$ be an arbitrary initial condition vector. Applying eq. (40) for $t, t-1, ..., t-p+1$, the components of the solution vector $\mathbf{y}_t$ associated with eq. (33) are given by:

$$\begin{array}{lcccccc} y_t & = & \xi_{t,r}^{(1)}y_r & + & \xi_{t,r}^{(2)}y_{r-1} & +...+ & \xi_{t,r}^{(p)}y_{r-p+1} \\ y_{t-1} & = & \xi_{t-1,r}^{(1)}y_r & + & \xi_{t-1,r}^{(2)}y_{r-1} & +...+ & \xi_{t-1,r}^{(p)}y_{r-p+1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ y_{t-p+1} & = & \xi_{t-p+1,r}^{(1)}y_r & + & \xi_{t-p+1,r}^{(2)}y_{r-1} & +...+ & \xi_{t-p+1,r}^{(p)}y_{r-p+1}. \end{array}$$

The above $p \times p$ system of linear equations can be expressed in a vector equation form as:

$$\mathbf{y}_t = \mathbf{\Xi}_{t,r} \cdot \mathbf{y}_r \quad \text{for } t \geq r. \tag{52}$$

A comparison of eqs. (51) and (52), on account of the uniqueness of the solution vector $\mathbf{y}_t$, implies that: $\mathbf{\Xi}_{t,r}\ \mathbf{y}_r = \mathbf{F}_{t,r}\ \mathbf{y}_r$ for all $\mathbf{y}_r \in \mathbb{C}^p$. It follows from Remark 1 that

$$\mathbf{\Xi}_{t,r} = \mathbf{F}_{t,r}, \tag{53}$$

as asserted. □

By virtue of eq. (53) the entries of $\mathbf{F}_{t,r}$ for $t \geq r+1$ are the banded Hessenbergians, whose elements are explicitly expressed in terms of the variable coefficients of eq. (33). As $\mathbf{F}_{t,r}$ is invertible, we conclude from eq. (53) that $\mathbf{\Xi}_{t,r}$ is invertible too. This statement recovers the result stated in Corollary 1. In the following Example we apply Theorem 3 to the second order VC-LDE(2).

Example 2. In this example we consider the second order homogeneous VC-LDE:

$$y_t = \phi_1(t)y_{t-1} + \phi_2(t)y_{t-2}.$$

Let $s \in \mathbb{Z}$ and $\phi_2(t) \neq 0$ for all $t \geq s+1$. The Definition in eqs. (35) and (32) is applied for $r = t-1, t-2, t-3$ to verify the identity in eq. (52), assuming that $r \geq s$.

i) If $r = t-1 \geq s$, then we conclude that: $\xi_{t,t-1} = \phi_1(t),\ \xi^{(2)}_{t,t-1} = \phi_2(t), \xi_{t-1,t-1} = 1,\ \xi^{(2)}_{t-1,t-1} = 0.$
The associated companion matrix is given by

$$\mathbf{F}_{t,t-1} = \mathbf{\Gamma}_t = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \xi_{t,t-1} & \xi^{(2)}_{t,t-1} \\ \xi_{t-1,t-1} & \xi^{(2)}_{t-1,t-1} \end{bmatrix} = \mathbf{\Xi}_{t,t-1}.$$

ii) If $r = t-2 \geq s$, then we conclude that:

$$\xi_{t,t-2} = \begin{vmatrix} \phi_1(t-1) & -1 \\ \phi_2(t) & \phi_1(t) \end{vmatrix},\ \xi^{(2)}_{t,t-2} = \begin{vmatrix} \phi_2(t-1) & -1 \\ 0 & \phi_1(t) \end{vmatrix},\ \xi_{t-1,t-2} = \phi_1(t-1),\ \xi^{(2)}_{t-1,t-2} = \phi_2(t-1).$$

The product of the first two companion matrices is given by

$$\mathbf{F}_{t,t-2} = \mathbf{\Gamma}_t\mathbf{\Gamma}_{t-1} = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \phi_1(t-1) & \phi_2(t-1) \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \phi_1(t)\phi_1(t-1) + \phi_2(t) & \phi_1(t)\phi_2(t-1) \\ \phi_1(t-1) & \phi_2(t-1) \end{bmatrix}$$

$$= \begin{bmatrix} \xi_{t,t-2} & \xi^{(2)}_{t,t-2} \\ \xi_{t-1,t-2} & \xi^{(2)}_{t-1,t-2} \end{bmatrix} = \mathbf{\Xi}_{t,t-2}.$$

iii) If $r = t-3 \geq s$, then we conclude that:

$$\xi_{t,t-3} = \begin{vmatrix} \phi_1(t-2) & -1 & 0 \\ \phi_2(t-1) & \phi_1(t-1) & -1 \\ 0 & \phi_2(t) & \phi_1(t) \end{vmatrix} = \phi_1(t-2)[\phi_1(t)\phi_1(t-1) + \phi_2(t)] + \phi_2(t-1)\phi_1(t),$$

$$\xi^{(2)}_{t,t-3} = \begin{vmatrix} \phi_2(t-2) & -1 & 0 \\ 0 & \phi_1(t-1) & -1 \\ 0 & \phi_2(t) & \phi_1(t) \end{vmatrix} = \phi_2(t-2)[\phi_1(t)\phi_1(t-1) + \phi_2(t)],$$

$$\xi_{t-1,t-3} = \begin{vmatrix} \phi_1(t-2) & -1 \\ \phi_2(t-1) & \phi_1(t-1) \end{vmatrix} = \phi_1(t-1)\phi_1(t-2) + \phi_2(t-1),$$

$$\xi^{(2)}_{t-1,t-3} = \begin{vmatrix} \phi_2(t-2) & -1 \\ 0 & \phi_1(t-1) \end{vmatrix} = \phi_1(t-1)\phi_2(t-2).$$

The product of the first three companion matrices is given by given by

$$\mathbf{F}_{t,t-3} = \mathbf{\Gamma}_t\mathbf{\Gamma}_{t-1}\mathbf{\Gamma}_{t-2} = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \phi_1(t-1) & \phi_2(t-1) \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \phi_1(t-2) & \phi_2(t-2) \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \phi_1(t)\phi_1(t-1) + \phi_2(t) & \phi_1(t)\phi_2(t-1) \\ \phi_1(t-1) & \phi_2(t-1) \end{bmatrix}\begin{bmatrix} \phi_1(t-2) & \phi_2(t-2) \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \phi_1(t-2)[\phi_1(t)\phi_1(t-1) + \phi_2(t)] + \phi_1(t)\phi_2(t-1) & \phi_2(t-2)[\phi_1(t)\phi_1(t-1) + \phi_2(t)] \\ \phi_1(t-1)\phi_1(t-2) + \phi_2(t-1) & \phi_1(t-1)\phi_2(t-2) \end{bmatrix}$$

$$= \begin{bmatrix} \xi_{t,t-3} & \xi^{(2)}_{t,t-3} \\ \xi_{t-1,t-3} & \xi^{(2)}_{t-1,t-3} \end{bmatrix} = \mathbf{\Xi}_{t,t-3}.$$

4. One Sided Green's Function

As the Green's function determinant-ratio formula (see [8], eq. (2.6), p. 39 or [22], eq. (2.11.7), p. 77) is independent of the choice of the fundamental solution set, having the set Ξ_s at our disposal, the Green's function is explicitly represented in Theorem 4 of this Section. Moreover, Theorem 5 shows that a domain restriction of the Green's function coincides with the corresponding restriction of the principal determinant function. This result enables us to express the general homogeneous solution of VC-LDEs(p) explicitly in terms of the Green's function in eq. (60) below. Some fundamental properties of the Green's function are also recovered. We start our discussion with the analogous representation of the Green's matrix.

Let $(t,r) \in \mathbb{Z}_s^2$. The Green's matrix (see [8], p. 14) associated with the difference operator in eq. (34) is a two variable function defined as via the companion matrix product $\mathbf{F}_{t,s}$ as follows:

$$\mathbf{G}_{t,r} = \mathbf{F}_{t,s}\ \mathbf{F}_{r,s}^{-1}. \tag{54}$$

As $t \geq s$ and $r \geq s$, Theorem 3 entails that $\mathbf{F}_{t,s} = \mathbf{\Xi}_{t,s}$ and $\mathbf{F}_{r,s} = \mathbf{\Xi}_{r,s}$, therefore eq. (54) can be written in terms of Casorati matrices as:

$$\mathbf{G}_{t,r} = \mathbf{\Xi}_{t,s}\ \mathbf{\Xi}_{r,s}^{-1}. \tag{55}$$

Following Miller (see [8], p. 39), the Green's function $H(t,r)$ associated the difference operator in eq. (34) is defined to be the entry in the upper left-hand corner of $\mathbf{G}_{t,r}$, that is

$$H(t,r) \stackrel{\text{def}}{=\!=} \mathbf{e}_1\ \mathbf{G}_{t,r}\ \mathbf{e}_1', \tag{56}$$

where $\mathbf{e}_1$ is the row unit vector: $\mathbf{e}_1 = [1, 0, ..., 0]$. An extension of the above Definition of $H(t,r)$ to cover the values $(t,r) \in \mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ is given in Theorem 4 below.

In Lemma 2 and Theorems 4 and 5 below, we provide an explicit representation of the Green's function in terms of banded Hessenbergians. Let $t \geq s$. We define the sets: $\mathfrak{X}_t = \{(t,s), (t,s+1), ..., (t,t)\}$ and $\mathfrak{Y}_s = \bigcup_{t\geq s} \mathfrak{X}_t$. It follows that $(t,r) \in \mathfrak{Y}_s$ if and only if $(t,r) \in \mathbb{Z}_s \times \mathbb{Z}_s$ and $r \leq t$, whence $\mathfrak{Y}_s \subset \mathbb{Z}_s \times \mathbb{Z}_s$.

Lemma 2. *Let $H(t,r)|_{\mathfrak{Y}_s}$ be the restriction of the Green's function, $H(t,r)$, to $\mathfrak{Y}_s$. Then $H(t,r)|_{\mathfrak{Y}_s} = \xi_{t,r}$. In particular, if $t = r$, then $H(t,t) = H(t,t)|_{\mathfrak{Y}_s} = \xi_{t,t} = 1$, that is a well known property of the Green's function. If $t \geq r+1$, then $H(t,r)|_{\mathfrak{Y}_s}$ can be represented by a banded Hessenbergian, that is the principal determinant function*:

$$H(t,r)|_{\mathfrak{Y}_s} = \begin{vmatrix} \phi_1(r+1) & -1 & & & & & \\ \phi_2(r+2) & \phi_1(r+2) & \ddots & & & & \\ \vdots & \vdots & \ddots & \ddots & & & \\ \phi_p(r+p) & \phi_{p-1}(r+p) & \ddots & \ddots & & & \\ & \vdots & \ddots & \ddots & \ddots & & \\ & \phi_p(r+p+1) & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & \phi_p(t-1) & \phi_{p-1}(t-1) & \cdots & -1 \\ & & & & \phi_p(t) & \cdots & \phi_1(t) \end{vmatrix} \tag{57}$$

Proof. Starting with the Definition in eq. (54) the following equalities hold:

$$\begin{array}{rcl} \mathbf{G}_{t,r} & = & \mathbf{F}_{t,s}\mathbf{F}_{r,s}^{-1} \\ \text{(apply twice the Definition in eq. (54))} & = & (\mathbf{\Gamma}_t\mathbf{\Gamma}_{t-1}...\mathbf{\Gamma}_{r+1}\mathbf{\Gamma}_r\mathbf{\Gamma}_{r-1}...\mathbf{\Gamma}_{s+1})(\mathbf{\Gamma}_r\mathbf{\Gamma}_{r-1}...\mathbf{\Gamma}_{s+1})^{-1} \\ \text{(by an elementary property of invertible matrices)} & = & \mathbf{\Gamma}_t\mathbf{\Gamma}_{t-1}...\mathbf{\Gamma}_{r+1}\mathbf{\Gamma}_r\mathbf{\Gamma}_{r-1}...\mathbf{\Gamma}_{s+1}\mathbf{\Gamma}_{s+1}^{-1}...\mathbf{\Gamma}_{r-1}^{-1}\mathbf{\Gamma}_r^{-1} \\ \text{(since } s \leq r \leq t) & = & \mathbf{\Gamma}_t\mathbf{\Gamma}_{t-1}...\mathbf{\Gamma}_{r+1} \\ \text{(apply the Definition in eq. (50))} & = & \mathbf{F}_{t,r} \\ \text{(by Theorem 3)} & = & \mathbf{\Xi}_{t,r}. \end{array}$$

Applying the above result to the Definition of the Green's function in eq. (56), on account of eq. (39), we conclude that: $H(t,r) = \mathbf{e}_1\ \mathbf{G}_{t,r}\ \mathbf{e}_1' = \mathbf{e}_1\ \mathbf{\Xi}_{t,r}\ \mathbf{e}_1' = \xi_{t,r}$ for all $(t,r) \in \mathfrak{Y}_s$, as asserted. If $t = r$, then the above result and the Definition of $\xi_{t,r}$ in eqs. (36), applied for $m = 1$, entail that $H(t,t) = H(t,t)|_{\mathfrak{Y}_s} = \xi_{t,t} = 1$. Finally, if $t \geq r+1$, then eq. (57) follows from the Definition of $\xi_{t,r}$ below eq. (42). □

An alternative indirect proof to Lemma 2 is presented at the end of Appendix A. In the following Theorem, we use the set Ξ_s, along with the Definition in eq. (56) to re-establish the determinant-ratio formula of the Green's function over the domain $\mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ (see the previously cited reference), but now in a fully explicit form expressed directly in terms of the elements of Ξ_s and therefore of the variable coefficients of eq. (29).

Theorem 4. *The Green's function $H(t,r)$ for $(t,r) \in \mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ associated the difference operator in eq.* (34) *can be explicitly expressed as a ratio of determinants:*

$$H(t,r) = \begin{vmatrix} \xi^{(1)}_{t,s} & \xi^{(2)}_{t,s} & \dots & \xi^{(p)}_{t,s} \\ \xi^{(1)}_{r-1,s} & \xi^{(2)}_{r-1,s} & \dots & \xi^{(p)}_{r-1,s} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{r-p+1,s} & \xi^{(2)}_{r-p+1,s} & \dots & \xi^{(p)}_{r-p+1,s} \end{vmatrix} \begin{vmatrix} \xi^{(1)}_{r,s} & \xi^{(2)}_{r,s} & \dots & \xi^{(p)}_{r,s} \\ \xi^{(1)}_{r-1,s} & \xi^{(2)}_{r-1,s} & \dots & \xi^{(p)}_{r-1,s} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{r-p+1,s} & \xi^{(2)}_{r-p+1,s} & \dots & \xi^{(p)}_{r-p+1,s} \end{vmatrix}^{-1}. \tag{58}$$

Proof. We remark that the elements of the first row of the two matrices involved in eq. (58), have the same cofactors and therefore the cofactor expansion of the first of the above determinants, expanded along its first row, can be expressed as:

$$\begin{vmatrix} \xi^{(1)}_{t,s} & \xi^{(2)}_{t,s} & \dots & \xi^{(p)}_{t,s} \\ \xi^{(1)}_{r-1,s} & \xi^{(2)}_{r-1,s} & \dots & \xi^{(p)}_{r-1,s} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{r-p+1,s} & \xi^{(2)}_{r-p+1,s} & \dots & \xi^{(p)}_{r-p+1,s} \end{vmatrix} = \xi^{(1)}_{t,s} Cof[\xi^{(1)}_{r,s}] + \xi^{(2)}_{t,s} Cof[\xi^{(2)}_{r,s}] + \dots + \xi^{(p)}_{t,s} Cof[\xi^{(p)}_{r,s}]. \tag{59}$$

Let $(t,r) \in \mathbb{Z}_s \times \mathbb{Z}_s$. In view of eq. (56) and using the well known cofactor formula of the inverse matrix $\mathbf{\Xi}^{-1}_{r,s}$, we have:

$$\begin{aligned}
H(t,r) &= \mathbf{e}_1 \, \mathbf{G}_{t,r} \mathbf{e}'_1 \\
&= \mathbf{e}_1 \, \mathbf{\Xi}_{t,s} \, \mathbf{\Xi}^{-1}_{r,s} \, \mathbf{e}'_1 \\
&= \mathbf{e}_1 \begin{bmatrix} \xi^{(1)}_{t,s} & \dots & \xi^{(p)}_{t,s} \\ \xi^{(1)}_{t-1,s} & \dots & \xi^{(p)}_{t-1,s} \\ \vdots & \vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{t-p+1,s} & \dots & \xi^{(p)}_{t-p+1,s} \end{bmatrix} \begin{bmatrix} \xi^{(1)}_{r,s} & \dots & \xi^{(p)}_{r,s} \\ \xi^{(1)}_{r-1,s} & \dots & \xi^{(p)}_{r-1,s} \\ \vdots & \vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{r-p+1,s} & \dots & \xi^{(p)}_{r-p+1,s} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \mathbf{e}_1 \begin{bmatrix} \xi^{(1)}_{t,s} & \dots & \xi^{(p)}_{t,s} \\ \xi^{(1)}_{t-1,s} & \dots & \xi^{(p)}_{t-1,s} \\ \vdots & \vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{t-p+1,s} & \dots & \xi^{(p)}_{t-p+1,s} \end{bmatrix} \left(\frac{1}{|\mathbf{\Xi}_{r,s}|} \begin{bmatrix} Cof[\xi^{(1)}_{r,s}] & \dots & Cof[\xi^{(1)}_{r-p+1,s}] \\ Cof[\xi^{(2)}_{r,s}] & \dots & Cof[\xi^{(2)}_{r-p+1,s}] \\ \vdots & \vdots\vdots\vdots & \vdots \\ Cof[\xi^{(p)}_{r,s}] & \dots & Cof[\xi^{(p)}_{r-p+1,s}] \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \frac{\mathbf{e}_1}{|\mathbf{\Xi}_{r,s}|} \begin{bmatrix} \xi^{(1)}_{t,s} & \xi^{(2)}_{t,s} & \dots & \xi^{(p)}_{t,s} \\ \xi^{(1)}_{t-1,s} & \xi^{(2)}_{t-1,s} & \dots & \xi^{(p)}_{t-1,s} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{t-p+1,s} & \xi^{(2)}_{t-p+1,s} & \dots & \xi^{(p)}_{t-p+1,s} \end{bmatrix} \begin{bmatrix} Cof[\xi^{(1)}_{r,s}] \\ Cof[\xi^{(2)}_{r,s}] \\ \vdots \\ Cof[\xi^{(p)}_{r,s}] \end{bmatrix} \\
&= \frac{[1,0,\dots,0]}{|\mathbf{\Xi}_{r,s}|} \begin{bmatrix} \xi^{(1)}_{t,s} Cof[\xi^{(1)}_{r,s}] & + & \xi^{(2)}_{t,s} Cof[\xi^{(2)}_{r,s}] & + \dots + & \xi^{(p)}_{t,s} Cof[\xi^{(p)}_{r,s}] \\ \xi^{(1)}_{t-1,s} Cof[\xi^{(1)}_{r,s}] & + & \xi^{(2)}_{t-1,s} Cof[\xi^{(2)}_{r,s}] & + \dots + & \xi^{(p)}_{t-1,s} Cof[\xi^{(p)}_{r,s}] \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{t-p+1,s} Cof[\xi^{(1)}_{r,s}] & + & \xi^{(2)}_{t-p+1,s} Cof[\xi^{(2)}_{r,s}] & + \dots + & \xi^{(p)}_{t-p+1,s} Cof[\xi^{(p)}_{r,s}] \end{bmatrix} \\
&= |\mathbf{\Xi}_{r,s}|^{-1} \left(\xi^{(1)}_{t,s} Cof[\xi^{(1)}_{r,s}] + \xi^{(2)}_{t,s} Cof[\xi^{(2)}_{r,s}] + \dots + \xi^{(p)}_{t,s} Cof[\xi^{(p)}_{r,s}] \right).
\end{aligned}$$

Comparing eq. (59) with the above last expression of $H(t,r)$ for $(t,r) \in \mathbb{Z}_s \times \mathbb{Z}_s$, eq. (58) follows. Moreover, for any $m \in [\![1,p]\!]$, $H(s+1-m,r)$ in eq. (58) is well defined on the extended domain $\mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ and is given by:

$$H(s+1-m,r) = \frac{\xi^{(1)}_{s+1-m,s} Cof[\xi^{(1)}_{r,s}] + \xi^{(2)}_{s+1-m,s} Cof[\xi^{(2)}_{r,s}] + \dots + \xi^{(p)}_{s+1-m,s} Cof[\xi^{(p)}_{r,s}]}{|\mathbf{\Xi}_{r,s}|}. \qquad \square$$

Taking into account that $\{\xi^{(m)}_{\cdot,s}\}_{1 \le m \le p}$ are fundamental solutions defined for $t \in \mathbb{Z}_{s+1-p}$ (see Theorem 2), some crucial values of $H(t,r)$ on $\mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ are verified below (see Theorem 5 and the discussion below Corollary 2).

In the following Theorem, we further enlarge the domain $\mathfrak{Y}_s$ of Lemma 2, showing that $H(t,r)$ coincides with the principal determinant function $\xi_{t,r}$ on the enlarged domain: $\mathcal{Z}_{s+1-p} = \bigcup_{t \geq s+1-p} \mathfrak{T}_{t-1+p}$, where $\mathfrak{T}_{t-1+p} = \{(t,s),(t,s+1)...,(t,t-1+p)\}$. Equivalently, $(t,r) \in \mathcal{Z}_{s+1-p}$ if and only if (iff for short) $t \in \mathbb{Z}_{s+1-p}$ and $s \leq r \leq t-1+p$. As $\mathfrak{X}_t \subset \mathfrak{T}_{t-1+p}$ for all $t \geq s+p-1$, it follows that $\mathfrak{Y}_s \subseteq \mathcal{Z}_{s+1-p}$. In particular, if $t = s$, then

$$\mathfrak{T}_{s-1+p} = \{(s,s),...,(s,s-1+p)\} \supseteq \{(s,s)\} = \mathfrak{X}_s.$$

Moreover, if $t = s+1-p$, then $\mathfrak{T}_{t-1+p} = \mathfrak{T}_s = \{(s+1-p,s)\}$, while $\mathfrak{X}_{s+1-p} = \emptyset$, whenever $p > 1$, that is $\mathfrak{X}_{s+1-p} \subset \mathfrak{T}_s$. If $p=1$, then $t=s$, whence $\mathfrak{T}_{t+1-p} = \mathfrak{T}_s = \{(s,s)\}\} = \mathfrak{X}_s$.

Theorem 5. *Let $H(t,r)|_{\mathcal{Z}_{s+1-p}}$ be the restriction of the Green's function to $\mathcal{Z}_{s+1-p}$. Then $H(t,r)|_{\mathcal{Z}_{s+1-p}} = \xi_{t,r}$.*

Proof. The Definition in eq. (35), applied for $m=1$, implies that $\xi_{t,r} = 0$, whenever $r > t$ (or $\xi_{t,t+j} = 0$, whenever $j \geq 1$) for all $t \in \mathbb{Z}_{s+1-p}$. In view of Lemma 2, it suffices to show that $H(t,r) = 0$ on the set $\mathcal{Z}_{s+1-p} \setminus \mathfrak{Y}_s$. Formally $(t,r) \in \mathcal{Z}_{s+1-p} \setminus \mathfrak{Y}_s$ iff $t \in \mathbb{Z}_{s+1-p}$ and $t < r \leq t-1+p$ (or $1 \leq r-t \leq p-1$). Setting $i = r-t$, then $(t,r) = (t,t+i)$ for some $i \in [\![1,p-1]\!]$. This yields an equivalent statement to the Definition (ESD) of $\mathcal{Z}_{s+1-p} \setminus \mathfrak{Y}_s$, that is

$$\text{ESD}: \quad (t,t+i) \in \mathcal{Z}_{s+1-p} \setminus \mathfrak{Y}_s \text{ for some } i \in [\![1,p-1]\!] \text{ iff } t \in \mathbb{Z}_{s+1-p} \text{ and } 1 \leq i \leq p-1.$$

For any $i \in [\![1,p-1]\!]$, we define the set $\mathfrak{J}_i = \{(s-i,s),(s-i+1,s+1),...,(t,t+i),...)\}$. Thus $(t,t+i) \in \mathfrak{J}_i$ for $i \in [\![1,p-1]\!]$ iff $t \in \mathbb{Z}_{s-i}$. We show next that $\mathcal{Z}_{s+1-p} \setminus \mathfrak{Y}_s = \bigcup_{j=1}^{p-1} \mathfrak{J}_j$. Let $(t,r) \in \bigcup_{j=1}^{p-1} \mathfrak{J}_j$. Then there exists some $i \in [\![1,p-1]\!]$ such that $(t,r) \in \mathfrak{J}_i$ and by the Definition of $\mathfrak{J}_i$: $t \in \mathbb{Z}_{s-i}$ and $r = t+i$. As $\mathbb{Z}_{s+1-p} = \bigcup_{j=1}^{p-1} \mathbb{Z}_{s-j}$ (for a proof see Proposition A5 in the Appendix), we infer that $t \in \mathbb{Z}_{s-i} \subseteq \mathbb{Z}_{s+1-p}$ with $i \in [\![1,p-1]\!]$, whence ESD yields: $(t,t+i) = (t,r) \in \mathcal{Z}_{s+1-p} \setminus \mathfrak{Y}_s$. As a consequence $\bigcup_{j=1}^{p-1} \mathfrak{J}_j \subseteq \mathcal{Z}_{s+1-p} \setminus \mathfrak{Y}_s$. On the other hand, in view of ESD, if $(t,t+i) \in \mathcal{Z}_{s+1-p} \setminus \mathfrak{Y}_s$ for some $1 \leq i \leq p-1$, then it follows from $(t+i) \in \mathbb{Z}_s$ that $t+i \geq s$, i.e., $t \geq s-i$, whence $t \in \mathbb{Z}_{s-i}$. From the Definition of $\mathfrak{J}_i$ we conclude that $(t,t+i) \in \mathfrak{J}_i \subseteq \bigcup_{j=1}^{p-1} \mathfrak{J}_j$, whence $\mathcal{Z}_{s+1-p} \setminus \mathfrak{Y}_s \subseteq \bigcup_{j=1}^{p-1} \mathfrak{J}_j$. As a consequence $\mathcal{Z}_{s+1-p} \setminus \mathfrak{Y}_s = \bigcup_{j=1}^{p-1} \mathfrak{J}_j$, and it therefore suffices to establish that that $H(t,r) = 0$ on the set $\bigcup_{i=1}^{p-1} \mathfrak{J}_i$, that is $H(t,t+i) = 0$ for any $i \in [\![1,p-1]\!]$ and any $t \in \mathbb{Z}_{s-i}$. The result follows from the fact that for any $i \in [\![1,p-1]\!]$ and any $t \in \mathbb{Z}_{s-i}$, the numerator of $H(t,t+i)$ in eq. (58) is zero, that is

$$\begin{vmatrix} \xi_{t,s}^{(1)} & \xi_{t,s}^{(2)} & \cdots & \xi_{t,s}^{(p)} \\ \xi_{t+i-1,s}^{(1)} & \xi_{t+i-1,s}^{(2)} & \cdots & \xi_{t+i-1,s}^{(p)} \\ \vdots & \vdots & \vdots & \vdots \\ \xi_{t+i-p+1,s}^{(1)} & \xi_{t+i-p+1,s}^{(2)} & \cdots & \xi_{t+i-p+1,s}^{(p)} \end{vmatrix} = 0,$$

since its first row coincides with one of its remaining rows for any $i = 1,2,...,p-1$, while its denominator is nonzero, i.e., $|\Xi_{t+i,s}| \neq 0$, since $t+i \geq s$, whence $\Xi_{t+i,s}$ is invertible (see Corollary 1). □

Corollary 2. *The general solution of eq.* (33) *(or the general homogeneous solution of eq.* (29)*) can be expressed in terms of the Green's function, the varying coefficients and the sequence of prescribed values $\{y_{r+1-m}\}_{1 \leq m \leq p}$ as*

$$y_t = \sum_{m=1}^{p} \sum_{i=1}^{p+1-m} \phi_{m-1+i}(r+i) H(t,r+i) y_{r+1-m} \quad \textit{for all } t \geq r+1. \tag{60}$$

Proof. The range of values of i in the second sum of eq. (47) is: $1 \leq i \leq p+1-m$. Taking into account that $t \geq r+1 \geq s+1$ and $1 \leq i \leq p+1-m \leq p$ (since $m \geq 1$), the following chain of inequalities holds:

$$s \leq s+1 \leq r+1 \leq r+i \leq r+p+1-m \leq r+p = (r+1)-1+p \leq t-1+p. \tag{61}$$

Thus $s \leq r+i \leq t-1+p$ for any $i \in [\![1,p-1]\!]$. It turns out that $t \in \mathbb{Z}_{s+1-p}$ and $s \leq r+i \leq t-1+p$, whenever $1 \leq i \leq p+1-m$, whence $(t,r+i) \in \mathcal{Z}_{s+1-p}$. Now Theorem 5 allows us to replace $\xi_{t,r+i}$ in eq. (47) with $H(t,r+i)$, which, in turn, implies that eq. (60) holds true, as claimed. □

In the proof of Theorem 5, we have established a property of the Green's function that is $H(t,r) = 0$ for all $(t,r) \in (\mathcal{Z}_{s+1-p} \setminus \mathfrak{Y}_s)$, showing its equivalence to a well known result, that is $H(t,t+i) = 0$ for any $1 \leq i \leq p-1$ and any $t \in \mathbb{Z}_{s-i}$ (see [8] eqs. (2.12), p. 41, applied for $q = p$, $s = a+q-1$ and $i = k$). Proposition A7 in the Appendix, recovers an additional property of the Green's function, that is $H(t,t+p) = \frac{1}{\phi_p(t+p)}$ for all $t \in \mathbb{Z}_{s+1-p}$ (see the above cited reference, additionally applied with $a_q(t) = -\phi_p(t)$). As $\xi_{t,t+p} = 0$ (see eq. (35), applied for $m=1$), we conclude that $H(t,t+p) \neq \xi_{t,t+p}$.

The computational time complexity of the Green's function involved in eq. (60) is linear. This is due to the identification $H(t,r)|_{\mathcal{Z}_{s+1-p}} = \xi_{t,r}$ (see Theorem 5, combined with the fact that the Gaussian elimination process computing banded determinants uses approximately $\dfrac{k(p+1)^2}{4}$ multiplications, where k is the order of the matrix and $(p+1)$ is the bandwidth of the matrix (see [21]). Accordingly, the time complexity of the Green's function, involved in the solution of VC-LDEs(p) is $O(k)$. This is computationally tractable and comparable with the time complexity of algorithms computing the same restriction of the Green's function by recursion.

5. Explicit Green's Function Solution Representation

In this Section, the banded Hessenbergian representation of the Green's function restriction on $\mathcal{Z}_{s+1-p}$, in Theorem 5, along with an analogous particular solution representation (see Proposition 5 below) are employed to obtain an explicit expression to the general solution for nonhomogeneous VC-LDEs(p), given by eq. (29), solely in terms of the Green's function, the variable coefficients and the forcing terms (see eq. (65)). Furthermore, we show the full equivalence between the aforementioned Green's function solution representation and the single determinant representation of the solution, established by Kittappa in [3].

5.1. Particular Solution

In the following Proposition, we provide the Hessenbergian representation of the solution associated with zero initial values.

Proposition 5. *The particular solution of eq.* (29)*, taking on the initial values* $y_r = y_{r-1} = ... = y_{r-p+1} = 0$*, can be expressed as a Hessenbergian function of* $t \geq r+1$ *for a fixed* $r \geq s$*:*

$$y_t^{par} = \begin{vmatrix} v_{r+1} & -1 & & & & \\ v_{r+2} & \phi_1(r+2) & & & & \\ \vdots & \vdots & \ddots & & & \\ v_{r+p-m+1} & \phi_{p-m}(r+p-m+1) & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ v_{r+p+1} & \phi_p(r+p+1) & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ v_{t-1} & & & \phi_{p-1}(t-1) & \cdots & -1 \\ v_t & & & \phi_p(t) & \cdots & \phi_1(t) \end{vmatrix}. \tag{62}$$

An equivalent formula to y_t^{par} *in eq.* (62) *expressed in terms of the principal determinant function and the forcing terms is given by:*

$$y_t^{par} = \sum_{i=1}^{t-r} v_{r+i}\xi_{t,r+i}. \tag{63}$$

Proof. As the cofactor of an entry $\phi_m(t)$ in the last row of eq. (62) is y_{t-m}^{par}, expanding the determinant in eq. (62) along the last row we obtain $y_t^{par} = \sum_{m=1}^{p} \phi_m(t) y_{t-m}^{par} + v_t$, which shows that y_t^{par} solves eq. (29). Let us now apply eq. (62) for $t = r+1, ..., r+p$. We have:

$$y_{r+1}^{par} = v_{r+1}, \quad y_{r+2}^{par} = \phi_1(r+2) y_{r+1}^{par} + v_{r+2}, \ ..., \ y_{r+p}^{par} = \sum_{m=1}^{p-1} \phi_m(r+p) y_{r+p-m}^{par} + v_{r+p}.$$

As a consequence, for any $i \geq 1$, we can write

$$y_{r+i}^{par} = \sum_{m=1}^{i-1} \phi_m(r+i) y_{r+i-m}^{par} + v_{r+i} + \sum_{m=i}^{p} \phi_m(r+i) y_{r+i-m},$$

whenever $y_{r+i-m} = 0$ for any m such that $i \leq m \leq p$. In particular, if $i = 1$, then $y_{r+1-m} = 0$ for all m such that $1 \leq m \leq p$ and $y_{r+1}^{par} = v_{r+1}$. Thus setting $y_{r+1-m} = 0$ for any $m \in [\![1, p]\!]$ in eq. (29), the latter equation is satisfied by y_t^{par} associated with zero initial values, that is $\{y_{r+1-p} = 0, ..., y_r = 0\}$. Therefore, the result follows from the uniqueness of the initial value problem. Finally, expanding the determinant in eq. (62) along the first column, the expression in eq. (63) follows immediately. □

5.2. General Nonhomogeneous Solution

Let us call y_t^{hom} the general homogeneous solution given by eq. (60). Adding eqs. (60) and (63), we obtain the general nonhomogeneous solution $y_t = y_t^{hom} + y_t^{par}$ for $t \geq r+1$ of eq. (29), explicitly in terms of the principal determinant function $\xi_{t,r}$, the varying coefficients $\phi_m(t)$, the forcing terms v_t, and the prescribed values y_{r+1-m} for $1 \le m \le p$, as formulated below:

$$y_t = \sum_{m=1}^{p} \sum_{i=1}^{p+1-m} \phi_{m-1+i}(r+i)\xi_{t,r+i} y_{r+1-m} + \sum_{i=1}^{t-r} \xi_{t,r+i} v_{r+i}. \tag{64}$$

Taking into account that $1 \le i \le t-r$ (in the second summation term of eq. (64)), it follows that $r+1 \le r+i \le t$, thus Lemma 2 allows us to use the identification: $H(t, r+i) = \xi_{t,r+i}$. Also the first term of the right-hand side of eq. (64) is the homogeneous solution part, given by eq. (60), thus we can rewrite eq. (64) as:

$$y_t = \sum_{m=1}^{p} \sum_{i=1}^{p-m+1} \phi_{m-1+i}(r+i) H(t, r+i) y_{r+1-m} + \sum_{i=1}^{t-r} H(t, r+i) v_{r+i} \quad \text{for all } t \ge r+1. \tag{65}$$

5.3. Equivalent Solution Representations

The equivalence between the Green's function explicit representations of the general solution to VC-LDEs(p) and that obtained by Kittapa in [3] is demonstrated in the following Proposition.

Proposition 6. *The Green's function solution representation in eq.* (65) *is equivalent to the single determinant solution representation of eq.* (29) *established in* [3].

Proof. Replacing the homogeneous solution part of eq. (65) (or eq. (64)) with its equivalent expression in eq. (40), we can rewrite eq. (65) as:

$$y_t = \sum_{m=1}^{p} \xi_{t,r}^{(m)} y_{r+1-m} + \sum_{i=1}^{t-r} \xi_{t,r+i} v_{r+i} \quad \text{for } t \ge r+1. \tag{66}$$

Applying the Hessenbergian expression of $\xi_{t,r}^{(m)}$ in eq. (35) and the Hessenbergian expression of the particular solution in eq. (62) to eq. (66), the latter solution representation of eq. (29), can be expressed in more detail as

$$y_t = \sum_{m=1}^{p} y_{r+1-m} \begin{vmatrix} \phi_m(r+1) & -1 & & & & \\ \phi_{m+1}(r+2) & \phi_1(r+2) & & & & \\ \vdots & \vdots & & & & \\ \phi_p(r+p-m+1) & \phi_{p-m}(r+p-m+1) & \ddots & & & \\ & \vdots & & & & \\ & \phi_p(r+p+1) & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \phi_{p-1}(t-1) & \cdots & -1 \\ & & & \phi_p(t) & \cdots & \phi_1(t) \end{vmatrix}$$

$$+ \begin{vmatrix} v_{r+1} & -1 & & & & \\ v_{r+2} & \phi_1(r+2) & & & & \\ \vdots & \vdots & \ddots & & & \\ v_{r+p-m+1} & \phi_{p-m}(r+p-m+1) & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ v_{r+p+1} & \phi_p(r+p+1) & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ v_{t-1} & & & \phi_{p-1}(t-1) & \cdots & -1 \\ v_t & & & \phi_p(t) & \cdots & \phi_1(t) \end{vmatrix}.$$

As a result of the multi-linearity of determinants in columns, the right-hand side of the above expression of y_t

takes a single determinant form as:

$$
y_t = \begin{vmatrix}
\sum_{m=1}^{p} y_{r+1-m}\phi_m(r+1) + v_{r+1} & -1 & & & & \\
\sum_{m=1}^{p} y_{r+1-m}\phi_{m+1}(r+2) + v_{r+2} & \phi_1(r+2) & & & & \\
\vdots & \vdots & & & & \\
\sum_{m=1}^{p} y_{r+1-m}\phi_p(r+1+p-m) + v_{r+1+p-m} & \phi_{p-m}(r+1+p-m) & \ddots & & & \\
\vdots & \vdots & \ddots & \ddots & & \\
v_{r+p+1} & \phi_p(r+p+1) & \ddots & \ddots & \ddots & \\
\vdots & & \ddots & \ddots & \ddots & \\
v_{t-1} & & & \phi_{p-1}(t-1) & \cdots & -1 \\
v_t & & & \phi_p(t) & \cdots & \phi_1(t)
\end{vmatrix} . \tag{67}
$$

The latter expression coincides with the single determinant solution representation in [3]. □

The equivalence result in Proposition 6 enables us to introduce an alternative scheme to the generation of the fundamental set of solutions Ξ_r, by using as starting point the single determinant solution representation in place of the IGE, as follows: Applying eq. (67) with $v_{r+i} = 0$ for all $i \geq 1$ and assigning for each fixed $m \in [\![1, p]\!]$, the initial conditions $y_{r+1-m} = 1$ and $y_{r+1-j} = 0$, whenever $m \neq j$, we recover the sequences $\xi^{(m)}_{\cdot,r}$ constructed by the IGE in Subsection 3.1 and established in Subsection 3.2.

6. Compact Representations

In this Section we apply the Leibnizian and nested sum representations of Hessenbergians to derive compact representations for the domain restriction of the Green's function $H(t, r)$ to $\mathcal{Z}_{s+1-p}$ (see Subsections 6.1 and 6.2 respectively). Moreover, compact representations for the elements of the companion matrix product and the elements of the Green's function determinant-ration formula are provided in Subsections 6.3 and 6.4, respectively. A Leibnizian compact representation for the general solution of nonhomogeneous VC-LDEs(p) is obtained in Subsection 6.5 .

In what follows, we consider a $k \times k$ full lower Hessenberg matrix $[h_{i,j}]_{1\leq i,j\leq k}$ with superdiagonal elements $h_{i,i+1} = -1$. Applying eq. (37) for $m = 1$ we get:

$$
h_{i,j} = \begin{cases} \phi_{i-j+1}(r+i) & \text{if } 1 \leq i-j+1 \leq p \\ -1 & \text{if } i-j+1 = 0 \\ 0 & \text{elsewhere.} \end{cases} \tag{68}
$$

Under the assignment (68), the matrix $\mathbf{H}_{t-r}$ in eq. (3), applied for $k = t - r$, turns to a banded Hessenberg matrix, being identical to $\mathbf{\Phi}_{t,r}$ in eq. (32). A few distinct elements of $\mathbf{H}_{t-r}$ are provided below: $h_{1,1} = \phi_1(r + 1)$, $h_{1,2} = -1$, $h_{2,1} = \phi_2(r + 2)$, $h_{t-r,t-r} = \phi_1(t)$ and $h_{t-r,1} = \begin{cases} \phi_{t-r}(t) & \text{if } 1 \leq t-r \leq p \\ 0 & \text{if } t-r > p \end{cases}$.

6.1. Leibnizian Representation of the Green's Function Restriction

Applying the assignment in eq. (68) to eq. (28), we establish in 69 the Leibnizian compact representation of the the principal determinant function $\xi_{t,r}$. By virtue of Theorem 5, $\xi_{t,r}$ is a banded Hessenbergian representation the Green's function restriction $H(t, r)|_{\mathcal{Z}_{s+1-p}}$, whence:

$$
H(t,r)|_{\mathcal{Z}_{s+1-p}} = \xi_{t,r} = \begin{cases} \displaystyle\sum_{m=0}^{2^{t-r-1}-1} \prod_{i=1}^{t-r} \phi_{i-\sigma_{t-r,i}(m)+1}(r+i), & \text{if } \quad s \leq r < t \\ 1, & \text{if } \quad t = r \\ 0, & \text{elsewhere} \end{cases} \tag{69}
$$

6.2. Nested Sum Representation of the of the Green's Function Restriction

The nested sum representation of Hessenbergians, established in [6] (see eq. (19) in their Corollary 4.1.), can be expressed according to our notation in eq. (3) (using the adjustment: $k = n$, $h_{i,i+1} = -1$ and $h_{i,j} = b_{i,j}$, elsewhere), as:

$$\det(\mathbf{H}_k) = h_{k,1} + \sum_{j=2}^{k} \sum_{k_1=j}^{k} \sum_{k_2=j-1}^{k_1-1} ... \sum_{k_{j-1}=2}^{k_{j-2}-1} h_{k,k_1} \prod_{m=2}^{j-1} h_{k_{m-1}-1,k_m} h_{k_{j-1}-1,1}. \tag{70}$$

Applying the assignment in eq. (68) to eq. (70), the nested sum representation of $H(t,r)|_{\mathcal{Z}_{s+1-p}}$ (or $\xi_{t,r}$) takes the form:

$$H(t,r)|_{\mathcal{Z}_{s+1-p}} = \xi_{t,r} \tag{71}$$

$$= \begin{cases} \phi_{t-r}(t) + \sum_{j=2}^{t-r} \sum_{k_1=j}^{t-r} \sum_{k_2=j-1}^{k_1-1} ... \sum_{k_{j-1}=2}^{k_{j-2}-1} \phi_{t-r-k_1+1}(t) \prod_{m=2}^{j-1} \phi_{k_{m-1}-k_m}(r+k_{m-1}-1)\phi_{k_{j-1}-1}(r+k_{j-1}-1), & \text{if} \quad s \le r < t \\ 1, & \text{if} \quad t = r \\ 0, & \text{elsewhere} \end{cases}$$

Proceeding with the above mentioned assignment, the Green's function restriction $H(t,r)|_{\mathcal{Z}_{s+1-p}}$ can also be represented by Mallik's combinatorial formula in [4], as adjusted for Hessenbergians in [6] (see eq. (9) therein).

6.3. Companion Matrix Product

By virtue of Theorem 5, it follows from the chain of inequalities in eq. (61) that we can replace $\xi_{t,r+j}$ with the Green's function restriction $H(t,r+j)|_{\mathcal{Z}_{s+1-p}}$ in eq. (46) to obtain the expression:

$$\xi_{t,r}^{(m)} = \sum_{j=1}^{p+1-m} \phi_{m-1+j}(r+j)H(t,r+j). \tag{72}$$

Applying eq. (69) to (72), we conclude that $\xi_{t,r}^{(m)}$ are equipped with the following Leibnizian representations:

$$\xi_{t,r}^{(m)} = \sum_{j=1}^{p-m+1} \phi_{m-1+j} \sum_{q=0}^{2^{t-r-1-j}-1} \prod_{i=1}^{t-r-j} \phi_{i-\sigma_{t-r-j,i}(q)+1}(r+j+i). \tag{73}$$

As $\mathbf{F}_{t,r} = \mathbf{\Xi}_{t,r}$ whenever $t \ge r \ge s$ (see Theorem 3), the expressions in eq. (73) yield compact representations for the elements of the companion matrix product in eq. (50), respectively. The latter result is be compared with Theorem 2.1. in ([5]).

6.4. Green's Function Determinant-Ratio Formula

The expressions in eq. 73, applied for $r = s$, yield Leibnizian compact representations for the elements of Green's function determinant-ratio formula in eq. (58).

Similarly, applying the expression in eq. (71) to eq. (72) and using $r = s$, we obtain nested sum representations for the elements of Green's function determinant-ratio formula in eq. (58).

6.5. Leibnizian Solution Representation

Applying the expression in eq. (69) to eq. (66), the compact Leibnizian representation of the solution to eq. (29) for $t \ge r+1 \ge s+1$ and any sequence of prescribed values $\{y_{r+1-p}, ..., y_r\}$, takes the form:

$$\begin{aligned} y_t &= \sum_{m=1}^{p} y_{r+1-m} \sum_{j=1}^{p+1-m} \phi_{m-1+j}(r+j) \sum_{q=0}^{2^{t-r-j-1}-1} \prod_{i=1}^{t-r-j} \phi_{i-\sigma_{t-r-j,i}(q)+1}(r+j+i) \\ &+ \sum_{j=1}^{t-r} v_{r+j} \sum_{q=0}^{2^{t-r-j-1}-1} \prod_{i=1}^{t-r-j} \phi_{i-\sigma_{t-r-j,i}(q)+1}(r+j+i). \end{aligned} \tag{74}$$

Eq. (74) is to be compared with the nested sum representation of the general solution established in [6] (see eq. (17) therein). In Appendix C, both Algorithms 1 and 2 are applied along with the assignment in eq. (68) to verify the compact solution representation in eq. (74), evaluated for any given sequence of prescribed values.

7. Future Work

The results of this work can be extended in multiple directions. We highlight two of them:

Our results can be extended to cover an explicit representation to the solution of infinite order linear difference equations with constant or variable coefficients (ILDEs). This can be established by extending our results to cover linear difference equations of unbounded order, but of finite kernel index p (ULDE(p)) (see [18]). [4] An ULDE(p) is naturally derived as a p order truncation of the ILDE, yielding an approximation of p order to the original ILDE. The fundamental solution set obtained here can be similarly formulated considering full lower Hessenberg matrices in place of banded ones, as in eq. (43) of Lemma 1. As a consequence, the general solution of ULDEs(p) is given by eq. (64), using full Hessenbergians in place of $\xi_{t,r+i}$. In a similar manner the Leibnizian and nested sum representations of the solution can be directly derived. The corresponding solution of the ILDE turns out to be the limit of the associated ULDEs(p), as $p \to \infty$.

Our methodology can be generalized to the case of multivariate VC-LDEs(p), where square matrices with elements variable coefficients are used in place of scalar valued variable coefficients, as in eq. (29). Working on the algebra of non commutative rings, an explicit form to the general solution representation of multi-variate difference equations with variable matrix coefficients is obtained. This result has some remarkable consequences on the fundamental properties of multi-variate ARMA models.

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Appendices

In Appendix A, we provide proofs for the results reported in the main body of the paper. The infinite Gaussian elimination algorithm is presented in Appendix B. Finally, in Appendix C, we provide two algorithms translated into automatically executable computer programs. The first, constructs and verifies the Leibnizian compact representation of Hessenbergians in eq. (28). The second, constructs the Leibnizian representation of the Green's function $H(t, r)$ for $(t, r) \in \mathcal{Z}_{s+1-p}$, followed by the corresponding representation of the general solution of a VC-LDE(p).

Appendix A [Proofs]

Proposition A1. *i) The recurrence in eq.* (4) *can be equivalently expressed by eq.* (6). *ii) The number of non-trivial* SEPs *of Hessenbergians is* 2^{k-1}, *that is* $\mathrm{card}(\mathcal{E}_k) = 2^{k-1}$.

Proof. i) Applying the assignments $h_{i,j} = c_{i,j}$, whenever $j \neq i+1$ and $h_{i,i+1} = -c_{i,i+1}$ to (4) after some algebraic manipulations eq. (6) follows, as demonstrated below

$$\begin{aligned}
\det(\mathbf{H}_k) &= h_{k,k}\det(\mathbf{H}_{k-1}) + \sum_{i=1}^{k-1}(-1)^{k-i}h_{k,i}\prod_{j=i}^{k-1}h_{j,j+1}\det(\mathbf{H}_{i-1}) \\
&= c_{k,k}\det(\mathbf{H}_{k-1}) + \sum_{i=1}^{k-1}(-1)^{k-i}c_{k,i}\prod_{j=i}^{k-1}(-1)c_{j,j+1}\det(\mathbf{H}_{i-1}) \\
&= c_{k,k}\det(\mathbf{H}_{k-1}) + \sum_{i=1}^{k-1}(-1)^{k-i}c_{k,i}(-1)^{k-i}\prod_{j=i}^{k-1}c_{j,j+1}\det(\mathbf{H}_{i-1}) \\
&= c_{k,k}\det(\mathbf{H}_{k-1}) + \sum_{i=1}^{k-1}(-1)^{2(k-1)}\prod_{j=i}^{k-1}c_{k,i}c_{j,j+1}\det(\mathbf{H}_{i-1}) \\
&= c_{k,k}\det(\mathbf{H}_{k-1}) + \sum_{i=1}^{k-1}\prod_{j=i}^{k-1}c_{k,i}c_{j,j+1}\det(\mathbf{H}_{i-1}).
\end{aligned}$$

ii) Let $\mathrm{n}(r)$ be the number of distinct non-trivial SEPs associated with $\mathbf{H}_r$. Taking into account that $\mathrm{n}(0) = \mathrm{n}(1) = 1$, the recurrence in eq. (7) implies that $\mathrm{n}(r) = n(0) + n(1) + \sum_{i=2}^{r-1}\mathrm{n}(i)$ for all $r \geq 1$, provided that $\sum_{i=j}^{l} a(i) = 0$, whenever $j > i$. This can be rewritten as:

$$\mathrm{n}(r) = 1 + \sum_{i=1}^{r-1}\mathrm{n}(i) \quad \text{for all} \quad r \geq 1. \tag{A.1}$$

Working with the weak induction on $k \geq 1$ we shall show that $\mathrm{n}(k) = 2^{k-1}$ for all $k \geq 1$. The basis step $\mathrm{n}(1) = 2^0 = 1$ holds true. The induction hypothesis assumes that the statement $\mathrm{n}(k-1) = 2^{k-2}$ holds true. Starting with eq. (A.1), applied for $r = k$, we obtain:

$$\begin{aligned}
\mathrm{n}(k) &= 1 + \sum_{i=1}^{k-1} \mathrm{n}(i) \\
\text{(equivalently)} &= (1 + \sum_{i=1}^{k-2} \mathrm{n}(i)) + \mathrm{n}(k-1) \\
\text{(apply eq. (A.1) for } r = k-1) &= \mathrm{n}(k-1) + \mathrm{n}(k-1) \\
\text{(equivalently)} &= 2 \cdot \mathrm{n}(k-1) \\
\text{(by the induction Hyhpothesis)} &= 2 \cdot 2^{k-2} \\
\text{(equivalently)} &= 2^{k-1}
\end{aligned}$$

This satisfies the induction step, and the proof is completed. □

Proposition A2. *The standard IS, say $c_{i,j}$, of any initial string $C[i-1;\ell]$ is uniquely determined by the number, say m $(0 \leq m \leq i-1)$, of consecutive non-standard predecessors of $c_{i,j}$ and in this case $j = i - m$.*

Proof. The hypothesis entails that the initial string can be expressed as:

$$C[i-1;\ell] = c_{1\ell_1} ... c_{i-m-2,\ell_{i-m-2}} \underbrace{c_{i-m-1,\ell_{i-m-1}}}_{\text{standard}} \underbrace{c_{i-m,i-m+1} c_{i-m+1,i-m+2} ... c_{i-1,i}}_{m \text{ non}-\text{standard factors}}.$$

As $c_{i,j}$ is a standard IS of $C[i-1;\ell]$, we can write $j = i-n$ for some $n = 0, 1, 2, .., i-1$. In order to show that this standard IS of $C[i-1;\ell]$ is $c_{i,i-m}$ (or $n = m$), it suffices to show that none of the factors of $C[i-1;\ell]$ has column index $i - m$. First, the non-standard factors next to $c_{i-m-1,\ell_{i-m-1}}$ have column indices $i - m + 1, ..., i$. Thus $(i-m) \notin \{i-m+1, ..., i\}$. Moreover, as $c_{i-m-1,\ell_{i-m-1}}$ is standard, we infer that $\ell_{i-m-1} \neq i-m$, since otherwise $c_{i-m-1,\ell_{i-m-1}} = c_{i-m-1,i-m}$ which is non-standard. Finally if $\ell_{i-m-2} = i-m$, then $c_{i-m-2,\ell_{i-m-2}} = c_{i-m-2,i-m}$, which is a trivial entry, since $i-m-(i-m-2) = 2$. The same holds for all the preceding factors of $c_{i-m-2,\ell_{i-m-2}}$ and the result follows. □

Proposition A3. *The function $f_k : \mathcal{E}_k \mapsto \mathfrak{R}_k$ defined in eq. (8) is bijective.*

Proof. As the set $\mathfrak{R}_k$ and the set $\mathcal{E}_k$ have the same number of elements (2^{k-1}) it suffices to show that f_k is injective. Let us consider $Q = c_{1,\ell_1} c_{2,\ell_2} \ldots c_{k,\ell_k}$ and $P = c_{1,l_1} c_{2,l_2} \ldots c_{k,l_n}$ in $\mathcal{E}_k$ such that $f_k(C) = f_k(P)$. We need to show that $Q = P$ or equivalently that $\ell = l$. Let us call $f_k(C) = f_k(P) = \mathbf{r}$, where $\mathbf{r} = (r_1, r_2, ..., r_{k-1}, 1)$. We examine the following cases:

I) Let $r_i = 0$. The Definition of f_k implies that the ith non-trivial factor of C and P is non-standard. As there is only one such factor, that is the entry $(i, i+1)$, it must be the factor $c_{i,i+1}$. Thus $\ell_i = l_i = i + 1$.

II) Let $r_i = 1$. The Definition of f_k implies that the ith non-trivial factors of Q and P, say c_{i,ℓ_i} and c_{i,l_i}, are standard. Property 4 in Proposition 1, entails that c_{i,ℓ_i} and c_{i,l_i} are completely determined by the number of the consecutive non-standard predecessors of c_{i,ℓ_i} and c_{i,l_i}. The result follows from case I, which entails that both SEPs have identical non-standard factors occupying the same order positions.

Therefore in all cases $\ell_i = l_i$, whence $C = P$ as required. □

Proposition A4. *The function $\zeta_{k,i}(\mathbf{r})$ in defined in (13) can be expressed as an elementary integer function, which is given by :*

$$\zeta_{k,i}(\mathbf{r}) = r_i(i - \max_{0 \leq j < i}\{j \cdot r_j\}) - 1 \tag{A.2}$$

Proof. Let us call $z_{k,i}(\mathbf{r}) = r_i(i - \max_{0 \leq j < i}\{j \cdot r_j\}) - 1$, while $\zeta_{k,i}(\mathbf{r})$ is given by eq. (13). We shall show that $z_{k,i}(\mathbf{r}) = \zeta_{k,i}(\mathbf{r})$ for all $\mathbf{r} = (r_1, r_2, ..., r_i, ..., r_{k-1}, 1) \in \mathfrak{R}_k$. First we notice that if $i = 1$, then, in view of eq. (15), the equality $z_{k,1}(\mathbf{r}) = \zeta_{k,1}(\mathbf{r})$ holds true for all $\mathbf{r} \in \mathfrak{R}_k$. It remains to show that $z_{k,i}(\mathbf{r}) = \zeta_{k,i}(\mathbf{r})$ for all $i \geq 2$. In this case we have:

$$\max_{0 \leq j < i}\{j \cdot r_j\}) = \max\{0 \cdot r_0,\ 1 \cdot r_1, ..., (i-1) \cdot r_{i-1}\} = \max\{1 \cdot r_1, ..., (i-1) \cdot r_{i-1}\} = \max_{1 \leq j < i}\{j \cdot r_j\}).$$

Therefore in the case when $i \geq 2$, we can use the expression: $z_{k,i}(\mathbf{r}) = r_i(i - \max_{1 \leq j < i}\{j \cdot r_j\}) - 1$. We examine the following cases:

i) Let $r_i = 0$. Then a simple evaluation gives $z_{k,i}(\mathbf{r}) = -1 = \zeta_{k,i}(\mathbf{r})$.

ii) Let $r_i = 1$. We examine the following sub-cases:

a) Let $\max_{1\le j<i}\{j\cdot r_j\}=0$. Then $r_j=0$ for all j such that $1\le j<i$. Applying the formula of $z_{k,i}(\mathbf{r})$ we get $z_{k,i}(\mathbf{r})=i-1=\zeta_{k,i}(\mathbf{r})$.

b) Let $\max_{1\le j<i}\{j\cdot r_j\}=M$ and $M>0$. Then we can write:

$$\{j\cdot r_j\}_{1\le j<i}=\{1\cdot r_1,...,(M-1)r_{M-1},Mr_M,(M+1)r_{M+1},...,(i-1)r_{i-1}\}.$$

We shall show that $r_M=1$. On the contrary we assume that $r_M=0$. Then the following equality must hold

$$M=\max\{1r_1,...,(M-1)r_{M-1},0,(M+1)r_{M+1},...,(i-1)r_{i-1}\},$$

which is contradictory, because $M\notin\{1,2,...,M-1,0,M+1,...,i-1\}$.

In this case we further conclude that $r_{M+1}=r_{M+2}=...=r_{i-1}=0$; for if otherwise $\max_{1\le j<i}\{j\cdot r_j\}>M$. Therefore we conclude that: $\{j\cdot r_j\}_{1\le j\le i}=\{1r_1,...,Mr_M,0\}$. As the number of consecutive 0s between $r_M=1$ and $r_i=1$ is $i-M-1$, Definition (13) gives $\zeta_{k,i}(\mathbf{r})=i-M-1$. Also the formula of $z_{k,i}(\mathbf{r})$ yields $z_{k,i}(\mathbf{r})=r_i(i-M)-1=1(i-M)-1=i-M-1$, whence $z_{k,i}(\mathbf{r})=\zeta_{k,i}(\mathbf{r})$.

The proof of Proposition is complete. □

An Alternative Proof to Lemma 2: It is well known that the Green's function $H(t,r)$ for $t\ge r+1$ and fixed $r\ge s$ solves the homogeneous equation (33), assuming the initial values $H(r,r)=1$ and $H(r-i,r)=0$ for $i=1,2,...,p-1$ (see [23] Theorem 3.4.1 p. 87). Also by virtue of Proposition 2, applied for $m=1$, the principal determinant function $\xi_{t,r}$ solves eq. (33) for the same initial values while r is also fixed. Thus the result follows from the uniqueness of the solution of an initial value problem.

Proposition A5. *The following equality of sets holds:*

$$\mathbb{Z}_{s+1-p}=\bigcup_{j=1}^{p-1}\mathbb{Z}_{s-j}.$$

Proof. As $\mathbb{Z}_{s-i}\subseteq\mathbb{Z}_{s-(p-1)}$ for all $i\in[\![1,p-1]\!]$, it follows that $\mathbb{Z}_{s-(p-1)}=\bigcup_{j=1}^{p-1}\mathbb{Z}_{s-j}$ Now the equality follows from

$$\mathbb{Z}_{s+1-p}=\mathbb{Z}_{s-(p-1)}=\bigcup_{j=1}^{p-1}\mathbb{Z}_{s-j},$$

as asserted □

Proposition A6. *The Casoratian* $|\Xi_{t,r}|$ *defined in eq.* (39) *satisfies the first order linear difference equation:*

$$|\Xi_{t,r}|=(-1)^{p-1}\phi_p(t)|\Xi_{t-1,r}|. \tag{A.3}$$

Proof. If we replace the elements $\xi_{t,r}^{(m)}$ for $1\le m\le p$ in the first row of $|\Xi_{t,r}|$ with the right-hand side of the recurrence (45) $|\Xi_{t,r}|$ takes the form:

$$|\Xi_{t,r}|=\begin{vmatrix}\phi_p(t)\xi_{t-p,r}^{(1)}+...+\phi_1(t)\xi_{t-1,r}^{(1)} & ... & \phi_p(t)\xi_{t-p,r}^{(p)}+...+\phi_1(t)\xi_{t-1,r}^{(p)}\\ \xi_{t-1,r}^{(1)} & ... & \xi_{t-1,r}^{(p)}\\ \vdots & \vdots\vdots\vdots & \vdots\\ \xi_{t-p+1,r}^{(1)} & ... & \xi_{t-p+1,r}^{(p)}\end{vmatrix}. \tag{A.4}$$

Using the multi-linearity of determinants in rows, eq. (A.4) can be written as

$$\begin{aligned}|\Xi_{t,r}| &= \phi_p(t)\begin{vmatrix}\xi_{t-p,r}^{(1)} & ... & \xi_{t-p,r}^{(p)}\\ \xi_{t-1,r}^{(1)} & ... & \xi_{t-1,r}^{(p)}\\ \vdots & \vdots\vdots\vdots & \vdots\\ \xi_{t-p+1,r}^{(1)} & ... & \xi_{t-p+1,r}^{(p)}\end{vmatrix}+\phi_{p-1}(t)\begin{vmatrix}\xi_{t-p+1,r}^{(1)} & ... & \xi_{t-p+1,r}^{(p)}\\ \xi_{t-1,r}^{(1)} & ... & \xi_{t-1,r}^{(p)}\\ \vdots & \vdots\vdots\vdots & \vdots\\ \xi_{t-p+1,r}^{(1)} & ... & \xi_{t-p+1,r}^{(p)}\end{vmatrix}\\ &\quad +...+\phi_1(t)\begin{vmatrix}\xi_{t-1,r}^{(1)} & ... & \xi_{t-1,r}^{(p)}\\ \xi_{t-1,r}^{(1)} & ... & \xi_{t-1,r}^{(p)}\\ \vdots & \vdots\vdots\vdots & \vdots\\ \xi_{t-p+1,r}^{(1)} & ... & \xi_{t-p+1,r}^{(p)}\end{vmatrix}.\end{aligned}$$

The values of the determinants from the second term up to and including the last term of the right-hand side of the above equality are zero, since they have two identical rows, whence

$$|\boldsymbol{\Xi}_{t,r}| = \phi_p(t) \begin{vmatrix} \xi^{(1)}_{t-p,r} & \xi^{(2)}_{t-p,r} & \cdots & \xi^{(p)}_{t-p,r} \\ \xi^{(1)}_{t-1,r} & \xi^{(2)}_{t-1,r} & \cdots & \xi^{(p)}_{t-1,r} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{t-p+1,r} & \xi^{(2)}_{t-p+1,r} & \cdots & \xi^{(p)}_{t-p+1,r} \end{vmatrix}.$$

One needs $(p-1)$ successive row interchanges to move the first row to the last row position and the above equality can be written as

$$|\boldsymbol{\Xi}_{t,r}| = (-1)^{p-1}\phi_p(t) \begin{vmatrix} \xi^{(1)}_{t-1,r} & \xi^{(2)}_{t-1,r} & \cdots & \xi^{(p)}_{t-1,r} \\ \xi^{(1)}_{t-2,r} & \xi^{(2)}_{t-2,r} & \cdots & \xi^{(p)}_{t-2,r} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{t-p+1,r} & \xi^{(2)}_{t-p+1,r} & \cdots & \xi^{(p)}_{t-p+1,r} \\ \xi^{(1)}_{t-p,r} & \xi^{(2)}_{t-p,r} & \cdots & \xi^{(p)}_{t-p,r} \end{vmatrix} = (-1)^{p-1}\phi_p(t)|\boldsymbol{\Xi}_{t-1,r}|.$$

This completes the proof of Proposition. □

Proposition A7. *The Green's function associated with the difference operator in eq.* (34) *has the property:*

$$H(t, t+p) = \frac{1}{\phi_p(t+p)} \quad \textit{for} \quad t \in \mathbb{Z}_{s+1-p}.$$

Proof. Eq. (58) applied for $r = t + p$ yields:

$$H(t, t+p) = \frac{\begin{vmatrix} \xi^{(1)}_{t,s} & \xi^{(2)}_{t,s} & \cdots & \xi^{(p)}_{t,s} \\ \xi^{(1)}_{t+p-1,s} & \xi^{(2)}_{t+p-1,s} & \cdots & \xi^{(p)}_{t+p-1,s} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{t+1,s} & \xi^{(2)}_{t+1,s} & \cdots & \xi^{(p)}_{t+1,s} \end{vmatrix}}{\begin{vmatrix} \xi^{(1)}_{t+p,s} & \xi^{(2)}_{t+p,s} & \cdots & \xi^{(p)}_{t+p,s} \\ \xi^{(1)}_{t+p-1,s} & \xi^{(2)}_{t+p-1,s} & \cdots & \xi^{(p)}_{t+p-1,s} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{t+1,s} & \xi^{(2)}_{t+1,s} & \cdots & \xi^{(p)}_{t+1,s} \end{vmatrix}}.$$

Applying $(p-1)$ successive row interchanges to the numerator determinant of $H(t, t+p)$ in the above equality, its first row is moved to occupy the last row position, whence:

$$H(t, t+p) = (-1)^{p-1} \frac{\begin{vmatrix} \xi^{(1)}_{t+p-1,s} & \xi^{(2)}_{t+p-1,s} & \cdots & \xi^{(p)}_{t+p-1,s} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{t+1,s} & \xi^{(2)}_{t+1,s} & \cdots & \xi^{(p)}_{t+1,s} \\ \xi^{(1)}_{t,s} & \xi^{(2)}_{t,s} & \cdots & \xi^{(p)}_{t,s} \end{vmatrix}}{\begin{vmatrix} \xi^{(1)}_{t+p,s} & \xi^{(2)}_{t+p,s} & \cdots & \xi^{(p)}_{t+p,s} \\ \xi^{(1)}_{t+p-1,s} & \xi^{(2)}_{t+p-1,s} & \cdots & \xi^{(p)}_{t+p-1,s} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \xi^{(1)}_{t+1,s} & \xi^{(2)}_{t+1,s} & \cdots & \xi^{(p)}_{t+1,s} \end{vmatrix}} = (-1)^{p-1} \frac{|\boldsymbol{\Xi}_{t+p-1,s}|}{|\boldsymbol{\Xi}_{t+p,s}|}. \quad \text{(A.5)}$$

The Casoratian recurrence in eq. (A.3) takes the form

$$|\boldsymbol{\Xi}_{t+p,s}| = (-1)^{p-1}\phi_p(t+p)|\boldsymbol{\Xi}_{t+p-1,s}|,$$

or equivalently

$$\frac{1}{\phi_p(t+p)} = (-1)^{p-1} \frac{|\boldsymbol{\Xi}_{t+p-1,s}|}{|\boldsymbol{\Xi}_{t+p,s}|}. \quad \text{(A.6)}$$

As the right-hand side members of eqs. (A.5) and (A.6) coincide, the result follows. □

Appendix B [The Infinite Gaussian Elimination]

We present here the basic steps of the IGE implemented with right-most pivot elements the (-1)s in eq. (30). It constructs the rows of $\mathbf{FRREF(A)}$ along with the particular solution (see Subsection 3.1), yielding equivalent row-recurrences. At the end of this Appendix we give some supplementary results, as noticed in Example 1.

In what follows, the rows of the coefficient matrix, $\mathbf{A}$, in eq. (30) are denoted as $\mathbf{R}_i$ for $i \geq 1$. In the first algorithmic step $\mathbf{R}_1$ is normalized by multiplying $\mathbf{R}_1$ with (-1) so that the rightmost element, that is (-1), is reduced to 1, yielding: $\tilde{\mathbf{R}}_1 = (-1)\mathbf{R}_1$. The new row $\tilde{\mathbf{R}}_1$ replaces $\mathbf{R}_1$ and remains invariant during the forthcoming process. That is $\tilde{\mathbf{R}}_1$ is the first row of $\mathbf{FRREF(A)}$. In the second step the algorithm uses $\tilde{\mathbf{R}}_1$ as pivot row to eliminate the entry $\phi_1(r+2)$ of $\mathbf{R}_2$, lying in the same column and below the (rightmost) pivot entry 1 of $\tilde{\mathbf{R}}_1$. This is obtained by multiplying $(-\tilde{\mathbf{R}}_1)$ (or $\mathbf{R}_1$) with $\phi_1(r+2)$ and adding the result to $\mathbf{R}_2$. After normalization, the second step is described by: $\tilde{\mathbf{R}}_2 = -[\phi_1(r+2)(-\tilde{\mathbf{R}}_1) + \mathbf{R}_2]$. The new row $\tilde{\mathbf{R}}_2$ replaces $\mathbf{R}_2$, yielding the second row of $\mathbf{FRREF(A)}$. In the third step the algorithm uses $\tilde{\mathbf{R}}_1$ and $\tilde{\mathbf{R}}_2$ as pivot rows to eliminate the entries $\phi_2(r+3)$ and $\phi_1(r+3)$ of $\mathbf{R}_3$, respectively. After normalization, the new row $\tilde{\mathbf{R}}_3$ is given by: $\tilde{\mathbf{R}}_3 = -[\phi_2(r+3)(-\tilde{\mathbf{R}}_1) + \phi_1(r+3)(-\tilde{\mathbf{R}}_2) + \mathbf{R}_3]$. The new row $\tilde{\mathbf{R}}_3$ replaces $\mathbf{R}_3$ yielding the third row of $\mathbf{FRREF(A)}$. Proceeding in this way the algorithm constructs the rows of a FRREF of $\mathbf{A}$, which are given by

$$\tilde{\mathbf{R}}_i = \phi_1(r+i)\tilde{\mathbf{R}}_{i-1} + \phi_2(r+i)\tilde{\mathbf{R}}_{i-2} + ... + \phi_p(r+i)\tilde{\mathbf{R}}_{i-p} - \mathbf{R}_i \tag{B.1}$$

provided that $\tilde{\mathbf{R}}_i = \mathbf{0}$, whenever $1-p \leq i \leq 0$. The set $\{\tilde{\mathbf{R}}_i\}_{1-p\leq i\leq 0}$, consisting of p zero rows, can be viewed as a set of initial conditions associated with the recurrence in eq. (B.1), generating the rows of $\mathbf{FRREF(A)} = [\tilde{\mathbf{R}}_i]_{i\geq 1}$. For example if $i = 1$, then eq. (B.1) gives: $\tilde{\mathbf{R}}_1 = -\mathbf{R}_1$, since $\tilde{\mathbf{R}}_0 = \tilde{\mathbf{R}}_{-1} = ... = \tilde{\mathbf{R}}_{1-p} = \mathbf{0}$. If $i = 2, 3$, then eq. (B.1) gives:

$$\begin{aligned}
\tilde{\mathbf{R}}_2 =& \phi_1(r+2)\tilde{\mathbf{R}}_1 - \mathbf{R}_2 = -[\phi_1(r+2)(-\tilde{\mathbf{R}}_1) + \mathbf{R}_2] \\
\tilde{\mathbf{R}}_3 =& \phi_2(r+3)\tilde{\mathbf{R}}_1 + \phi_1(r+3)\tilde{\mathbf{R}}_2 - \mathbf{R}_3 = -[\phi_2(r+3)(-\tilde{\mathbf{R}}_1) + \phi_1(r+3)(-\tilde{\mathbf{R}}_2) + \mathbf{R}_3],
\end{aligned}$$

as expected, and so forth.

As an alternative, the recurrence in eq. (B.1) can be viewed as a VC-LDE(p) with initial condition sequences $\tilde{\mathbf{R}}_m$ for $1 \leq m \leq p$, constructed by the finite Gaussian elimination algorithm after a sequence of p steps, whereas the algorithm is implemented with rightmost pivoting and forcing terms $-\mathbf{R}_i$ for $i \geq 1$. Thereafter, the recurrence in eq. (B.1) generates the remaining rows: $\{\tilde{\mathbf{R}}_i\}_{i\geq p+1}$.

Next, we employ Example 1 to verify the first two zero outcomes of the matrix multiplication $\mathbf{A}\ [0, 1, \xi^{(1)}_{r+1,r}, \xi^{(1)}_{r+2,r}, ...]'$:

$$\begin{aligned}
[\phi_2(r+1), \phi_1(r+1), -1, 0, ...][0, 1, \xi^{(1)}_{r+1,r}, \xi^{(1)}_{r+2,r}, ...]' &= \phi_1(r+1) \cdot 1 + (-1)\phi_1(r+1) = 0, \\
[0, \phi_2(r+2), \phi_1(r+2), -1, 0, ...][0, 1, \xi^{(1)}_{r+1,r}, \xi^{(1)}_{r+2,r}, ...]' &= \phi_2(r+2) \cdot 1 + \phi_1(r+2)\phi_1(r+1) + \\
&\quad (-1)(\phi_1(r+1)\phi_1(r+2) + \phi_2(r+2)) = 0, \\
... \qquad & \qquad ...
\end{aligned}$$

A particular solution is also constructed by the IGE algorithm, by applying the same sequence of row elementary operations, used by the IGE for the row reduction of $\mathbf{A}$ to $\mathbf{FRREF(A)}$, but now to the sequence $\{-v_{r+i}\}_{i\geq 1}$. The algorithm gives rise to a recurrence, which similarly follows as in eq. (B.1)

$$\tilde{v}_{r+i} = \phi_1(r+i)\tilde{v}_{r+i-1} + \phi_2(r+i)\tilde{v}_{r+i-2} + ... + \phi_p(r+i)\tilde{v}_{r+i-p} - v_{r+i}, \quad i \geq 1,$$

taking on zero initial values, that is $\tilde{v}_{r+1-m} = 0$ for all $m \in [\![1, p]\!]$. The so constructed solution sequence $\{\tilde{v}_t\}_{t\geq r+1}$ is a particular solution, which is also represented explicitly by a Hessenbergian function that is $y^{par}_t = \tilde{v}_t$ for $t \geq r+1$ (see Proposition 5).

The general solution is a linear combination of the fundamental solutions with coefficients arbitrary initial condition values $y_{r+1-m} = a_m$ for $1 \leq m \leq p$ (that is the general homogeneous solution, see Proposition 3) plus the particular solution mentioned above (see eq. (64)).

Appendix C [Algorithms]

Two Algorithms are presented in this Appendix. The first, returns the Leibnizian representation of Hessenbergians given in eq. (28). The second, returns the restriction of the Green'r function $H(t, r)$, involved in the general homogeneous solution of VC-LDEs(p), which coincides with $\xi_{t,r}$ (see Theorem 5). This algorithm is completed by the construction of the general nonhomogeneous solution of eq. (65), expressed in terms of the Green's function $H(t, r)$ (or $\xi_{t,s}$). Both Algorithms are followed by automatically executable computer programs written in

Mathematica's symbolic language. The instructions of the algorithms follow the structure of the paper and use the corresponding formulas established in it. This can be viewed as a verification scheme for the validity of the results derived and used in the paper.

In order to run the first program one needs to insert the order k of the matrix. This is the only one external input, whereas the other inputs are internal instructions defined within the program and remain invariant in each new call of the program. Using Mathematica symbolic computation, the program returns an expression of eq. (28) exclusively in terms of the non-trivial entries $h_{i,j}$ of $\mathbf{H}_k$. This program is to be compared with corresponding routines evaluating Hessenbergians.

The functions $\mathfrak{z}_{k,i}(\mathbf{r})$ and $\tau_k(m)$ along with their composite $\sigma_{k,i}(m)$ are defined within the program expressing the corresponding formulas in the chosen language. In their program notation, the variable k (the order of the matrix) is omitted. Instead they are designated as $\mathfrak{z}_i, \tau, \sigma_i$, respectively, since all these functions are redefined for each new input of k.

In both algorithms each algorithmic step is followed by the corresponding instruction of the program, which is directly executable by Mathematica.

Algorithm 1 (Leibnizian representation of Hessenbergians)**.**

$$In[1]:\ \text{\$Assumptions} = k > 0\ \&\&\ k \in \text{Integers};$$

i) Enter the order of the Hessenberg matrix:

$$In[2]:\ k := ...$$

ii) Define the Hessenberg matrix $\mathbf{H}_k = (h_{i,j})_{i,j\in[[1,k]]}$ of order k:

$$In[3]:\ \mathbf{H}[k] := \text{Table}[\ \text{If}\ [j \leq i+1,\ h[i,j],\ 0], \{i,1,k\}, \{j,1,k\}]$$

iii) Define the entries $c_{i,j}$ of $\mathbf{H}_k$, according to eq. (5):

$$In[4]:\ c[i\_, j\_] :=\ \text{If}\ [j \neq i+1,\ \mathbf{H}[k][[i,j]],\ -\mathbf{H}[k][[i,j]]]$$

iv) Define the ith component, say $\tau_i(m)$, of $\tau(m)$ (given by eq. (23)) and assign $\tau_1(m) = 1$, whenever $i \notin [\![1, k-1]\!]$:

$$In[5]:\ \tau[i\_, m\_] :=\ \text{If}\ [1 \leq i \leq k-1, \lfloor m \div 2^{k-i-1} \rfloor - 2\lfloor \frac{\lfloor m \div 2^{k-i-1} \rfloor}{2} \rfloor, 1]$$

v) Define the composition of $\mathfrak{z}_i$ in eq. (16) and τ in eq. (23), using the function ζ_i in eq. (14), which, in turn, is constructed in a step by step procedure as follows:
a) Define the list of products $(j \cdot \tau(j,m))_{j=0,1,\ldots,i-1}$:

$$In[6]:\ \text{Prod}[i\_, m\_] := \text{Table}[\ j \times \tau[j,m], \{j, 0, i-1\}]$$

b) Evaluate the maximum value of $\text{Prod}[i,m]$ and group these values in lists $\text{M}[m] = \{\max(\text{Prod}[i,m],\ i \in [\![1,k]\!])\}$:

$$In[7]:\ \text{M}[m\_] := \text{Table}[\ \text{Max}[\text{Prod}[i,m]], \{i,1,k\}]$$

c) Define the function $Z(i,m) = \zeta_i \circ \tau(m)$ for $(i,m) \in [\![1,k]\!] \times \mathbb{I}_{k-1}$, according to eq. (14), that is $Z(i,m)$ is the number of consecutive zero predecessors of the factor $\tau(i,m)$:

$$In[8]:\ Z[i\_, m\_] := \tau[i,m] \times (i - \text{M}[m][[i]]) - 1$$

vi) Define $\sigma_i(m)$ in terms of $Z(i,m)$, defined as $\sigma_i(m) = \mathfrak{z}_i \circ \tau(m) = i - \zeta_i(\tau(m))$:

$$In[9]:\ \sigma[i\_, m\_] := i - Z[i,m]$$

vii) Define the Hessenbergian formula (28):

$$In[10]: \ \mathrm{Hsb}[k] := \sum_{m=0}^{2^{k-1}-1} \prod_{i=1}^{k} c[i, \sigma[i, m]]$$

viii) Expand the Hessenbergian formula:

$$In[11]: \ \mathrm{Expand}[\mathrm{Hsb}[k]]$$

ix) Check whether the equation $\mathrm{Hsb}[k] = \mathrm{Det}[\mathbf{H}[k]]$ holds true, where Det[ ] stands for Mathematica's symbolic evaluation of determinants:

$$In[12]: \ \mathrm{Hsb}[k] - \mathrm{Det}[\mathbf{H}[k]] == 0$$

As an example, setting $k = 4$ and running the above program, it returns Hsb[4]:

$$\begin{aligned} Out[1] = \quad & -h[1,2]h[2,3]h[3,4]h[4,1] + h[1,1]h[2,3]h[3,4]h[4,2] \\ & +h[1,2]h[2,1]h[3,4]h[4,3] - h[1,1]h[2,2]h[3,4]h[4,3] \\ & +h[1,2]h[2,3]h[3,1]h[4,4] - h[1,1]h[2,3]h[3,2]h[4,4] \\ & -h[1,2]h[2,1]h[3,3]h[4,4] + h[1,1]h[2,2]h[3,3]h[4,4] \end{aligned}$$

The program replies to the instruction (ix): "TRUE". This can be repeated by any value of $k \geq 2$, and the program replies "TRUE".

The second Algorithm computes the Green's function restriction stated in Theorem 5 as a Hessenbergian, followed be the construction of the general solution of eq. (29) in terms of the Green function in eq. (65).

Algorithm 2 (Green's function and the general solution of VC-LDEs(p)).

$$In[1]: \ \$\mathrm{Assumptions} = p > 0 \ \&\& \ p \in \mathrm{Integers} \ \&\& \ s \in \mathrm{Integers} \ \&\& \ r \in \mathrm{Integers} \ \&\& \ t \in \mathrm{Integers};$$

i) Enter the order of the linear difference equation:

$$In[2]: \ p := ...$$

ii) Enter the value of the variable $r \geq s$:

$$In[3]: \ r := ...$$

iii) Enter the value of the variable t such that $t > r$:

$$In[4]: \ t := ...$$

iv) Replace the entries of $\mathbf{H}_k$ with the entries of $\mathbf{\Phi}_{t,r}$ according to the assignment 68:

$$In[5]: \ h[i\_, j\_] := \mathrm{Which}[i == j-1, -1, 1 \leq i-j+1 \leq p, \phi_{i-j+1}[r+i], \mathrm{True}, 0]$$

v) Define the principal matrix as a function of n, m for $n > m$:

$$In[6]: \ \mathbf{\Phi}[n\_, m\_] := \mathrm{Table}[h[i,j], \{i, m+1-r, n-r\}, \{j, m+1-r, n-r\}]$$

vi) Define the Green's function restriction:
$H(n,j) = \xi_{n,j}$ for $n > j$, $H(n,j) = 0$ for $n < j$, $H(n,j) = 1$ for $n = j$:

$$In[7]: \ H[n\_, j\_] := \mathrm{Which}[n < j, 0, n == j, 1, n == j+1, \phi_1[j+1], n \geq j+2, \mathrm{Det}[\mathbf{\Phi}_{n,j}]]$$

vii) Define the general solution formula in eq. (65) (or eq. (64)) with initial condition values $\{y_{r-p+1}, ..., y_r\}$, and forcing terms v_{r+i}, as a function of n:

$$In[8]:\ y[n\_] := \sum_{m=1}^{p} \sum_{i=1}^{p+1-m} \phi_{m-1+i}(r+i)H(n, r+i)y_{r+1-m} + \sum_{i=1}^{n-r} H(n, r+i)v_{r+i}.$$

viii) Apply the Definition of the Green's function in (vi) for $n = t$ and $j = r$:

$$In[9]:\ \text{Expand}[H[t, r]]$$

ix) Apply the general solution with $n = t$:

$$In[10]:\ \text{Expand}[y[t]]$$

As an illustrative example, setting $p = 2$, $t = 5$, $s = 1$ and $r = 2$ and running the above program, it returns the following expression

$$Out[1] := \ \phi_1(3)\phi_1(4)\phi_1(5) + \phi_1(5)\phi_2(4) + \phi_1(3)\phi_2(5).$$

This is an expansion of the Green's function $H(5, 2) = \xi_{5,2}$ associated with the second order VC-LDE.

The solution y_5 of the initial value problem $y_2 = a, y_1 = b$ with forcing terms v_3, v_4, v_5 is also recoved by the program yielding:

$$Out[2] := \phi_1(3)\phi_1(4)\phi_1(5)a + \phi_2(4)\phi_1(5)a + \phi_1(3)\phi_2(5)a + \phi_1(4)\phi_1(5)\phi_2(3)b + \phi_2(3)\phi_2(5)b + v_4\phi_1(5) + v_3\phi_1(4)\phi_1(5) + v_3\phi_2(5) + v_5.$$

This result is in accord with the solution expansion obtained directly by recursion.