Passive Investing and the Rise of Mega-Firms

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Abstract

We study how passive investing affects asset prices. Flows into passive funds raise disproportionately the prices of the largest stocks in the index, while also making them more volatile. If, in addition, stocks are mispriced because of noise traders, then passive flows raise the most the prices of the overvalued stocks among the index's largest. Passive flows drive the aggregate market up even when they are entirely due to a switch from active to passive. Underlying these results is that passive flows make prices more sensitive to idiosyncratic future cashflows. We provide empirical evidence in support of our model's mechanisms.

JEL: G10, G11, G12, G23

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1 Introduction

One of the most important capital-market developments of the past thirty years has been the growth of passive investing. Passive funds, such as index mutual funds and index exchange-traded funds (ETFs), track market indices and charge lower fees than active funds. In 1993, passive funds invested in US stocks managed $23 billion of assets. That was 3.7% of the combined assets managed by active and passive funds invested in US stocks, and 0.44% of the US stock market as a whole. By 2021, passive assets had risen to $8.4 trillion. Moreover, passive funds had overtaken active funds: passive assets had risen to 53% of combined active and passive, and to 16% of the stock market. The S&P500 index attracts the bulk of passive investing: as of 2021, 42% of assets managed by index mutual funds invested in US stocks were tracking that index.[1]

The growth of passive investing has stimulated academic and policy interest in how passive investing affects asset prices and the real economy. In this paper we show that flows into passive funds tracking capitalization-weighed indices raise disproportionately the prices of the largest stocks in the indices. If, for example, passive funds tracking the S&P500 index receive inflows, then the largest S&P500 stocks will experience higher returns than smaller S&P500 stocks. The same largest stocks will experience an increase in return volatility and sensitivity to cashflow shocks. If, in addition, stocks are mispriced because of noise traders, then passive flows will raise disproportionately the prices of the overvalued stocks among the index's largest. Our theory implies that passive investing is not neutral, but reduces primarily the financing costs of the largest firms and makes the size distribution of firms more skewed.

To explain the intuition for our results, we first describe how flows into passive funds tracking the market portfolio affect prices in a CAPM world. If passive flows are due to increased stock-market participation, then their effect is to drive down the market risk premium. Therefore, stock prices rise, and the effect is more pronounced for high CAPM-beta stocks. Since small stocks have higher CAPM beta than large stocks (Fama and French (1992)), their prices rise more. If instead

[1] The data come from the 2022 Investment Company Institute (ICI) Factbook (Figure 2.9 and Tables 11 and 42), and from `https://data.worldbank.org/indicator/CM.MKT.LCAP.CD?locations=US`. We identify passive funds with the index funds in the ICI Factbook, and identify more generally passive investing with indexing throughout this paper. The growth of passive investing is even more dramatic when accounting for the decline in active share and rise of closet indexing (Petajisto (2013)), and for the passive investments of pension funds, sovereign-wealth funds and other institutions that are made outside mutual funds and ETFs.

passive flows are due to a switch from active to passive, then they have no effect on stock prices. Indeed, active funds hold the market portfolio because it is efficient, and so do passive funds.

The simple CAPM argument describes correctly how passive flows affect price levels holding price volatility constant. It fails to account, however, for the flows' effect on volatility, which feeds back into levels. To illustrate the feedback effect, we introduce noise traders, and later return to a CAPM world without them. Suppose that a stock, e.g., Tesla, is in high demand by noise traders, and that the additional demand generated by passive flows is such that smart-money investors must sell the stock short in equilibrium. The high demand not only raises the stock's price, but also makes it more sensitive to cashflow shocks. Indeed, following a positive cashflow shock, the stock accounts for a larger fraction of market movements. Therefore, smart-money investors find themselves with a riskier short position and become more willing to unwind it, amplifying the effect of the cashflow shock. Higher price sensitivity implies higher volatility, making smart-money investors even more willing to unwind their position, and causing the price to rise further, become even more sensitive to cashflow shocks, and so on. This feedback effect is quantitatively significant for large stocks because their idiosyncratic risk is non-negligible.

Section 2 presents the model. Agents can invest in a riskless asset and in multiple stocks over an infinite horizon. The riskless rate is exogenous and constant over time. Each stock's dividend flow per share is the sum of a constant and of a systematic and an idiosyncratic component that follow independent square-root processes. The square-root specification implies that the volatility of dividends per share increases with the dividend level, a property that is realistic and key to our results. Agents can be experts or non-experts. Experts can invest in all assets without constraints. They can be interpreted as investors who follow active strategies using stocks, mutual funds or hedge funds. Non-experts can invest in the riskless asset and in a capitalization-weighted index. They can be interpreted as investors in passive funds. Experts and non-experts maximize a mean-variance objective over instantaneous changes in wealth. Noise traders can also be present, and hold a number of shares of each stock that is exogenous and constant over time.

Section 3 solves for equilibrium prices. A stock's price is an affine function of the systematic and the idiosyncratic component of the stock's dividends. The coefficient multiplying each component increases as the supply held by experts declines. Thus, as described previously, an increase in the demand of non-experts or noise traders not only raises the stock price, but also makes it more sensitive to dividend shocks. Key to this result is that the volatility of dividends per share increases with the dividend level.

Section 4 illustrates our results using a calibrated example, and Section 5 derives them for general parameter values. In the calibrated example, the systematic component of dividends is assumed to be larger, relative to the other components, for smaller stocks, implying that CAPM beta decreases with stock size. Yet, flows into passive funds due to increased stock-market participation (the measure of non-experts increases holding that of experts constant) generate the highest price increase for the largest stocks. This result holds even in a CAPM world without noise traders.

The intuition in a CAPM world is as follows. Flows into passive funds do not affect the present value of the idiosyncratic component of dividends of small or mid-size stocks: since these stocks account for a negligible fraction of market movements, their idiosyncratic dividends are discounted at the riskless rate. Passive flows impact small and mid-size stocks by raising the present value of their systematic dividends. Since small stocks have the highest CAPM beta, their prices rise more than of mid-size stocks. Prices of large stocks rise the most because passive flows raise the present value of not only their systematic but also their idiosyncratic dividends—and because the latter effect is larger than the contribution of idiosyncratic dividends to CAPM beta (which for large stocks is small but not negligible). The role of idiosyncratic dividends becomes important because of the effect that flows have on volatility. Flows render the present value of systematic dividends more sensitive to shocks, and the increase in systematic price volatility attenuates the increase in the present value. Flows also render the present value of idiosyncratic dividends of large stocks more sensitive to shocks, but attenuation is weaker. This is because the increase in idiosyncratic price volatility pertains to a long position in one stock rather than in the aggregate market.

The increase in the idiosyncratic price volatility of large stocks, and its feedback effect on the price, underlie many of our other results as well. First, passive flows raise the return volatility of large stocks, while having a negligible effect on the volatility of small stocks. Second, the effects of passive flows on the largest relative to smaller stocks within an index are more pronounced the narrower the index is (e.g., the S&P500 versus the Russell 3000). This is because passive flows into a narrower index generate higher demand for each stock in the index. Third, in the presence of noise traders, the effects of passive flows are more pronounced for the overvalued stocks among the index's largest. Indeed, since experts hold short positions in overvalued stocks, the feedback effect of idiosyncratic volatility on the price switches from attenuation to amplification. Fourth, with noise traders or a narrow index, flows have an asymmetric effect in the cross-section, driving the aggregate market up even when they are entirely due to a switch from active to passive (the measure of non-experts increases and the measure of experts decreases by the same amount).

Our final result concerns additions of individual stocks to an index. These can be interpreted as passive flows into individual stocks rather than into an aggregate index. We show that larger or more overvalued stocks experience higher price increases when they are added to an index.

Section 6 provides empirical evidence in support of our model's mechanisms. We take the index to be the S&P500, and passive flows to be into index mutual funds and index ETFs tracking it. Our flow data are quarterly, from 1996 to 2020. During quarters when index funds receive high inflows, the largest stocks in the index outperform the index. During the same quarters, index concentration, as measured by combined weight of the ten largest stocks, or by standard deviation of index weights, or by Herfindahl index, increases. Following the same quarters, the idiosyncratic return volatility of large stocks increases, and does so twice as much as for smaller stocks. Finally, large stocks experience higher returns than smaller stocks when they are added to the index.

Our paper is related to a literature that examines the implications of passive investing for asset prices and market efficiency. That literature builds on Grossman and Stiglitz (1980, GS), in which informed and uninformed investors trade with noise traders. Informed and uninformed investors in GS can be interpreted as active and passive fund managers, respectively. A switch from active to passive reduces market efficiency and can exacerbate the mispricing caused by noise traders.[2] The interpretation of GS investors as fund managers is developed in Garleanu and Pedersen (2018), in which investors search for informed managers, and the efficiency of the search market for managers affects the efficiency of the asset market.[3] We add to that literature by showing that flows into

[2]Pastor and Stambaugh (2012) and Stambaugh (2014) explain an increase in market efficiency, as reflected in a decline in active funds' expected returns, by the increase in the assets that active funds manage and by the decline in noise trading, respectively.

[3]In Subrahmanyam (1991), the introduction of a market index facilitates passive investing and lowers liquidity for the assets that comprise the index. Related mechanisms are at play in Bhattacharya and O'Hara (2018) and Cong and Xu (2016) who study how ETFs affect market efficiency and liquidity, Bond and Garcia (2022) who study the effects of lowering the costs of passive investing, and Haddad, Huebner, and Loualiche (2022) who study how passive

passive funds have more pronounced effects on larger or more overvalued stocks.

Our paper is also related to a literature on the limits of arbitrage. That literature builds on De Long, Shleifer, Summers, and Waldmann (1990), in which arbitrageurs with short horizons trade with noise traders, and Shleifer and Vishny (1997), in which arbitrageurs are financially constrained. A key takeaway from that literature is that the effects of noise-trader demand on asset prices exceed those in a CAPM world, and assets with few arbitrageurs or high idiosyncratic risk are more affected.[4] We show that larger stocks can surprisingly be more affected by shocks to investor demand (passive flows).

Also related is a literature linking asset prices to institutional flows. On the theoretical side, Brennan (1993), Kapur and Timmermann (2005), Cuoco and Kaniel (2011) and Basak and Pavlova (2013) show that fund managers' concerns with relative performance induce them to buy assets in the index, causing their prices to rise.[5] We show that the effects of such buying pressure are more pronounced for the largest stocks in the index. Our model is closest to Buffa, Vayanos, and Woolley (2022, BVW), who examine how constraints on fund managers' deviations from market indices affect equilibrium prices. We depart from BVW by introducing correlation across assets and a size distribution.

Empirical papers linking asset prices to institutional flows mostly focus on institutions as a whole or on actively managed mutual funds, while we examine flows into passive funds.[6] Our paper is closer to a literature that examines how changes in index composition affect asset prices. Harris and Gurel (1986) and Shleifer (1986) find that when stocks are added to the S&P500 index, their prices rise, with the effect being partly temporary.[7] We add to that literature by showing that index additions have stronger effects on the prices of larger or more overvalued stocks.

investing affects the elasticity of asset demand curves.

[4] Vayanos and Woolley (2013) show explicitly the effects of idiosyncratic risk in a model with multiple risky assets.

[5] Kashyap, Kovrijnykh, Li, and Pavlova (2021) explore the implications of this effect for real investment.

[6] See, for example, Badrinath, Kale, and Noe (1995), Sias and Starks (1997), Nofsinger and Sias (1999), Wermers (1999) Griffin, Harris, and Topaloglu (2003), Sias, Starks, and Titman (2006), Coval and Stafford (2007), Dasgupta, Prat, and Verardo (2011) and Lou (2012). Papers focusing on passive flows include Goetzmann and Massa (2003), who find that investors sell index mutual funds after market declines and these flows are correlated with contemporaneous index returns, and Ben-David, Franzoni, and Moussawi (2018), who find that trading by index ETFs tend to destabilize the prices of the stocks they hold.

[7] For the effects of index additions, deletions and redefinitions, see also Beneish and Whaley (1996), Lynch and Mendenhall (1997), Kaul, Mehrotra, and Morck (2000), Wurgler and Zhuravskaya (2002), Chen, Noronha, and Singal (2004), Barberis, Shleifer, and Wurgler (2005), Greenwood (2005, 2008), Boyer (2011), Petajisto (2011), Chang, Hong, and Liskovich (2015), Pandolfi and Williams (2019), and Pavlova and Sikorskaya (2022).

2 Model

Time t is continuous and goes from zero to infinity. The riskless rate is exogenous and equal to $r > 0$. There are N risky assets, which we interpret as stocks. Stock $n = 1, .., N$ pays a dividend flow D_{nt} per share and is in supply of $\eta_n > 0$ shares. The dividend flow of stock n is

$$D_{nt} = \bar{D}_n + b_n D^s_t + D^i_{nt}, \tag{2.1}$$

the sum of a constant component $\bar{D}_n \geq 0$, a systematic component $b_n D^s_t$ and an idiosyncratic component D^i_{nt}. The systematic component is the product of a systematic factor D^s_t times a factor loading $b_n \geq 0$. The systematic factor follows the square-root processes

$$dD^s_t = \kappa^s \left(\bar{D}^s - D^s_t\right) dt + \sigma^s \sqrt{D^s_t} dB^s_t, \tag{2.2}$$

where $(\kappa^s, \bar{D}^s, \sigma^s)$ are positive constants and B^s_t is a Brownian motion. The idiosyncratic component follows the square-root process

$$dD^i_{nt} = \kappa^i_n \left(\bar{D}^i_n - D^i_{nt}\right) dt + \sigma^i_n \sqrt{D^i_{nt}} dB^i_{nt}, \tag{2.3}$$

where $\{\kappa^i_n, \bar{D}^i_n, \sigma^i_n\}_{n=1,..,N}$ are positive constants and $\{B^i_{nt}\}_{n=1,..,N}$ are Brownian motions that are mutually independent and independent of B^s_t. By possibly redefining factor loadings, we set the long-run mean $\bar{D}^s$ of the systematic factor to one. By possibly redefining the supply η_n, we set the long-run mean $\bar{D}_n + b_n + \bar{D}^i_n$ of the dividend flow of stock n to one for all n. With these normalizations, we can write the dividend flow of stock n as

$$D_{nt} = 1 + b_n(D^s_t - 1) + (D^i_{nt} - \bar{D}^i_n). \tag{2.4}$$

The square-root specification (2.2) and (2.3) ensures that dividends and prices are always positive. It also implies that the volatility of dividends per share increases with the dividend level, a property that is realistic and key to our results. A geometric Brownian motion specification for dividends, which is commonly used in the literature, would also imply these properties. We adopt the square-root specification because it yields closed-form solutions.

Denoting by S_{nt} the price of stock n, the stock's return per share in excess of the riskless rate is

$$dR^{sh}_{nt} \equiv D_{nt}dt + dS_{nt} - rS_{nt}dt, \tag{2.5}$$

and the stock's return per dollar in excess of the riskless rate is

$$dR_{nt} \equiv \frac{dR^{sh}_{nt}}{S_{nt}} = \frac{D_{nt}dt + dS_{nt}}{S_{nt}} - rdt. \tag{2.6}$$

We refer to dR^{sh}_t as share return, omitting that it is in excess of the riskless rate. We refer to dR_t as return, omitting that it is per dollar and in excess of the riskless rate.

Agents are competitive and form overlapping generations living over infinitesimal time intervals. Each generation includes experts and non-experts. Experts observe dividend flows, and can invest in the riskless asset and in the stocks without constraints. These agents can be interpreted as investors who follow active strategies using stocks, mutual funds or hedge funds. Non-experts do not observe dividend flows, and can invest in the riskless asset and in a stock portfolio that tracks an index. These agents can be interpreted as investors in passive funds.[8]

In addition to experts and non-experts, noise traders can be present. These agents generate an exogenous demand for each stock, which is smaller than the supply coming from the issuing firm. For tractability, we take the demand by noise traders to be constant over time when expressed in number of shares. A constant demand can capture slowly mean-reverting market sentiment. When noise traders are absent, or when their demand is proportional to issuer supply in the cross-section, experts and non-experts hold the same portfolio of stocks in equilibrium. When instead noise-trader demand is non-proportional to issuer supply, experts hold a superior portfolio. Our main result on how the price impact of passive flows depends on stock size does not require noise traders. The presence of noise traders strengthens that result and yields additional implications.

The index includes all stocks or a subset of them. It is capitalization-weighted over the stocks

[8] Investors' choice to invest in active or passive funds can result from trading off the superior returns of active funds with their higher fees, in the spirit of Grossman and Stiglitz (1980) and Garleanu and Pedersen (2018).

that it includes, i.e., weights them proportionately to their market capitalization. We refer to the included and the non-included stocks as index and non-index stocks, respectively. We denote by $\mathcal{I}$ the subset of index stocks, by $\mathcal{I}^c$ its complement and by η'_n the number of shares of stock n included in the index. Since the index is capitalization-weighted over the stocks that it includes, η'_n for $n \in \mathcal{I}$ is constant over time and is proportional to the number of shares η_n coming from the issuer. By possibly rescaling the index, we set $\eta'_n = \eta_n$ for $n \in \mathcal{I}$. For $n \in \mathcal{I}^c$, $\eta'_n = 0$.

We denote by W_{1t} and W_{2t} the wealth of an expert and a non-expert, respectively, by z_{1nt} and z_{2nt} the number of shares of stock n that these agents hold, and by μ_1 and μ_2 these agents' measure. A non-expert thus holds $z_{2nt} = \lambda\eta'_n$ shares of stock n, where λ is a proportionality coefficient that the agent chooses optimally. We denote by $u_n < \eta_n$ the number of shares of stock n held by noise traders. The special case where noise traders are absent corresponds to $u_n = 0$ for all n.

Experts and non-experts born at time t are endowed with wealth W. Their budget constraint is

$$dW_{it} = \left(W - \sum_{n=1}^{N} z_{int}S_t\right) rdt + \sum_{n=1}^{N} z_{int}(D_t dt + dS_t) = Wrdt + \sum_{n=1}^{N} z_{int}dR^{sh}_{nt}, \tag{2.7}$$

where dW_{it} is the infinitesimal change in wealth over their life, $i = 1$ for experts, and $i = 2$ for non-experts. They have mean-variance preferences over dW_{it}. These preferences can be derived from any VNM utility u, as can be seen from the second-order Taylor expansion

$$u(W + dW_{it}) = u(W) + u'(W)dW_{it} + \frac{1}{2}u''(W)dW^2_{it} + o(dW^2_{it}). \tag{2.8}$$

Experts, who observe $\{D_{nt}\}_{n=1,..,N}$, maximize the conditional expectation of (2.8). This is equivalent to maximizing

$$\mathbb{E}_t(dW_{1t}) - \frac{\rho}{2}\mathbb{V}\text{ar}_t(dW_{1t}) \tag{2.9}$$

with $\rho = -\frac{u''(W)}{u'(W)}$, because infinitesimal dW_{1t} implies that $\mathbb{E}_t(dW^2_{1t})$ is equal to $\mathbb{V}\text{ar}_t(dW_{1t})$ plus smaller-order terms. Non-experts, who do not observe $\{D_{nt}\}_{n=1,..,N}$, maximize the unconditional expectation of (2.8). This is equivalent to maximizing

$$\mathbb{E}(dW_{2t}) - \frac{\rho}{2}\mathbb{V}\text{ar}(dW_{2t}), \tag{2.10}$$

because infinitesimal dW_{2t} implies that $\mathbb{E}(dW_{2t}^2)$ is equal to $\mathbb{V}\text{ar}(dW_{2t})$ plus smaller-order terms.

3 Equilibrium

We look for an equilibrium where the price S_{nt} of stock n is

$$S_{nt} = \bar{S}_n + b_n S^s(D_t^s) + S_n^i(D_{nt}^i), \tag{3.1}$$

the sum of the present value $\bar{S}_n$ of dividends from the constant component, the present value $b_n S^s(D_t^s)$ of dividends from the systematic component, and the present value $S_n^i(D_{nt}^i)$ of dividends from the idiosyncratic component. Assuming that the functions $(S^s(D_t^s), S_n^i(D_{nt}^i))$ are twice continuously differentiable, we can write the share return dR_{nt}^{sh} as

$$\begin{aligned} dR_{nt}^{sh} &= (\bar{D}_n + b_n D_t^s + D_{nt}^i)dt + (b_n dS^s(D_t^s) + dS_n^i(D_{nt}^i)) - r\left(\bar{S}_n + b_n S^s(D_t^s) + S_n^i(D_{nt}^i)\right)dt \\ &= \mu_{nt}dt + b_n\sigma^s\sqrt{D_t^s}(S^s)'(D_t^s)dB_t^s + \sigma_n^i\sqrt{D_{nt}^i}(S_n^i)'(D_{nt}^i)dB_{nt}^i, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \mu_{nt} \equiv \frac{\mathbb{E}_t(dR_{nt}^{sh})}{dt} &= \bar{D}_n - r\bar{S}_n \\ &+ b_n\left[D_t^s + \kappa^s(1 - D_t^s)(S^s)'(D_t^s) + \frac{1}{2}(\sigma^s)^2 D_t^s (S^s)''(D_t^s) - rS^s(D_t^s)\right] \\ &+ \left[D_{nt}^i + \kappa_n^i(\bar{D}_n^i - D_{nt}^i)(S_n^i)'(D_{nt}^i) + \frac{1}{2}(\sigma_n^i)^2 D_{nt}^i (S_n^i)''(D_{nt}^i) - rS_n^i(D_{nt}^i)\right], \end{aligned} \tag{3.3}$$

and the second step in (3.2) follows from (2.2), (2.3) and Ito's lemma.

Using (2.7) and (3.2), we can write the objective (2.9) of experts as

$$\sum_{n=1}^{N} z_{1nt}\mu_{nt} - \frac{\rho}{2}\left[\left(\sum_{n=1}^{N} z_{1nt}b_n\right)^2 (\sigma^s)^2 D_t^s[(S^s)'(D_t^s)]^2 + \sum_{n=1}^{N} z_{1nt}^2(\sigma_n^i)^2 D_{nt}^i[(S_n^i)'(D_{nt}^i)]^2\right] dt, \quad (3.4)$$

Experts maximize (3.4) over positions $\{z_{1nt}\}_{n=1,..,N}$. Their first-order condition is

$$\mu_{nt} = \rho\left[b_n\left(\sum_{m=1}^{N} z_{1mt}b_m\right)(\sigma^s)^2 D_t^s[(S^s)'(D_t^s)]^2 + z_{1nt}(\sigma_n^i)^2 D_{nt}^i[(S_n^i)'(D_{nt}^i)]^2\right]. \quad (3.5)$$

Using (2.7), (3.2) and $z_{2nt} = \lambda\eta_n'$, we can write the objective (2.10) of non-experts as

$$\sum_{n=1}^{N} \lambda\eta_n'\mu_n - \frac{\rho}{2}\lambda^2\left[\left(\sum_{i=1}^{N} \eta_n' b_n\right)^2 (\sigma^s)^2\mathbb{E}\left[D_t^s[(S^s)'(D_t^s)]^2\right] + \sum_{n=1}^{N}\left(\eta_n'\right)^2(\sigma_n^i)^2\mathbb{E}\left[D_{nt}^i[(S_n^i)'(D_{nt}^i)]^2\right]\right], \quad (3.6)$$

where $\mu_n \equiv \frac{\mathbb{E}(dR_{nt}^{sh})}{dt} = \mathbb{E}(\mu_{nt})$. Non-experts maximize (3.6) over λ. The first-order condition is

$$\sum_{n=1}^{N} \eta_n'\mu_n = \rho\lambda\left[\left(\sum_{n=1}^{N} \eta_n' b_n\right)^2 (\sigma^s)^2\mathbb{E}\left[D_t^s[(S^s)'(D_t^s)]^2\right] + \sum_{n=1}^{N}\left(\eta_n'\right)^2(\sigma_n^i)^2\mathbb{E}\left[D_{nt}^i[(S_n^i)'(D_{nt}^i)]^2\right]\right]. \quad (3.7)$$

Market clearing requires that the demand of experts, non-experts and noise traders equals the supply coming from issuers:

$$\mu_1 z_{1nt} + \mu_2\lambda\eta_n' + u_n = \eta_n. \quad (3.8)$$

Solving for $z_{1nt} = \frac{\eta_n - \mu_2\lambda\eta_n' - u_n}{\mu_1}$, and substituting into the first-order condition (3.5) of experts, we find

$$\mu_{nt} = \rho\left[b_n\left(\sum_{m=1}^{N} \frac{\eta_m - \mu_2\lambda\eta_m' - u_m}{\mu_1} b_m\right)(\sigma^s)^2 D_t^s[(S^s)'(D_t^s)]^2 + \frac{\eta_n - \mu_2\lambda\eta_n' - u_n}{\mu_1}(\sigma_n^i)^2 D_{nt}^i[(S_n^i)'(D_{nt}^i)]^2\right]. \quad (3.9)$$

We look for functions $(S^s(D^s_t), S^i_n(D^i_{nt}))$ that are affine in their arguments,

$$S^s(D^s_t) = a^s_0 + a^s_1 D^s_t, \tag{3.10}$$

$$S^i_n(D^i_{nt}) = a^i_{n0} + a^i_{n1} D^i_{nt}, \tag{3.11}$$

for positive constants $(a^s_0, a^s_1, \{a^i_{n0}, a^i_{n1}\}_{n=1,..,N})$. Substituting (3.3), (3.10) and (3.11) into (3.9), we can write (3.9) as

$$\begin{aligned}
&\bar{D}_n - r\bar{S}_n + b_n\left[D^s_t + \kappa^s a^s_1(1 - D^s_t) - r(a^s_0 + a^s_1 D^s_t)\right] \\
&+ \left[D^i_{nt} + \kappa^i_n a^i_{n1}(\bar{D}^i_n - D^i_{nt}) - r(a^i_{n0} + a^i_{n1} D^i_{nt})\right] \\
&= \rho\left[b_n\left(\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1} b_m\right)(\sigma^s a^s_1)^2 D^s_t + \frac{\eta_n - \mu_2\lambda\eta'_n - u_n}{\mu_1}(\sigma^i_n a^i_{n1})^2 D^i_{nt}\right].
\end{aligned} \tag{3.12}$$

Identifying terms in D^s_t, yields a quadratic equation that determines a^s_1. Identifying terms in D^i_{nt}, yields a quadratic equation that determines a^i_{n1}. Identifying the remaining terms, yields $\bar{S}_n + b_n a^s_0 + a^i_{n0}$. Substituting $(a^s_1, \{a^i_{n1}\}_{n=1,..,N})$ into the first-order condition (3.7) of non-experts, yields an equation for λ, whose solution completes our characterization of the equilibrium.

Proposition 3.1. *In equilibrium, the price of stock n is*

$$S_{nt} = \frac{\bar{D}_n + b_n\kappa^s a^s_1 + \kappa^i_n a^i_{n1}\bar{D}^i_n}{r} + b_n a^s_1 D^s_t + a^i_{n1} D^i_{nt}, \tag{3.13}$$

where

$$a^s_1 = \frac{2}{r + \kappa^s + \sqrt{(r+\kappa^s)^2 + 4\rho\left(\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1} b_m\right)(\sigma^s)^2}}, \tag{3.14}$$

$$a^i_{n1} = \frac{2}{r + \kappa^i_n + \sqrt{(r+\kappa^i_n)^2 + 4\rho\frac{\eta_n - \mu_2\lambda\eta'_n - u_n}{\mu_1}(\sigma^i_n)^2}}, \tag{3.15}$$

and $\lambda > 0$ solves

$$\left(\sum_{m=1}^{N} \eta'_m b_m\right)\left(\sum_{m=1}^{N} (\eta_m - u_m) b_m\right)(\sigma^s a_1^s)^2 + \sum_{m=1}^{N} \eta'_m (\eta_m - u_m)(\sigma_m^i a_{m1}^i)^2 \bar{D}_m^i$$
$$= \lambda(\mu_1 + \mu_2)\left[\left(\sum_{m=1}^{N} \eta'_m b_m\right)^2 (\sigma^s a_1^s)^2 + \sum_{m=1}^{N} \left(\eta'_m\right)^2 (\sigma_m^i a_{m1}^i)^2 \bar{D}_m^i\right]. \tag{3.16}$$

The price depends on $(\mu_1, \mu_2, \sigma^s, \{b_m, \sigma_m^i, \eta_m, \eta'_m, u_m\}_{m=1,..,M})$ only through $\left(\sum_{m=1}^{N} \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m\right)(\sigma^s)^2$ and $\frac{\eta_n - \mu_2 \lambda \eta'_n - u_n}{\mu_1}(\sigma_n^i)^2$, and is decreasing and convex in the latter two variables.

The price of stock n depends on two measures of supply: systematic supply and idiosyncratic supply. Systematic supply is $\left(\sum_{m=1}^{N} \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m\right)(\sigma^s)^2$, the aggregate risk-adjusted supply of all stocks that each expert holds in equilibrium. The supply of stock m held by all experts combined is equal to the supply η_m coming from the issuer, minus the demand $\mu_2 \lambda \eta'_m$ and u_m coming from non-experts and noise traders, respectively. It is expressed in per-expert terms by dividing by the measure μ_1 of experts, is risk-adjusted by multiplying by the factor loading b_m of stock m and by the diffusion parameter σ^s of the systematic factor, and is aggregated across all stocks. Idiosyncratic supply is $\frac{\eta_n - \mu_2 \lambda \eta'_n - u_n}{\mu_1}(\sigma_n^i)^2$, the risk-adjusted supply of stock n that each expert holds in equilibrium. Risk adjustment is made by multiplying by the diffusion parameter σ_n^i of the idiosyncratic component of stock n's dividends.

An increase in systematic or idiosyncratic supply causes the price of stock n to drop. This is the usual risk-premium channel. An increase in systematic or idiosyncratic supply has the additional effect that the price of stock n becomes less sensitive, in absolute terms, to shocks to the respective component of dividends. The intuition is as follows. A positive shock to dividends not only raises expected future dividends but also makes them riskier. (The square-root specification implies that the diffusion coefficient of dividends per share increases with the level of dividends.) If the supply held by experts is positive, i.e., experts hold a long position, then the increase in risk makes them more willing to unwind their position by *selling* stock n. This results in a smaller price increase compared to the case where supply is zero. If supply is negative, i.e., experts hold a short position, then the increase in risk makes them more willing to unwind their position by *buying* stock n. This results in a larger price increase compared to the case where supply is zero.

4 Calibrated Example

We next characterize how flows into passive funds affect stock prices. In this section we illustrate our results using a calibrated example. In Section 5 we consider general parameter values.

4.1 Parameter Values

The model parameters are the riskless rate r, the number N of stocks, the parameters $(\kappa^s, \bar{D}^s, \sigma^s)$ and $(b_n, \kappa^i_n, \bar{D}^i_n, \sigma^i_n)_{n=1,..,N}$ of the dividend processes, the supply parameters $(\eta_n, \eta'_n, u_n)_{n=1,..,N}$, the measures (μ_1, μ_2) of experts and non-experts, and the risk-aversion coefficient ρ.

We set the sum $\mu_1 + \mu_2$ to one in the baseline case. This is a normalization because we can redefine ρ. We set ρ to one. This is also a normalization because we can redefine the numeraire in the units of which wealth is expressed. Since the dividend flow is normalized by $\bar{D}_n + b_n + \bar{D}^i_n = 1$, redefining the numeraire amounts to rescaling the numbers of shares $(\eta_n, \eta'_n, u_n)_{n=1,..,N}$. We set the riskless rate r to 3%.

We assume that in the baseline case $\mu_1 = 0.9$ and $\mu_2 = 0.1$, i.e., non-experts hold 10% of total wealth. We examine how stock prices change when μ_2 is raised to 0.6, i.e., non-experts' wealth rises six-fold. We consider two polar cases for experts' wealth. The first polar case is when flows into passive funds are due to increased participation by households in asset markets, and not to a switch from active to passive. In that case, experts' wealth does not change and μ_1 remains equal to 0.9. Non-experts' wealth becomes two-thirds ($\frac{0.6}{0.9}$) of experts' wealth, and total investable wealth rises by 50% ($\frac{0.9+0.6}{1}$). The second polar case is when flows into passive funds are due to a switch by households from active to passive. In that case, total investable wealth does not change and $\mu_1 + \mu_2$ remains equal to one. Non-experts' wealth becomes 50% larger than experts' wealth ($\frac{0.6}{0.4}$). We can derive all cases in-between the two polar cases by setting $\mu_1 = 0.9 - \zeta \times 0.5$, where $\zeta \in [0, 1]$ is equal to zero in the first polar case and to one in the second polar case.

We calibrate the number N of stocks and the supply η_n coming from issuers based on the distribution of firms' market capitalization in the US stock market. Averaging market capitalization across the ten largest US firms yields approximately one trillion dollars (per firm). That average scales down by a factor of approximately five when computed across the next 50 firms, then by five again when computed across the next 250 firms, then by five again when computed across the next 1250 firms, and then by five again when computed across the next 1250 firms.[9] Consequently, we assume that there are ten stocks in supply of $3125 \times \eta$ shares each, 50 stocks in supply of $625 \times \eta$ shares each, 250 stocks in supply of $25 \times \eta$ shares each, 1250 stocks in supply of $5 \times \eta$ shares each, and 1250 stocks in supply of η shares each. We refer to the smallest stocks as size group 1 and to the largest stocks as size group 5. The distribution of firm size within size groups 2 to 5 conforms approximately to a power law with exponent one, in line with the literature (Axtell (2001), Gabaix (2016)).

We consider two cases about noise trader demand. The baseline case is that noise-trader demand is equal to zero for all stocks. The second case is that noise-trader demand u_n is equal to zero for one-half of the stocks in each size group, and is equal to a constant fraction $\Delta u > 0$ of the supply coming from issuers for the remaining half. The former stocks are the low-demand ones and the latter stocks are the high-demand ones. We set $\Delta u = 0.4$, i.e., noise traders demand 40% of the available supply of the high-demand stocks ($u_n = 0.4 \times \eta_n$).

We consider two cases about index composition. The baseline case is that the index includes all stocks and is thus the true market portfolio, i.e., $\eta'_n = \eta_n$ for all n. The second case is that the index includes only the stocks in the top three size groups, i.e., $\eta'_n = \eta_n$ for the 310 stocks in size groups 3, 4 and 5, and $\eta'_n = 0$ for the 2500 stocks in size groups 1 and 2. Under the second assumption, the index is a large-stock index such as the S&P500.

We set the mean-reversion parameters κ^s and $\{\kappa^i_n\}_{n=1,..,N}$ to a common value κ. We set the long-run means $\bar{D}^i_n$ and diffusion parameters $\{\sigma^i_n\}_{n=1,..,N}$ of the idiosyncratic components to common values $\bar{D}^i$ and σ^i, respectively. The stationary distribution of D^i_{nt} is gamma with support $(0, \infty)$

[9] As of 13 March 2022, the average market capitalization across the ten largest US firms was $1.01 trillion; across the next 50 firms was $207 billion; across the next 250 firms was $48.1 billion; across the next 1250 firms was $6.71 billion; and across the next $1250 firms was $815 million. See `https://companiesmarketcap.com/usa/largest-companies-in-the-usa-by-market-cap/`.

and density

$$f(D_{nt}^i) = \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)}(D_{nt}^i)^{\alpha^i - 1} e^{-\beta^i D_{nt}^i}, \tag{4.1}$$

where

$$\alpha^i \equiv \frac{2\kappa \bar{D}^i}{(\sigma^i)^2},$$
$$\beta^i \equiv \frac{2\kappa}{(\sigma^i)^2},$$

and Γ is the Gamma function. The distribution of D_t^s is also gamma, with density given by (4.1) in which D_{nt}^i is replaced by D_t^s, α^i by $\alpha^s \equiv \frac{2\kappa \bar{D}^s}{(\sigma^s)^2} = \frac{2\kappa}{(\sigma^s)^2}$, and β^s by $\beta^s \equiv \frac{2\kappa}{(\sigma^s)^2}$. We set $\frac{\sigma^i}{\sqrt{\bar{D}^i}} = \frac{\sigma^s}{\sqrt{\bar{D}^s}} = \sigma^s$. This ensures that the distributions of D_t^s and D_{nt}^i are the same when scaled by their long-run means: $\frac{D_{nt}^i}{\bar{D}^i}$ has the same distribution as $\frac{D_t^s}{\bar{D}^s} = D_t^s$.

We allow for correlation between size and systematic risk. We assume that the value of the loading b_n on the systematic factor for stocks in size group $m = 1, .., 5$ is $b_n = \bar{b} - (m-3)\Delta b \geq 0$. The relationship between size and systematic risk is negative when Δb is positive, and vice-versa.

The parameters left to calibrate are $(\kappa, \bar{D}^i, \bar{b}, \Delta b, \sigma^s, \eta)$. We calibrate them based on stocks' expected return, return variance, CAPM beta, and CAPM R-squared (fraction of return variance explained by index movements). We compute unconditional versions of these moments. We use the values in the baseline case as calibration targets. The formulas are in Appendix B and the values in the baseline case are in Table 1.

The effects of changing κ on return moments and other numerical results are similar to those of changing the remaining parameters. We set $\kappa = 4\%$.

The values of $(\bar{D}^i, \bar{b}, \Delta b)$ must satisfy $\bar{b} + (m-3)\Delta b + \bar{D}^i \leq 1$ for all $m = 1, .., 5$ because of $\bar{D}_n \geq 0$ and the normalization $\bar{D}_n + b_n + \bar{D}^i = 1$. Inequality $\bar{b} + (m-3)\Delta b + \bar{D}^i \leq 1$ for all $m = 1, .., 5$ is equivalent to $\bar{b} + 2|\Delta b| + \bar{D}^i \leq 1$. We assume that the latter inequality holds as an equality for the stocks with largest b_n. This minimizes the constant component $\bar{D}_n \geq 0$ (which becomes zero for the largest b_n stocks). Minimizing $\bar{D}_n$ maximizes return variances by maximizing leverage, and brings them closer to their empirical counterparts.

We choose Δb to be positive, consistent with the empirical negative relationship between size and CAPM beta. We set $\Delta b = 0.025$, to generate a spread in CAPM betas between small and large stocks of 0.40: CAPM beta averages 1.35 for the stocks in size group 1, and 0.95 for the stocks in size group 5. This is in line with the spread of 0.45 in Fama and French (1992): CAPM beta averages 1.42 for the stocks in size deciles 1 and 2, and 0.97 for the stocks in size deciles 9 and 10.

We determine the relative size of $\bar{b}$ and $\bar{D}^i$ based on CAPM R-squared. We set $\bar{b} = 0.85$ and $\bar{D}^i = 0.10$, to generate a CAPM R-squared that averages to 22.69% across all stocks, and to 26.83% when stocks are weighted by number of shares. By comparison, the average adjusted R-squared from a CAPM regression with monthly returns and a five-year lookback window across all CRSP stocks in our sample period is 16.7% and the market-capitalization weighted average is 27.1%. Lowering the R-squared (by lowering $\bar{b}$ or raising $\bar{D}^i$) strengthens our results.

We determine the supply parameter η based on stocks' expected returns (in excess of the riskless rate). We set $\eta = 0.00003$, to generate expected returns across size groups that lie between 4-6%. Expected return ranges from 5.61% for the stocks in size group 1 to 4.09% for the stocks in size group 5.

We determine the diffusion parameter σ^s based on stocks' return variances. Raising σ^s (and σ^i through $\frac{\sigma^i}{\sqrt{\bar{D}^i}} = \sigma^s$) has a non-monotone effect on variances. For given values of D_t^s and $\{D_{nt}^i\}_{n=1,..,N}$, variances rise. At the same time, the stationary distributions of D_t^s and $\{D_{nt}^i\}_{n=1,..,N}$ shift more weight towards very small or very large values, for which variances are low. We choose σ^s to maximize return variances. Return volatility (square root of the variance) ranges from 21.12% for stocks in size group 1 to 11.58% for stocks in size group 5. These values are about half of their empirical counterparts. The discrepancy is partly due to discount-rate shocks in our model being perfectly correlated with cashflow shocks and attenuating them. (Since experts hold a long position in the systematic component of dividends, they become more willing to sell stocks following a positive shock to that component and the resulting increase in risk.) Allowing for independent discount-rate shocks would increase return volatilities. Our results strengthen when raising volatilities (i.e., when raising σ^s towards its volatility-maximizing value).

4.2 No Noise Traders

Table 1 shows return moments in the baseline case, in which there are no noise traders, the index includes all stocks, and non-experts hold 10% of total wealth. Expected return, return volatility, and market beta decline when moving from the smallest to the largest size group. The decline in CAPM beta is built into our calibrated example because we set Δb to a positive value. The decline in expected return reflects the decline in CAPM beta because without noise traders the conditional CAPM holds in our model. The decline in return volatility reflects partly the decline in CAPM beta. It also reflects that shocks to the idiosyncratic component of dividends have larger effects on the prices of small stocks. This is because Proposition 3.1 implies that the price is more sensitive to idiosyncratic dividend shocks when idiosyncratic supply is small. Because idiosyncratic dividend shocks have larger effects on small stocks, CAPM R-squared rises when moving from small to large stocks, consistent with the empirical evidence.

Table 1: Return Moments.

Size Group	**Expected Return (%)**	**Return Volatility (%)**	**CAPM Beta**	**CAPM R^2 (%)**
1 (Smallest)	5.61	21.12	1.35	22.68
2	4.94	18.19	1.16	22.45
3	4.45	16.01	1.02	22.70
4	4.17	13.98	0.95	25.79
5 (Largest)	4.09	11.58	0.95	37.21

Table 2 shows how flows into passive funds affect stock prices. We compute the percentage change in the price S_{nt} of stock n assuming that the systematic component D^s_t and idiosyncratic component D^i_{nt} of dividends are equal to their long-run means, $\bar{D}^s = 1$ and $\bar{D}^i_n$, respectively. Since the price is linear in D^s_t and D^i_{nt}, its value for $(D^s_t, D^i_{nt}) = (1, \bar{D}^i_n)$ is its unconditional average $\mathbb{E}(S_{nt})$.[10]

The second and third columns of Table 2 report the percentage price change when μ_2 is raised

[10] Computing the unconditional average of the percentage change in the price instead of the percentage change in the unconditional average of the price yields similar results. We use the percentage change in the unconditional average of the price because (4.4) and (4.5) become simpler and more comparable.

to 0.6 and μ_1 is held equal to 0.9. This corresponds to increased participation in the stock market through passive funds. The second column assumes that the index includes all stocks, and the third column assumes that only size groups 3, 4 and 5 are included. The fourth and fifth columns are counterparts of the second and third columns when μ_2 is raised to 0.6 and μ_1 is lowered to 0.4. This corresponds to a switch from active to passive.

Table 2: Percentage Price Change Following Flows into Passive Funds.

Size Group	**Increase in Market Participation**		**Switch from Active to Passive**	
	All Stocks in Index	**Size Groups 3-5 in Index**	**All Stocks in Index**	**Size Groups 3-5 in Index**
1 (Smallest)	6.51	6.36	0	-0.52
2	5.60	5.32	0	-1.05
3	5.44	5.70	0	1.08
4	6.54	7.62	0	3.97
5 (Largest)	7.71	9.90	0	7.23

Table 2 shows our main results. Consider first the case where flows into passive funds are due to increased participation in the stock market, and where the index includes all stocks. As shown in the second column of Table 2, stock prices increase. Moreover, the effect is J-shaped with size: the percentage price increase becomes smaller when moving from size group 1 to size group 3, becomes larger when moving from size group 3 to size group 5, and is largest for size group 5.

To explain the intuition for the J-shape, we return to the general formulas derived in Section 3. Passive flows amount to raising $\mu_2\lambda$ (the measure μ_2 of non-experts times the fraction λ of the index that they hold). Equation (3.1) implies that the percentage price change of stock n due to passive flows is

$$\frac{1}{S_{nt}}\frac{\partial S_{nt}}{\partial(\mu_2\lambda)} = \frac{b_n\frac{\partial S^s(\bar{D}^s)}{\partial(\mu_2\lambda)} + \frac{\partial S^i_n(\bar{D}^i_n)}{\partial(\mu_2\lambda)}}{\bar{S}_n + b_n S^s(\bar{D}^s) + S^i_n(\bar{D}^i_n)}. \tag{4.2}$$

For small and mid-size stocks (size groups 1 to 3), the present value $S^i_n(\bar{D}^i_n)$ of the idiosyncratic component of dividends is almost insensitive to passive flows, i.e., $\frac{\partial S^i_n(\bar{D}^i_n)}{\partial(\mu_2\lambda)} \approx 0$. Indeed, since any small or mid-size stock accounts for a negligible fraction of market movements, its idiosyncratic dividends are discounted at the riskless rate r independently of passive flows. Passive flows affect small and mid-size stocks because they raise the present value $b_n S^s(\bar{D}^s)$ of the systematic component of dividends. Since that present value rises more for stocks with higher b_n, and thus with higher CAPM beta, (4.2) implies that passive flows have a smaller effect on mid-size than on small stocks. This explains the decreasing part of the J-shape. The explanation for the increasing part is that since large stocks account for a non-negligible fraction of market movements, their idiosyncratic dividends are discounted at a rate higher than r. Passive flows lower that discount rate, thus raising the present value $S_n^i(\bar{D}_n^i)$ of idiosyncratic dividends. Since that effect is absent for mid-size stocks, (4.2) implies that passive flows can have a smaller effect on mid-size than on large stocks.

The above explanation leaves two questions open. First, why is the effect of passive flows on the present value of idiosyncratic dividends of large stocks so sizeable as to overcome the effect of CAPM beta? Second and related, since idiosyncratic dividends of large stocks contribute to those stocks' CAPM beta (and are accordingly discounted at a rate higher than r) why is the effect of passive flows not subsumed by beta? In particular, why is the effect of passive flows J-shaped with size, while beta decreases with size? According to the simple CAPM argument in the Introduction, the effect of passive flows should depend only on beta and be an increasing function of it.[11]

To answer both questions, we distinguish between a partial effect of passive flows that holds price volatility constant, and a total effect that includes the change in volatility. We compute the partial effect on the price of stock n by calculating how (a_1^s, a_{n1}^i) in the left-hand side of (3.12) change when $\mu_2\lambda$ changes and (a_1^s, a_{n1}^i) in the right-hand side remains constant. This yields the partial effect because the left-hand side of (3.12) corresponds to expected return and the right-hand

[11] Formally, consider a two-period CAPM world, in which stock n pays expected dividend $\bar{D}_n$ and has CAPM beta β_n. The stock's expected return is $r + \beta_n \mathrm{MRP}$, where r and MRP are the riskless rate and market risk premium, respectively. The price of stock n is

$$S_n = \frac{\bar{D}_n}{1 + r + \beta_n \mathrm{MRP}}.$$

Since flows into passive funds lower MRP, their effect is proportional in the cross-section to

$$\frac{1}{S_n}\frac{\partial S_n}{\partial(-\mathrm{MRP})} = \frac{\beta_n}{1 + r + \beta_n \mathrm{MRP}}, \tag{4.3}$$

and is increasing in β_n.

side to volatility. Using (3.1), (3.12) and (3.13), we find

$$\frac{1}{S_{nt}}\left.\frac{\partial S_{nt}}{\partial(\mu_2\lambda)}\right|_{\substack{\text{constant}\\ \text{volatility}}} = \frac{\rho}{\mu_1 S_{nt}}\left[b_n\left(\sum_{m=1}^{N}\eta'_m b_m\right)(\sigma^s a_1^s)^2\bar{D}^s + \eta'_n(\sigma_n^i a_{n1}^i)^2\bar{D}_n^i\right]$$
$$= \frac{\rho}{\mu_1}\mathbb{C}\text{ov}_t\left(dR_{nt}, \sum_{m=1}^{N}\eta'_m dR_{mt}^{sh}\right), \tag{4.4}$$

where the second step follows from (3.2), (3.10) and (3.11), and the covariance is evaluated for $(D_t^s, D_{nt}^i) = (\bar{D}^s, \bar{D}_n^i)$. Equation (4.4) shows that the partial effect of passive flows is equal to the covariance between the return of the stock n and the return of the index. That covariance is, in turn, proportional to stock n's conditional CAPM beta. The partial effect of passive flows thus depends only on beta, as per the simple CAPM argument in the Introduction.

The partial effect of passive flows can be further decomposed into an effect due to the reduction in systematic supply and an effect due to the reduction in idiosyncratic supply. The reduction in systematic supply raises the present value $b_n S^s(\bar{D}^s)$ of the systematic component of dividends, and is proportional to the covariance between stock n and the index that arises because of that component. Likewise, the reduction in idiosyncratic supply raises the present value $S_n^i(\bar{D}_n^i)$ of the idiosyncratic component of dividends and is proportional to the covariance that arises because of that component. The systematic and idiosyncratic covariance correspond to the first and second term, respectively, in the square bracket in (4.4). The idiosyncratic covariance is much smaller than the systematic covariance, even for the largest stocks: in Table 2, it is smaller by a factor of approximately twenty for the stocks in size group 5.

We next turn to the total effect of passive flows, which includes the change in volatility. We compute the total effect on the price of stock n by allowing (a_1^s, a_{n1}^i) in the left-hand side of (3.12) to change when $\mu_2\lambda$ changes. Using (3.13)-(3.15), we find

$$\frac{1}{S_{nt}}\frac{\partial S_{nt}}{\partial(\mu_2\lambda)} = \frac{\rho}{\mu_1 S_{nt} r}\left[\frac{b_n\left(\sum_{m=1}^{N}\eta'_m b_m\right)(\sigma^s a_1^s)^2}{\sqrt{1+\frac{4\rho(\sigma^s)^2}{(r+\kappa^s)^2}\left(\sum_{m=1}^{N}\frac{\eta_m-\mu_2\lambda\eta'_m-u_m}{\mu_1}b_m\right)}} + \frac{\eta'_n(\sigma_n^i a_{n1}^i)^2\bar{D}_n^i}{\sqrt{1+\frac{4\rho(\sigma_n^i)^2}{(r+\kappa_n^i)^2}\frac{\eta_n-\mu_2\lambda\eta'_n-u_n}{\mu_1}}}\right]. \tag{4.5}$$

The systematic and the idiosyncratic covariance are present in the numerator of the first and second term, respectively, in the square bracket in (4.5). They receive different weights, however, as shown in the denominator, with the weight given to the idiosyncratic covariance being larger. This explains formally why the effect of passive flows is not subsumed into CAPM beta, and why the flows' effect on the present value of idiosyncratic dividends can overcome the effect of beta.

The intuition is as follows. Since passive flows reduce systematic and idiosyncratic supply, they render the price of stock n more sensitive to dividend shocks (Proposition 3.1). The resulting increase in stock n's volatility lowers the experts' willingness to hold the stock and attenuates the stock's price rise. Crucially, the attenuation effect is weaker for idiosyncratic supply than for systematic supply. This is because the increase in idiosyncratic price volatility pertains to a long position in one stock rather than in the aggregate market. Because of the weaker attenuation, the weighted idiosyncratic covariance in Table 2 is smaller than the weighted systematic covariance by a factor of only two for the stocks in size group 5.

Consider next the case where the index includes only stocks in size groups 3, 4 and 5 (and flows into passive funds are still due to increased participation in the stock market). As shown in the third column of Table 2, the percentage price increase remains non-monotone when moving across size groups. Relative to the case where the index includes all stocks, the effect rises more sharply with size when moving from size group 3 to size group 5. This is because non-experts establish larger positions in the more restricted set of stocks, causing the reduction in idiosyncratic supply to be larger.

Consider finally the case where flows into passive funds are due to a switch from active to passive. When the index includes all stocks, stock prices do not change. This is because experts and non-experts hold the same portfolio, which is the index. When instead the index includes only stocks in size groups 3, 4 and 5, prices drop for size groups 1 and 2, and rise for size groups 3, 4 and especially 5. Moreover, the effect is asymmetric in the sense that the price rises exceed the price drops in absolute value, and the aggregate market rises. The asymmetry is surprising. Indeed, non-experts hold a portfolio (the index) that approximates the aggregate market, and are equally risk-averse as experts. Therefore, a substitution of experts by non-experts should have almost no effect on the exposure of each expert to the aggregate market. As a result, the compensation that experts require to hold aggregate-market risk should remain approximately the same, and the aggregate market should not rise.

The aggregate market rises for the same reason as why the effect of passive flows is not subsumed by CAPM beta. A switch from active to passive has almost no effect on the present value of idiosyncratic dividends of stocks in size groups 1, 2 and 3 because these dividends are discounted at the riskless rate r. On the other hand, the present value of idiosyncratic dividends of stocks in size groups 4 and 5 rises, and the effect is larger than the contribution of idiosyncratic covariance to beta. Therefore, even though the exposure of each expert to the aggregate market remains approximately the same, the aggregate market rises.

Flows into passive funds affect not only stock prices but also expected returns and return volatilities. Table 3 shows the effect on volatilities. Volatilities do not change when the index includes all stocks and flows into passive funds are due to a switch from active to passive. In all other cases, volatilities do not change for the small size groups but rise significantly for size groups 4 and especially 5. As we explain in Section 5, key to this result is that passive flows render the prices of large stocks more sensitive to shocks to the idiosyncratic component of dividends, while price sensitivity does not change for small stocks.

Table 3: Change in Return Volatility Following Flows into Passive Funds.

Size Group	**Baseline Return Volatility**	**Change in Return Volatility**			
		Increase in Market Participation		**Switch from Active to Passive**	
		All Stocks in Index	**Size Groups 3-5 in Index**	**All Stocks in Index**	**Size Groups 3-5 in Index**
1 (Smallest)	21.12	-0.04	-0.04	0	0
2	18.19	0.11	0.11	0	-0.03
3	16.01	0.22	0.23	0	0.06
4	13.98	0.39	0.46	0	0.28
5 (Largest)	11.58	0.65	0.83	0	0.66

The effects of passive flows in Table 2 are smaller than recent estimates of asset demand elasticity. For example, Gabaix and Koijen (2020) find that flows equal to 1% of stock market capitalization raise the market by 5%. In Table 2 instead, an increase in the measure μ_2 of non-experts from

0.1 to 0.6, which raises the total measure $\mu_1 + \mu_2$ of experts and non-experts by 50%, raises stock prices by only 5-8%. A important reason for the discrepancy is that our experts are unconstrained, while many active funds in practice have constraints limiting their deviations from indices. Additionally, non-experts do not fully invest in stocks (so an increase in their measure does not translate one-to-one into increased investment in stocks), and the return volatility in our calibrated example is smaller than in reality. Accounting for these considerations, the effects and distortions of passive flows could be larger than those we compute.

4.3 Noise Traders

Table 4 is the counterpart of Table 1 with noise traders. Stocks within each size group are split equally across those without noise traders and those for which noise traders hold 40% of the supply coming from issuers. This yields ten groups of stocks. The effects across size groups are similar to those in Table 1. The effects within size groups depend on size. Within size groups 1 and 2, expected return and volatility are independent of noise-trader demand. Within size groups 3, 4 and 5 instead, expected return declines and volatility rises when moving from low to high noise-trader demand.

Table 4: Return Moments with Noise Traders.

Size Group	**Noise-Trader Demand**	**Expected Return (%)**	**Return Volatility (%)**	**Market Beta**	**CAPM R^2 (%)**
1 (Smallest)	Low	5.17	21.10	1.34	24.95
	High	5.17	21.10	1.34	24.93
2	Low	4.58	18.25	1.16	24.78
	High	4.58	18.25	1.16	24.69
3	Low	4.16	16.10	1.03	25.11
	High	4.13	16.16	1.02	24.70
4	Low	3.91	14.10	0.96	28.40
	High	3.84	14.31	0.95	26.88
5 (Largest)	Low	3.86	11.75	0.95	40.06
	High	3.73	12.19	0.94	36.72

Table 4 implies that the risk-return relationship is positive across size groups but is negative within large stocks. The positive risk-return relationship across size groups is driven by fundamentals: high CAPM beta of small stocks implies high expected return and high volatility. The negative risk-return relationship within large stocks is driven by noise-trader demand: high demand implies low expected return and high volatility. Intuitively, noise-trader demand affects a stock's price through the present value of the idiosyncratic component of dividends. High demand lowers idiosyncratic supply, raising the present value and the price, and lowering expected return. High demand also raises return volatility because it renders the price more sensitive to shocks to idiosyncratic dividends. The effects of noise-trader demand are present only for large stocks because idiosyncratic dividends of small stocks are discounted at the riskless rate r regardless of demand.

Table 5 is the counterpart of Table 2 with noise traders. When flows into passive funds are due to increased participation in the stock market, their effect varies across size groups in a manner similar to Table 2. The effect within size groups 1, 2 and 3 is independent of noise-trader demand. Within size groups 4 and 5 instead, flows have a larger effect on the prices of high-demand stocks. The partial effect that flows have holding volatility constant does not depend on noise-trader demand. (CAPM beta is approximately independent of demand.) Demand influences instead the total effect of flows, which includes the change in volatility. Because stocks in high demand are in low idiosyncratic supply, the attenuation effect caused by the increase in price sensitivity is weaker.

Table 5: Percentage Price Change Following Flows into Passive Funds with Noise Traders.

Size Group	**Noise-Trader Demand**	**Increase in Market Participation**		**Switch from Active to Passive**	
		All Stocks in Index	**Size Groups 3-5 in Index**	**All Stocks in Index**	**Size Groups 3-5 in Index**
1 (Smallest)	Low	6.97	6.83	-0.07	-0.87
	High	6.97	6.83	0.01	-0.80
2	Low	5.98	5.75	-0.18	-1.33
	High	5.97	5.73	0.13	-1.04
3	Low	5.66	5.84	-0.61	-0.18
	High	5.65	5.85	0.64	1.25
4	Low	6.36	7.12	-1.57	0.45
	High	6.72	7.77	2.28	6.78
5 (Largest)	Low	7.13	8.54	-2.09	0.91
	High	8.94	12.17	4.81	31.95

When flows into passive funds are due to a switch from active to passive, they affect prices even in the case where the index includes all stocks. Prices drop for stocks in low noise-trader demand, and rise for stocks in high demand. Moreover, the effect is asymmetric in the sense that the price rises exceed the price drops in absolute value, within size groups 3, 4 and 5, and across the aggregate market. Stocks in low noise-trader demand drop because they are in high demand by experts, so a substitution of experts by non-experts lowers their net demand. Conversely, stocks in high noise-trader demand rise because they are in low demand by experts, so a substitution raises their net demand. The asymmetry arises for a similar reason as in Table 2. A switch from active to passive raises significantly the present value of idiosyncratic dividends of stocks that are in size groups 4 and 5 and in high demand by noise traders. Experts either hold a small long position in these stocks, in which case the attenuation effect is weak, or a short position, in which case attenuation turns into amplification. In our calibrated example, amplification arises in the case where the index includes only size groups 3, 4 and 5. That case corresponds to the fifth column in Table 5, which shows a particularly large price rise for high-demand stocks in size group 5.

4.4 Index Additions

We next compute the change in a stock's price and return volatility when the stock is added to the index. Passive flows in that case are only into that specific stock, while passive flows in Sections 4.2 and 4.3 are into each stock in the index. Table 6 reports the percentage price change of stock n and the change in the stock's return volatility when the stock is added to the index. We assume that there are noise traders, the measure of experts is 0.9 and the measure of non-experts is 0.6. We consider both the case where the index before the addition includes all stocks except stock n, and the case where the index before the addition includes all stocks in size groups 3, 4 and 5 except stock n.

Adding a stock to the index raises the stock's price and return volatility. The effect is almost zero for small stocks, but grows with size and becomes significant for large stocks. The intuition is the same as for the effect of noise-trader demand (Table 4). Index additions lower idiosyncratic supply, raising the present value of idiosyncratic dividends and the price. Index additions also raise return volatility because they render the price more sensitive to shocks to idiosyncratic dividends.

Table 6: Effects of Index Additions.

Size Group	Noise-Trader Demand	Percentage Price Change		Change in Return Volatility	
		All Stocks in Index	Size Groups 3-5 in Index	All Stocks in Index	Size Groups 3-5 in Index
1 (Smallest)	Low	0.04	0.06	0.00	0.00
	High	0.04	0.06	0.00	0.00
2	Low	0.18	0.26	0.01	0.01
	High	0.19	0.26	0.01	0.01
3	Low	0.72	1.03	0.04	0.05
	High	0.77	1.10	0.04	0.05
4	Low	2.03	2.98	0.14	0.20
	High	2.64	3.92	0.17	0.25
5 (Largest)	Low	2.66	4.14	0.23	0.36
	High	5.03	8.42	0.42	0.67

Holding size constant, index additions have larger effects on the price and return volatility of stocks in high demand by noise traders. The intuition is similar as why passive flows have a larger effect on high-demand stocks (Table 5). Index additions raise the price and render it more sensitive to shocks to idiosyncratic dividends. Because stocks in high noise-trader demand are in low idiosyncratic supply, the attenuation effect caused by the increase in price sensitivity is weaker.

5 General Results

In this section we derive results for general parameter values that parallel the results shown in the calibrated example. We first examine how the effect of passive flows on stock prices depends on stock characteristics.

Proposition 5.1. *Consider stocks n and n' with $(\kappa^i_n, \bar{D}^i_n, \sigma^i_n) = (\kappa^i_{n'}, \bar{D}^i_{n'}, \sigma^i_{n'})$ and $(n, n') \in \mathcal{I} \times \mathcal{I}$ or $(n, n') \in \mathcal{I}^c \times \mathcal{I}^c$. When μ_2 increases holding μ_1 constant, stock n experiences a larger percentage price increase than stock n', for all D^s_t and $D^i_{nt} = D^i_{n't}$, in the following cases:*

(i) $b_n > b_{n'}$ *and* $(\eta_n, u_n) = (\eta_{n'}, u_{n'})$*, under the sufficient condition*

$$\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} > \frac{\eta'_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}. \tag{5.1}$$

(ii) $\eta_n > \eta_{n'}$ *and* $(b_n, \frac{u_n}{\eta_n}) = (b_{n'}, \frac{u_{n'}}{\eta_{n'}})$*, under the sufficient condition* $(n,n') \in \mathcal{I}^c \times \mathcal{I}^c$*; or* $(n,n') \in \mathcal{I} \times \mathcal{I}$*,* $\mu_2\lambda + \frac{u_n}{\eta_n} \le 1$ *and*

$$(r+\kappa_n^i)^2 \ge 2(\sqrt{2}-1)\rho\frac{\eta_n(1-\mu_2\lambda)-u_n}{\mu_1}(\sigma_n^i)^2; \tag{5.2}$$

or $(n,n') \in \mathcal{I} \times \mathcal{I}$*,* $\mu_2\lambda + \frac{u_n}{\eta_n} > 1$ *and*

$$1 + \frac{\rho(\mu_2\lambda + \frac{u_n}{\eta_n} - 1)}{\mu_1}\left(\frac{3\eta_{n'}(\sigma_n^i a_{n'1}^i)^2}{2-(r+\kappa_n^i)a_{n'1}^i} - \frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s}\right) \ge 0. \tag{5.3}$$

(iii) $u_n > u_{n'}$ *and* $(b_n, \eta_n) = (b_{n'}, \eta_{n'})$*, under the sufficient condition* $(n,n') \in \mathcal{I} \times \mathcal{I}$ *and*

$$\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} \le \frac{3\eta_n(\sigma_n^i a_{n'1}^i)^2}{2-(r+\kappa_n^i)a_{n'1}^i}. \tag{5.4}$$

When μ_2 *increases and* μ_1 *decreases by a corresponding fraction* $\phi \in (0,1]$*, the results in Case (ii) with* $(n,n') \in \mathcal{I} \times \mathcal{I}$ *and* $\mu_2\lambda + \frac{u_n}{\eta_n} > 1$*, and in Case (iii), hold, provided additionally that* $n \in \operatorname{argmax}_m \frac{u_m}{\eta_m}$ *and that* $\phi < 1$ *or* $\mathcal{I} \subsetneq \{1,..,N\}$ *or* $\#\{\frac{u_m}{\eta_m} : m \in \{1,..,N\}\} > 1$*.*

When passive flows are due to increased participation in the stock market, Proposition 5.1 shows the following results. First, flows generate a larger percentage price increase for stocks that load more on the systematic factor (larger b_n), holding all else (including size) constant. This result requires that flows impact the present value of the systematic component of dividends more, in percentage terms, than the present value of the idiosyncratic component. This condition is intuitive because the discount rate is larger for the systematic component, and holds for all stocks in our calibrated example. Second, passive flows generate a larger percentage price increase for larger stocks (larger η_n), holding all else constant. This result requires an upper bound on stock size when experts hold a long position in equilibrium ($\mu_2\lambda + \frac{u_n}{\eta_n} > 1$), and a lower bound on size when experts hold a short position, with the requirements being always satisfied when the position of experts approaches zero. Third, passive flows generate a larger percentage price increase for stocks that are in higher demand by noise traders (larger u_n), holding all else constant. This result requires that stocks are sufficiently large. Table 5 shows that the result can indeed reverse for small stocks.

The effects of size and noise-trader demand carry through to the case where passive flows are partly due to a switch from active to passive ($0 < \phi < 1$), or are purely due to such a switch but the index does not include all stocks ($\mathcal{I} \subsetneq \{1, .., N\}$) or noise traders do not hold the market portfolio (the set $\{\frac{u_m}{\eta_m} : m \in \{1, .., N\}\}$ is not a singleton). The effects of passive flows increase with size for those stocks that are in the index ($n \in \mathcal{I}$) and in high demand by noise-traders ($n \in \text{argmax}_m \frac{u_m}{\eta_m}$). Holding size constant, the effects of passive flows are higher for the high-demand stocks ($n \in \text{argmax}_m \frac{u_m}{\eta_m}$). We next examine how the effect of passive flows on return volatilities depends on stock characteristics.

Proposition 5.2. *Consider stocks n and n' with $(b_n, \kappa_n^i, \bar{D}_n^i, \sigma_n^i) = (b_{n'}, \kappa_{n'}^i, \bar{D}_{n'}^i, \sigma_{n'}^i)$ and $(n, n') \in \mathcal{I} \times \mathcal{I}$. When μ_2 increases holding μ_1 constant, stock n experiences a rise in return volatility and stock n' experiences a decline for $D_t^s = \bar{D}^s = 1$, $D_{nt}^i = D_{n't}^i = \bar{D}_n^i$ and an interval of values of $\bar{D}_n = 1 - b_n - \bar{D}_n^i$, in the following cases:*

(i) $\eta_{n'} \approx 0$, η_n satisfies (5.4) for $n' = n$, and $\frac{u_n}{\eta_n} = \frac{u_{n'}}{\eta_{n'}}$, under the sufficient condition

$$\max\left\{ \frac{2\left(\frac{(r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i}{(r+\kappa_n^i) b_n (\sigma^s)^2 a_1^s} - 1\right)(r+\kappa_n^i) a_{n1}^i \bar{D}_n^i}{3 + \frac{(\sigma_n^i a_{n1}^i)^2 \bar{D}_n^i}{(b_n \sigma^s a_1^s)^2}}, 0 \right\} < \left(\frac{(r+\kappa^s)(\sigma_n^i)^2}{(r+\kappa_n^i)^2 b_n (\sigma^s)^2 a_1^s} - 1 \right) \bar{D}_n^i. \quad (5.5)$$

(ii) $u_n > u_{n'}$ and $\eta_n = \eta_{n'}$, under the sufficient conditions

$$\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2 - (r+\kappa^s) a_1^s} \geq \frac{\psi \eta_n (\sigma_n^i a_{n1}^i)^2}{2 - (r+\kappa_n^i) a_{n1}^i}, \quad (5.6)$$

$$(r+\kappa^s) b_n a_1^s < \frac{2(\psi - 1)\left(\frac{(r+\kappa^s)(\sigma_n^i)^2 a_{n'1}^i}{(r+\kappa_n^i) b_n (\sigma^s)^2 a_1^s} - 1\right)(r+\kappa_n^i) a_{n1}^i \bar{D}_n^i}{\psi + \frac{(\sigma_n^i a_{n1}^i)^2 \bar{D}_n^i}{(b_n \sigma^s a_1^s)^2}} \quad (5.7)$$

and (5.4), for some scalar $\psi > 1$.

Return volatility of both stocks rises for values of $\bar{D}_n$ above the interval, and declines for values below the interval. When μ_2 increases and μ_1 decreases by a corresponding fraction $\phi \in (0,1]$, the result in Case (ii) holds, provided additionally that $n \in \text{argmax}_m \frac{u_m}{\eta_m}$ and that $\phi < 1$ or $\mathcal{I} \subsetneq \{1,..,N\}$ or $\#\{\frac{u_m}{\eta_m} : m \in \{1,..,N\}\} > 1$.

When passive flows are due to increased participation in the stock market, Proposition 5.2 shows that they are more likely to raise the return volatility of larger stocks and of stocks that are in higher demand by noise traders. Higher likelihood is in the sense that there exists a parameter interval within which volatility rises for large or high-demand stocks and declines for small or low-demand stocks. Outside that interval, volatility moves in the same direction for all stocks.

The parameter that defines the interval is the constant component $\bar{D}_n$ of dividends. When $\bar{D}_n$ is large, passive flows raise volatilities of all stocks. Intuitively, flows raise the present value of the systematic and the idiosyncratic component of dividends and render them more sensitive to shocks. Volatility rises if the sensitivity of the price divided by the price increases. If $\bar{D}_n$ is large, then the percentage change in the price is small since the present value of the constant component of dividends does not rise in response to flows. Volatility rises because the percentage change in price sensitivity, which does not involve $\bar{D}_n$, is larger.

For large stocks, the effect of flows on the present value of the idiosyncratic component of dividends, and on that component's sensitivity to shocks, is significant enough to cause volatility to rise even for smaller values of $\bar{D}_n$. The same is true for high-demand stocks, provided that these stocks are also large ((5.4) is met). In both cases, the return volatility caused by the idiosyncratic component must be large enough relative to that caused by the systematic component ((5.5) and (5.7) are met). Unlike the price level results, the volatility results can fail to hold for extreme values of D^s_t and D^i_{nt}, and are shown when D^s_t and D^i_{nt} are equal to their long-run means. The effect of size carries through to the case where passive flows are due to a switch from active to passive, under the same conditions as in Proposition 5.1.

We finally examine how the effect of index additions on stock prices depends on stock characteristics. We assume that when a stock n is added to the index, a stock $\hat{n}$ with identical characteristics

$(b_n, \kappa^i_n, \bar{D}^i_n, \sigma^i_n, \eta_n, u_n) = (b_{\hat{n}}, \kappa^i_{\hat{n}}, \bar{D}^i_{\hat{n}}, \sigma^i_{\hat{n}}, \eta_{\hat{n}}, u_{\hat{n}})$ is taken out of the index. This ensures that aggregate quantities, such as the discount rate of the systematic component of dividends, do not change after the addition.[12] Proposition 5.3 shows that index additions generate a larger percentage price increase for stocks that are larger or in higher demand by noise traders.

Proposition 5.3. *Consider stocks n and n' with $(b_n, \kappa^i_n, \bar{D}^i_n, \sigma^i_n) = (b_{n'}, \kappa^i_{n'}, \bar{D}^i_{n'}, \sigma^i_{n'})$. Stock n experiences a larger percentage price increase than stock n' when it is added in the index, for all D^s_t and $D^i_{nt} = D^i_{n't}$, in the following cases:*

(i) $\eta_n > \eta_{n'}$ *and* $\frac{u_n}{\eta_n} = \frac{u_{n'}}{\eta_{n'}}$*, under the sufficient condition*

$$(r + \kappa^i_n)^2 \geq 2(\sqrt{2} - 1)\rho \frac{\eta_n - u_n}{\mu_1}(\sigma^i_n)^2. \tag{5.8}$$

(ii) $u_n > u_{n'}$ *and* $\eta_n = \eta_{n'}$.

6 Empirical Evidence

In this section we show that our model's predictions on the relationship between passive flows and stock size hold in the data. We take the index to be the S&P500, and passive flows to be into US listed index mutual funds and index ETFs tracking it. The S&P500 index accounts for the bulk of passive investing in US stocks: index mutual funds tracking the S&P500 index account for 47% to 87% of assets of all index mutual funds invested in US stocks in our sample. We refer to index mutual funds and index ETFs tracking the S&P500 index as S&P500 index funds.

6.1 Data and Descriptive Statistics

Our data on stock returns, market capitalization, and the composition of the S&P500 index come from the Center for Research in Security Prices (CRSP). Our data on net assets of S&P500 index mutual funds come from the Investment Company Institute (ICI). Our data on net assets of S&P500

[12] Our assumption also ensures that the number of stocks in the index does not change after the addition. By contrast, Table 6 in the calibrated example assumes that when a stock is added to the index, no stock is taken out, so the number of stocks in the index increases by one. The data in Table 6 remain almost the same if the number of stocks in the index is assumed to not change after the addition.

index ETFs come from CRSP. We include in our analysis only plain-vanilla ETFs, excluding alternative ETFs such as leveraged ETFs, inverse ETFs and buffered ETFs. Our ETF sample consists of the SPDR S&P500 ETF Trust, the iShares Core S&P500 ETF, and the Vanguard S&P500 Index Fund ETF, which collectively account for almost all of the plain-vanilla S&P500 ETF market. Our sample begins in the second quarter of 1996 and ends in the fourth quarter of 2020.

Table 7 reports descriptive statistics. The descriptive statistics in Panel A concern aggregate variables sourced at a quarterly frequency. The descriptive statistics in Panel B concern firm-level variables pertaining to all S&P500 firms and sourced at a quarterly frequency. The descriptive statistics in Panel C concern firm-level variables pertaining to episodes where firms were added to the index. There are 426 index-addition episodes during our sample period. All variables in Panel A except the last (VIX) and all variables in Panel C are multiplied by 100.

Table 7: Descriptive Statistics

	Mean	Standard Deviation	25th Percentile	Median	75th Percentile	Skewness	Kurtosis
Panel A: Aggregate Variables							
$R^{ew}_{Large-Index}$	-0.15	1.56	-1.06	-0.39	0.72	0.52	0.96
$R^{vw}_{Large-Index}$	-0.17	1.86	-1.27	-0.20	1.12	-0.17	0.50
$PassiveFlow$	0.05	0.09	0.01	0.05	0.10	0.33	3.62
$\Delta Top10$	0.51	3.87	-1.98	0.41	3.17	0.20	0.69
$\Delta Dispersion$	0.51	3.58	-1.90	0.45	2.37	0.41	1.23
ΔH	0.81	5.61	-2.85	0.58	3.55	0.52	1.57
VIX	20.36	7.59	14.57	19.31	24.92	1.80	6.03
Panel B: Firm-Level Variables for All Firms							
$TotVol$	-4.01	0.50	-4.36	-4.05	-3.71	0.43	0.70
$IdioVol$	-4.28	0.50	-4.64	-4.31	-3.95	0.34	0.40
Panel C: Firm-Level Variables for Index-Addition Episodes							
$CAR^{m}_{a,e-1}$	3.66	7.67	-0.84	2.53	7.09	1.28	9.92
$CAR^{m}_{e-1,e}$	1.04	4.36	-1.31	0.41	2.42	1.70	8.38
$CAR^{m}_{e,e+5}$	-1.12	5.65	-3.32	-0.66	1.69	-0.89	5.27
$CAR^{FFm}_{a,e-1}$	3.50	7.08	-0.83	2.56	6.72	0.81	8.28
$CAR^{FFm}_{e-1,e}$	0.88	4.36	-1.28	0.39	2.19	1.54	8.43
$CAR^{FFm}_{e,e+5}$	-0.98	5.09	-3.23	-0.71	1.73	-0.93	4.01
$Cap/\$SP500IndexCap$	0.08	0.08	0.05	0.06	0.09	8.41	106.10

The first two rows in Panel A concern the return of large stocks in the index in excess of the index return. The large-stock portfolio consists of the top decile of S&P500 stocks based on market capitalization. Deciles are formed at the end of any given quarter (implying that the large-stock portfolio is rebalanced quarterly). The first row concerns the equal-weighted quarterly excess return of the large-stock portfolio, $R^{ew}_{Large-Index}$, and the second row concerns the value-weighted quarterly excess return, $R^{vw}_{Large-Index}$. The means of $R^{ew}_{Large-Index}$ and $R^{vw}_{Large-Index}$ are -0.15% and -0.17%, respectively, implying that large stocks underperformed the index over our sample period. The standard deviations are 1.56% and 1.86%, respectively.

The third row in Panel A concerns passive flows. We measure flows into S&P500 index funds in any given quarter by the ratio of S&P500 index fund net assets to index market capitalization (i.e., combined capitalization of all S&P500 stocks) minus the same ratio in the previous quarter:

$$PassiveFlow_t = \frac{\$SP500IndexAssets_t}{\$SP500IndexCap_t} - \frac{\$SP500IndexAssets_{t-1}}{\$SP500IndexCap_{t-1}}.$$

The mean of passive flow is 0.05% quarterly, implying that the ratio of S&P500 index fund net assets to S&P500 market capitalization grew by approximately 5% during our sample period. The standard deviation of passive flow is 0.09%.

The fourth, fifth and sixth rows in Panel A concern three measures of index concentration: the combined weight of the top ten stocks in the index, denoted by $Top10$, the standard deviation of index weights across all S&P500 stocks, denoted by $Dispersion$, and the Herfindahl index of index weights across all S&P500 stocks, denoted by H. The descriptive statistics concern the first difference of the logarithm of the three variables. Index concentration has been growing during our sample period, by rates ranging from 0.51% to 0.81% per quarter.[13] The seventh row in Panel A concerns VIX, the CBOE volatility index.

The first row in Panel B concerns total volatility ($TotVol$) and the second row concerns idiosyncratic volatility ($IdioVol$). Total volatility is the standard deviation of daily stock returns in any given quarter. Idiosyncratic volatility is the standard deviation of daily residual returns from the Fama-French three-factor model. The descriptive statistics concern the logarithm of the two variables.

The first six rows in Panel C concern the returns of the stocks that are added to the index. We partition the period around each addition episode into three sub-periods. Two dates defining the

[13]The growth in index concentration is consistent with the underperformance of large stocks because large firms issued more new shares.

sub-periods are the announcement date, after the market close of which the addition is announced, and the effective date, after the market close of which the addition is implemented. The first sub-period ranges from the announcement date to one trading day before the effective date. The second sub-period ranges from one trading day before the effective date to the effective date. The third sub-period ranges from the effective date to five trading days after that date. We compute cumulative abnormal return (CAR) within each sub-period, from the close of the starting date to the close of the ending date.

The first, second and third rows in Panel C concern returns during the first, second, and third sub-period, respectively, adjusted for market movements by subtracting the market return. The fourth, fifth and sixth rows in Panel C concern returns during the same sub-periods adjusted using the Fama-French three-factor model augmented with a momentum factor (FFm). The two adjustment methods yield similar results.

During the first sub-period, the price of the stock that is added to the index rises on average, in anticipation of the demand by index funds. The mean abnormal return is 3.66% using the market adjustment and 3.50% using the FFm adjustment. During the second sub-period, the price of the stock rises further on average, as index funds buy the stock. The mean abnormal return is 1.04% using the market adjustment and 0.88% using the FFm adjustment. During the third sub-period, the price of the stock drops on average, as the market absorbs the demand imbalance. The mean abnormal return is -1.12% using the market adjustment and -0.98% using the FFm adjustment.

The seventh row in Panel C concerns the market capitalization of the stocks that are added to the index. To make capitalization comparable across index-addition episodes, we divide the capitalization of the added stock by the capitalization of the index at the end of the month before the addition announcement. The mean of the resulting variable, $Cap/\$SP500IndexCap$, is 0.08% and the standard deviation is 0.08%. The kurtosis is high (106.10) because while most stocks that are added to the index have capitalizations similar to the smaller stocks in the index, a few stocks are large.[14] Our tests on index additions account for the high kurtosis.

[14] An example of a large addition is Tesla. Its capitalization was $387 billion on the announcement date, 16 November 2020, and rose to $659 billion on the effective date, 18 December 2020. On the effective date, Tesla had the sixth largest capitalization among S&P500 firms. The large rise in Tesla's capitalization between announcement and effective date is consistent with our model.

6.2 Tests

6.2.1 Passive Flows and Excess Returns

Table 8 reports results from regressing the excess return of large stocks on passive flows. For ease of interpretation, we standardize $PassiveFlow$ to a mean of zero and a standard deviation of one. We denote the resulting variable by $\widehat{PassiveFlow}$. We use the same notation for VIX and for the three measures of S&P500 index concentration in Table 9. In both Tables 8 and 9, the t-statistics, in parentheses, are based on Newey-West heteroskedasticity- and autocorrelation-consistent standard errors with three lags. Our findings are robust to increasing the number of lags.

Consistent with our model, the relationship between passive flows and excess returns of large stocks is positive and significant statistically and economically. In the univariate regressions in Columns (1) and (2), a one-standard-deviation increase in $\widehat{PassiveFlow}$ is associated with an increase in the quarterly excess returns of large stocks by 0.55%. This is approximately one-third of the quarterly excess returns' standard deviation in Table 7. The t-statistics are around 3.60. Statistical and economic significance are similar when controlling for the contemporaneous and one-quarter lagged return of the S&P500 index and for VIX, in Columns (3) and (4).

Table 8: Passive Flows and Excess Returns on Large Stocks

	(1)	(2)	(3)	(4)
	$R^{ew}_{Large-Index}$	$R^{vw}_{Large-Index}$	$R^{ew}_{Large-Index}$	$R^{vw}_{Large-Index}$
$\widehat{PassiveFlow}$	0.00549	0.00550	0.00523	0.00525
	(3.60)	(3.67)	(4.14)	(3.64)
R_{Index}			-0.0374	-0.0203
			(-1.69)	(-0.70)
$L.R_{Index}$			-0.0104	0.00773
			(-0.41)	(0.36)
$\widehat{VIX}$			0.00201	0.00271
			(1.35)	(1.31)
Constant	-0.00146	-0.00166	-0.000197	-0.00134
	(-0.90)	(-0.79)	(-0.10)	(-0.52)
Observations	99	99	99	99
Adjusted R^2	0.124	0.087	0.206	0.123

Table 9 reports results from regressing changes in S&P500 index concentration on passive flows. Consistent with our model, the relationship between passive flows and changes to all three measures of concentration is positive and significant statistically and economically. In the univariate regressions in Columns (1)–(3), a one-standard-deviation increase in $PassiveFlow$ is associated with an increase in the concentration measures by 0.23-0.24 standard deviations. Statistical and economic significance are similar when controlling for the contemporaneous and lagged return of the S&P500 index and for VIX, in Columns (4)–(6).

Table 9: Passive Flows and Index Concentration

	(1)	(2)	(3)	(4)	(5)	(6)
	$\widehat{\Delta Top10}$	$\widehat{\Delta Dispersion}$	$\widehat{\Delta H}$	$\widehat{\Delta Top10}$	$\widehat{\Delta Dispersion}$	$\widehat{\Delta H}$
$\widehat{PassiveFlow}$	0.244	0.239	0.230	0.235	0.233	0.224
	(2.30)	(2.02)	(1.92)	(2.74)	(2.46)	(2.35)
R_{Index}				-0.520	0.0127	0.0905
				(-0.34)	(0.01)	(0.06)
$L.R_{Index}$				0.476	0.508	0.574
				(0.43)	(0.53)	(0.60)
$\widehat{VIX}$				0.239	0.274	0.283
				(1.73)	(1.98)	(2.01)
Observations	99	99	99	99	99	99
Adjusted R^2	0.060	0.057	0.053	0.121	0.126	0.125

6.2.2 Passive Flows and Return Volatility

Our model predicts that passive flows should raise the volatility of the largest stocks in the S&P500 index, while the effect should be weaker or negative for smaller stocks. To test for this prediction, we perform panel regressions of stock return volatility on one-quarter lagged $PassiveFlow$ interacted with a $Large$ firm indicator. The indicator is equal to one if the firm belongs to the top decile of S&P500 stocks based on market capitalization, and to zero otherwise. The regression results are reported in Table 10.

The dependent variable in the regressions is the logarithm of total volatility or of idiosyncratic volatility. The independent variables in Columns (1) and (2) are the interaction term, its two constituents separately, the one-quarter lagged index return, the logarithm of one-quarter lagged total or idiosyncratic volatility (to control for serial dependence in volatility), and firm fixed effects. In Columns (3) and (4), we introduce additionally time fixed effects to absorb the time-series variation, and drop lagged $PassiveFlow$ and index return. We conservatively double-cluster standard

Table 10: Passive Flows and Return Volatility

	(1)	(2)	(3)	(4)
	$TotVol$	$IdioVol$	$TotVol$	$IdioVol$
L.$PassiveFlow \times Large$	21.66	19.30	22.34	18.41
	(2.33)	(2.52)	(2.26)	(2.44)
L.$PassiveFlow$	20.51	20.64		
	(0.83)	(1.21)		
L.$Large$	-0.0354	-0.0471	-0.0401	-0.0668
	(-2.38)	(-2.84)	(-3.26)	(-4.81)
L.R_{Index}	-0.350	-0.356		
	(-1.41)	(-1.93)		
L.$TotVol$	0.610		0.530	
	(15.33)		(29.59)	
L.$IdioVol$		0.628		0.456
		(22.88)		(28.33)
Observations	45,737	45,737	45,737	45,737
Firm fixed effects	Yes	Yes	Yes	Yes
Time fixed effects	No	No	Yes	Yes
Adjusted R^2	0.559	0.600	0.777	0.712

errors by firm and time.

Consistent with our model, passive flows impact more strongly the return volatility of the largest stocks, and this effect is significant statistically and economically. In Column (1), a one-standard-deviation increase in $PassiveFlow$ is associated with a percentage increase in total volatility by 1.85% (=20.51×0.09%) for stocks outside the top decile, and this effect approximately doubles to 3.80% (=$(20.51 + 21.66) \times 0.09$%) for stocks in the top decile. Moreover, the incremental effect for large stocks is statistically significant while the effect for other stocks is not. Also consistent with our model, the effect of passive flows is not confined to total volatility but extends to idiosyncratic volatility: the coefficients of $PassiveFlow$ and of the interaction term in Column (2) are approximately equal to their counterparts in Column (1). Statistical significance is similar when adding time fixed effects, in Columns (3) and (4).

6.2.3 Index Additions

Table 11 reports results from regressing stock returns during index-addition episodes on stock size. The t-statistics, in parentheses, are based on White heteroskedasticity-robust standard errors.

Consistent with our model, size is positively related to CAR during the first and second sub-periods, and negatively during the third sub-period.[15] These relationships are significant statistically and economically. A one-standard deviation increase in $Cap/\$SP500IndexCap$, our measure of stock size, is associated with an increase in market-adjusted CAR during the first sub-period by 2.23% (=27.92 × 0.08%). This is almost two-thirds of the mean CAR during that sub-period. The corresponding increase in CAR during the second sub-period is 0.65% (=8.066 × 0.08%), almost two-thirds of the mean CAR during that sub-period, and the corresponding decrease in CAR during the third sub-period is 0.50% (=−6.234 × 0.08%). Results for the FFm-adjusted CAR are similar. Results are also similar when performing robust regression to account for the high kurtosis of $Cap/\$SP500IndexCap$.

Table 11: Index Additions and Firm Size

	(1)	(2)	(3)	(4)	(5)	(6)
	$CAR^m_{a,e-1}$	$CAR^m_{e-1,e}$	$CAR^m_{e,e+5}$	$CAR^{FFm}_{a,e-1}$	$CAR^{FFm}_{e-1,e}$	$CAR^{FFm}_{e,e+5}$
$Cap/\$SP500IndexCap$	27.92	8.066	-6.234	23.52	6.501	-7.433
	(7.28)	(2.38)	(-2.62)	(7.28)	(2.22)	(-2.47)
Constant	1.383	0.388	-0.610	1.588	0.346	-0.374
	(2.84)	(1.19)	(-1.74)	(3.62)	(1.12)	(-1.14)
Observations	426	426	426	426	426	426
Adjusted R^2	0.092	0.022	0.006	0.076	0.013	0.013

7 Conclusion

The dramatic growth of passive investing over the past thirty years has raised questions on how passive investing affects asset prices and the real economy. In this paper we show that passive investing is not neutral, in the sense that it does not reduce the financing costs of all firms equally. Nor does passive investing reduce the financing costs of the riskiest firms the most, as a simple CAPM argument would suggest. The firms whose financing costs decline the most when money flows into passive funds are instead the largest firms in the indices tracked by the funds. The same largest stocks experience an increase in return volatility and sensitivity to cashflow shocks. If, in

[15] Table 6 in the calibrated example does not report the change in expected return following index additions. Expected return declines, and more so for larger stocks.

addition, stocks are mispriced because of noise traders, then passive flows raise disproportionately the prices of the overvalued stocks among the indices' largest. Passive flows have an asymmetric effect in the cross-section, driving the aggregate market up even when they are entirely due to a switch from active to passive. Finally, larger or more overvalued stocks experience higher price increases when they are added to an index. Underlying our results is that passive flows make prices more sensitive to news about idiosyncratic future cashflows. We provide empirical evidence in support of our model's main mechanisms.

A natural extension of our work concerns the design of indices. Passive funds in our model track capitalization-weighted indices. While such indices are the most common in practice, it would be interesting to determine how flows into other types of indices, such as price-weighted or equal-weighted, affect prices. It would also be interesting to determine how indices should be designed to achieve welfare objectives. If the growth of passive funds reduces primarily the financing costs of the largest firms across industries, and market power is a concern, then should capitalization-weighting be moderated? Does a similar conclusion hold even in the absence of market power but if some of the largest firms benefit from low financing costs because they are overvalued (in which case the growth of passive funds amplifies the consequences of mispricing)? Should capitalization-weighting be moderated by imposing upper bounds on weights, as is the case for some sovereign-bond indices? Should it be moderated by giving more emphasis on earnings or other firm fundamentals? Is capitalization-weighting the best solution despite its drawbacks?

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Appendix

A Proofs

Proof of Proposition 3.1. The quadratic equation derived from (3.12) by identifying terms in D_t^s is

$$\rho\left(\sum_{m=1}^{N}\frac{\eta_m-\mu_2\lambda\eta_m'-u_m}{\mu_1}b_m\right)(\sigma^s a_1^s)^2+(r+\kappa^s)a_1^s-1=0. \tag{A.1}$$

The quadratic equation derived by identifying terms in D_{nt}^i is

$$\rho\frac{\eta_n-\mu_2\lambda\eta_n'-u_n}{\mu_1}(\sigma_n^i a_{n1}^i)^2+(r+\kappa_n^i)a_{n1}^i-1=0. \tag{A.2}$$

The equation derived by identifying the remaining terms is

$$\bar{D}_n-r\bar{S}_n+b_n\left(\kappa^s a_1^s-ra_0^s\right)+\kappa_n^i a_{n1}^i\bar{D}_n^i-ra_{n0}^i=0. \tag{A.3}$$

When $\sum_{m=1}^{N}(\eta_m-\mu_2\lambda\eta_m'-u_m)b_m\geq 0$, the left-hand side of (A.1) is increasing for positive values of a_1^s, and (A.1) has a unique positive solution, given by (3.14). When $\sum_{n=1}^{N}(\eta_m-\mu_2\lambda\eta_m'-u_m)b_n<0$, the left-hand side of (A.1) is hump-shaped for positive values of a_1^s, and (A.1) has either two positive solutions (including one double positive solution) or no solution. When two solutions exist, (3.14) gives the smaller of them, which is the continuous extension of the unique positive solution when $\sum_{m=1}^{N}(\eta_m-\mu_2\lambda\eta_m'-u_m)b_m>0$. Equation (3.15) gives the analogous solution of (A.2). Equation (A.3) yields

$$\bar{S}_n+b_n a_0^s+a_{n0}^i=\frac{\bar{D}_n+b_n\kappa^s a_1^s+\kappa_n^i a_{n1}^i\bar{D}_n^i}{r}. \tag{A.4}$$

Substituting (3.10), (3.11) and (A.4) into (3.1), we find (3.13).

Substituting $\mu_n = \mathbb{E}(\mu_{nt})$ and (3.9)-(3.11) into (3.7), we find

$$\left(\sum_{m=1}^{N}\eta'_m b_m\right)\left(\sum_{m=1}^{N}\frac{\eta_m-\mu_2\lambda\eta'_m-u_m}{\mu_1}b_m\right)(\sigma^s a_1^s)^2+\sum_{m=1}^{N}\eta'_m\left(\frac{\eta_m-\mu_2\lambda\eta'_m-u_m}{\mu_1}\right)(\sigma_m^i a_{m1}^i)^2\bar{D}_m^i$$
$$=\lambda\left[\left(\sum_{m=1}^{N}\eta'_m b_m\right)^2(\sigma^s a_1^s)^2+\sum_{m=1}^{N}\left(\eta'_m\right)^2(\sigma_m^i a_{m1}^i)^2\bar{D}_n^i\right], \tag{A.5}$$

which we can rewrite as (3.16). Since $\eta_m > u_m$ for all m, (3.16) implies $\lambda > 0$.

Equations (3.13)-(3.15) imply that the price depends on $(\mu_1, \mu_2, \sigma^s, \{b_m, \sigma_m^i, \eta_m, \eta'_m, u_m\}_{m=1,..,M})$ only through $\left(\sum_{m=1}^{N}\frac{\eta_m-\mu_2\lambda\eta'_m-u_m}{\mu_1}b_m\right)(\sigma^s)^2$ and $\frac{\eta_n-\mu_2\lambda\eta'_n-u_n}{\mu_1}(\sigma_n^i)^2$. The price is decreasing and convex in the latter two variables if a_1^s is decreasing and convex in $\left(\sum_{m=1}^{N}\frac{\eta_m-\mu_2\lambda\eta'_m-u_m}{\mu_1}b_m\right)(\sigma^s)^2$, and a_{n1}^i is decreasing and convex in $\frac{\eta_n-\mu_2\lambda\eta'_n-u_n}{\mu_1}(\sigma_n^i)^2$. These properties hold if the function

$$\Psi(z) \equiv \frac{1}{A+\sqrt{B+Cz}}$$

is decreasing and convex for $z \geq -\frac{B}{C}$, where (A, B, C) are positive constants. The function $\Psi(z)$ is decreasing because its derivative

$$\Psi'(z) = -\frac{C}{2\sqrt{B+Cz}}\frac{1}{\left(A+\sqrt{B+Cz}\right)^2}$$

is negative. Since, in addition, $\Psi'(z)$ is increasing, $\Psi(z)$ is convex.

An equilibrium exists if (A.5), in which a_1^s and $\{a_{n1}^i\}_{n=1,..,N}$ are implicit functions of λ defined by (3.14) and (3.15), respectively, has a solution. For all non-positive values of λ, both sides of (A.5) are well-defined because the non-negativity of $\sum_{m=1}^{N}(\eta_m - \mu_2\lambda\eta'_m - u_m)b_m$ and $\eta_n - \mu_2\lambda\eta'_n - u_n$ ensures that (3.14) and (3.15) have a solution for a_1^s and $\{a_{n1}^i\}_{n=1,..,N}$, respectively. Moreover, the right-hand side of (A.5) is positive, and exceeds the left-hand side which is non-positive. An equilibrium exists if both sides of (A.5) remain well-defined for a sufficiently large positive value of λ that renders them equal.

If an equilibrium exists, then it is unique. To show uniqueness, we treat (A.5) as an equation in the unknown $\mu_2\lambda$ instead of λ. Since the function $\Psi(z)$ is decreasing, (3.14) and (3.15) imply that the right-hand side of (A.5) is increasing in $\mu_2\lambda$. Equations (3.14) and (3.15) also imply that the left-hand side of (A.5) is decreasing in $\mu_2\lambda$ if the function

$$\Phi(z) \equiv \frac{z}{\left(A + \sqrt{B + Cz}\right)^2}$$

is increasing for $z \geq -\frac{B}{C}$, where (A, B, C) are positive constants. Showing that $\Phi(z)$ is increasing is equivalent to showing that

$$\hat{\Phi}(y) \equiv \frac{y^2 - B}{(A + y)^2}$$

is increasing for $y \equiv \sqrt{B + Cz} \geq 0$. The latter property follows because the functions $\hat{\Phi}_1(y) \equiv \frac{y}{A+y}$ and $\hat{\Phi}_2(y) \equiv -\frac{B}{(A+y)^2}$ are increasing for $y \geq 0$. Since the right-hand side of (A.5) is increasing in $\mu_2\lambda$ and the left-hand side is increasing, a solution $\mu_2\lambda$ of (A.5) is unique. □

Proof of Proposition 5.1. We first consider the case where μ_2 increases holding μ_1 constant. Since the right-hand side of (A.5) is increasing in $\mu_2\lambda$ and the left-hand side is decreasing, and since the right-hand side is decreasing in μ_2, $\mu_2\lambda$ increases in μ_2. Differentiating, we find

$$\begin{aligned}
\frac{1}{S_{nt}}\frac{\partial S_{nt}}{\partial \mu_2} &= \frac{\partial(\mu_2\lambda)}{\partial \mu_2}\frac{1}{S_{nt}}\frac{\partial S_{nt}}{\partial(\mu_2\lambda)} \\
&= \frac{\partial(\mu_2\lambda)}{\partial \mu_2}\frac{\rho}{\mu_1}\frac{b_n\frac{\partial a_1^s}{\partial(\mu_2\lambda)}(\kappa^s + rD_t^s) + \frac{\partial a_{n1}^i}{\partial(\mu_2\lambda)}(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)} \\
&= \frac{\partial(\mu_2\lambda)}{\partial \mu_2}\frac{\rho}{\mu_1}\frac{\frac{b_n\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2(\kappa^s + rD_t^s)}{\sqrt{(r+\kappa^s)^2 + 4\rho\left(\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1}b_m\right)(\sigma^s)^2}} + \frac{\eta'_n(\sigma_n^i a_{n1}^i)^2(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)}{\sqrt{(r+\kappa_n^i)^2 + 4\rho\frac{\eta_n - \mu_2\lambda\eta'_n - u_n}{\mu_1}(\sigma_n^i)^2}}}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)} \\
&= \frac{\partial(\mu_2\lambda)}{\partial \mu_2}\frac{\rho}{\mu_1}\frac{\frac{b_n\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s)^2(a_1^s)^3(\kappa^s + rD_t^s)}{2-(r+\kappa^s)a_1^s} + \frac{\eta'_n(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)}{2-(r+\kappa_n^i)a_{n1}^i}}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)} \equiv \frac{\partial(\mu_2\lambda)}{\partial \mu_2}\frac{\rho}{\mu_1}\mathcal{Z}.
\end{aligned}$$

where the second step follows from (3.13), the third step from (3.14) and (3.15), and the fourth step again from (3.14) and (3.15).

The result in Case (i) will follow if we show that $\mathcal{Z}$ increases in b_n holding $\sum_{m=1}^N \eta'_m b_m$ and

$\sum_{m=1}^{N} \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m$ constant. The derivative of $\mathcal{Z}$ with respect to b_n has the same sign as

$$\mathcal{Z}_b \equiv \frac{\left(\sum_{m=1}^{N} \eta'_m b_m\right) (\sigma^s)^2 (a_1^s)^3 (\kappa^s + r D_t^s)}{2 - (r + \kappa^s) a_1^s} \left[\bar{D}_n + a_{n1}^i (\kappa_n^i \bar{D}_n^i + r D_{nt}^i)\right] - \frac{\eta'_n (\sigma_n^i)^2 (a_{n1}^i)^3 (\kappa_n^i \bar{D}_n^i + r D_{nt}^i)}{2 - (r + \kappa_n^i) a_{n1}^i} a_1^s (\kappa^s + r D_t^s).$$

Since $\bar{D}_n \geq 0$, $\mathcal{Z}_b$ is positive if (5.1) holds.

The result in Case (ii) will follow if we show that $\mathcal{Z}$ increases in η_n holding $\sum_{m=1}^{N} \eta'_m b_m$, $\sum_{m=1}^{N} \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m$ and $\frac{u_n}{\eta_n} \equiv \hat{u}$ constant, and setting $\eta'_n = \eta_n$ if $(n, n') \in \mathcal{I} \times \mathcal{I}$ and $\eta'_n = 0$ if $(n, n') \in \mathcal{I}^c \times \mathcal{I}^c$. In the case $(n, n') \in \mathcal{I}^c \times \mathcal{I}^c$, (3.15), $\eta'_n = 0$ and $\eta_n > u_n$ imply that the denominator of $\mathcal{Z}$ decreases in η_n, and $\eta'_n = 0$ implies that the numerator is independent of η_n. Therefore, $\mathcal{Z}$ increases in η_n. In the case $(n, n') \in \mathcal{I} \times \mathcal{I}$, the derivative of $\mathcal{Z}$ with respect to η_n has the same sign as $\mathcal{Z}_{\eta 1} + \mathcal{Z}_{\eta 2}$, where

$$\mathcal{Z}_{\eta 1} \equiv \frac{\partial}{\partial \eta_n} \left[\frac{\eta_n (\sigma_n^i)^2 (a_{n1}^i)^3 (\kappa_n^i \bar{D}_n^i + r D_{nt}^i)}{2 - (r + \kappa_n^i) a_{n1}^i}\right] \left[\bar{D}_n + b_n a_1^s (\kappa^s + r D_t^s)\right] - \frac{b_n \left(\sum_{m=1}^{N} \eta'_m b_m\right) (\sigma^s)^2 (a_1^s)^3 (\kappa^s + r D_t^s)}{2 - (r + \kappa^s) a_1^s} \frac{\partial a_{n1}^i}{\partial \eta_n} (\kappa_n^i \bar{D}_n^i + r D_{nt}^i) \tag{A.6}$$

and

$$\mathcal{Z}_{\eta 2} \equiv \left[\frac{\partial}{\partial \eta_n} \left[\frac{\eta_n (\sigma_n^i)^2 (a_{n1}^i)^3 (\kappa_n^i \bar{D}_n^i + r D_{nt}^i)^2}{2 - (r + \kappa_n^i) a_{n1}^i}\right] a_{n1}^i - \frac{\eta_n (\sigma_n^i)^2 (a_{n1}^i)^3 (\kappa_n^i \bar{D}_n^i + r D_{nt}^i)^2}{2 - (r + \kappa_n^i) a_{n1}^i} \frac{\partial a_{n1}^i}{\partial \eta_n}\right]$$
$$= (a_{n1}^i)^2 \frac{\partial}{\partial \eta_n} \left[\frac{\eta_n (\sigma_n^i a_{n1}^i)^2 (\kappa_n^i \bar{D}_n^i + r D_{nt}^i)^2}{2 - (r + \kappa_n^i) a_{n1}^i}\right].$$

Equation (3.15) implies that $\mathcal{Z}_{\eta 2}$ is positive if the function

$$\Xi(z) \equiv \frac{z}{\left(A + \sqrt{B + Cz}\right) \sqrt{B + Cz}}$$

is increasing for $B + Cz \geq 0$, where (A, B) are positive constants. If $C \leq 0$, then $\Xi(z)$ is increasing because the numerator is increasing and the denominator is decreasing. If $C > 0$, then showing that $\Xi(z)$ is increasing is equivalent to showing that

$$\hat{\Xi}(y) \equiv \frac{y^2 - B}{(A+y)y}$$

is increasing for $y \equiv \sqrt{B+Cz} \geq 0$. The latter property follows because the functions $\hat{\Xi}_1(y) \equiv \frac{y}{A+y}$ and $\hat{\Xi}_2(y) \equiv -\frac{B}{(A+y)y}$ are increasing for $y \geq 0$. To show that $\mathcal{Z}_{\eta 1}$ is non-negative, we distinguish the cases $\mu_2\lambda + \hat{u} \leq 1$ and $\mu_2\lambda + \hat{u} > 1$.

In the case $\mu_2\lambda + \hat{u} \leq 1$, (3.15) and $\eta_n' = \eta_n$ imply that the partial derivative in the second line of (A.6) is non-positive. Equations (3.15) and $\eta_n' = \eta_n$ imply that the partial derivative in the first line of (A.6) is non-negative if the function

$$\Theta(z) \equiv \frac{z}{\left(A + \sqrt{B+Cz}\right)^2 \sqrt{B+Cz}}$$

is non-decreasing in z for $A \equiv r + \kappa_n^i$, $B \equiv (r + \kappa_n^i)^2$, $C \equiv 4\rho\frac{1-\mu_2\lambda-\hat{u}}{\mu_1}(\sigma_n^i)^2$ and $z = \eta_n$. The derivative $\Theta'(z)$ has the same sign as

$$\begin{aligned}
&\left(A+\sqrt{B+Cz}\right)^2\sqrt{B+Cz} - \frac{Cz}{\sqrt{B+Cz}}\left(A+\sqrt{B+Cz}\right)\sqrt{B+Cz} - \frac{Cz}{2\sqrt{B+Cz}}\left(A+\sqrt{B+Cz}\right)^2 \\
&= \frac{A+\sqrt{B+Cz}}{\sqrt{B+Cz}}\left[A\left(B+\frac{Cz}{2}\right) + \left(B - \frac{Cz}{2}\right)\sqrt{B+Cz}\right]. \qquad \text{(A.7)}
\end{aligned}$$

The sign of (A.7) is non-negative if

$$\begin{aligned}
&A^2\left(B+\frac{Cz}{2}\right)^2 \geq \left(B-\frac{Cz}{2}\right)^2(B+Cz) \\
&\Leftrightarrow B\left(B+\frac{Cz}{2}\right)^2 \geq \left(B-\frac{Cz}{2}\right)^2(B+Cz) \\
&\Leftrightarrow B^2 + BCz - \frac{(Cz)^2}{4} \geq 0 \\
&\Leftrightarrow B \geq \frac{\sqrt{2}-1}{2}Cz. \qquad \text{(A.8)}
\end{aligned}$$

Substituting for (B, C, z) and using $\hat{u} = \frac{u_n}{\eta_n}$, (A.8) becomes (5.2). Therefore, $Z_{\eta 1}$ is positive if (5.2) holds.

In the case $\mu_2\lambda + \hat{u} > 1$, (3.15) and $\eta'_n = \eta_n$ imply that a^i_{n1} increases in η_n and the partial derivative in the first line of (A.6) is positive. Since $\bar{D}_n \geq 0$, $\mathcal{Z}_{\eta 1}$ is positive if

$$\frac{\partial}{\partial \eta_n}\left[\frac{\eta_n(\sigma^i_n)^2(a^i_{n1})^3}{2-(r+\kappa^i_n)a^i_{n1}}\right] - \frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a^s_1)^2}{2-(r+\kappa^s)a^s_1}\frac{\partial a^i_{n1}}{\partial \eta_n} > 0. \tag{A.9}$$

Equation (A.9) holds under the sufficient condition

$$\begin{aligned}
&\frac{(\sigma^i_n)^2(a^i_{n1})^3}{2-(r+\kappa^i_n)a^i_{n1}} + \left[\frac{3\eta_n(\sigma^i_n a^i_{n1})^2}{2-(r+\kappa^i_n)a^i_{n1}} - \frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a^s_1)^2}{2-(r+\kappa^s)a^s_1}\right]\frac{\partial a^i_{n1}}{\partial \eta_n} \geq 0 \\
&\Leftrightarrow \frac{(\sigma^i_n)^2(a^i_{n1})^3}{2-(r+\kappa^i_n)a^i_{n1}} \\
&\quad + \left[\frac{3\eta_n(\sigma^i_n a^i_{n1})^2}{2-(r+\kappa^i_n)a^i_{n1}} - \frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a^s_1)^2}{2-(r+\kappa^s)a^s_1}\right]\frac{(\sigma^i_n)^2(a^i_{n1})^3}{2-(r+\kappa^i_n)a^i_{n1}}\frac{\rho(\mu_2\lambda+\hat{u}-1)}{\mu_1} \geq 0 \\
&\Leftrightarrow 1 + \frac{\rho(\mu_2\lambda+\hat{u}-1)}{\mu_1}\left[\frac{3\eta_n(\sigma^i_n a^i_{n1})^2}{2-(r+\kappa^i_n)a^i_{n1}} - \frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a^s_1)^2}{2-(r+\kappa^s)a^s_1}\right] \geq 0.
\end{aligned} \tag{A.10}$$

Equation (A.10) coincides with (5.3) for $\eta_{n'}$ instead of η_n. Since a^i_{n1} increases in η_n, (5.3) ensures that (A.10) holds for all $\eta \in [\eta_{n'}, \eta_n]$, which in turn ensures that $Z_{\eta 1}$ is non-negative for all $\eta \in [\eta_{n'}, \eta_n]$.

The result in Case (iii) will follow if we show that $\mathcal{Z}$ increases in u_n holding $\sum_{m=1}^N \eta'_m b_m$ and $\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1} b_m$ constant. Since $(n, n') \in \mathcal{I} \times \mathcal{I}$, the derivative of $\mathcal{Z}$ with respect to u_n has the same sign as $\mathcal{Z}_{u1} + \mathcal{Z}_{u2}$, where

$$\begin{aligned}
\mathcal{Z}_{u1} \equiv\ & \frac{\partial}{\partial u_n}\left[\frac{\eta_n(\sigma^i_n)^2(a^i_{n1})^3(\kappa^i_n\bar{D}^i_n + rD^i_{nt})}{2-(r+\kappa^i_n)a^i_{n1}}\right]\left[\bar{D}_n + b_n a^s_1(\kappa^s + rD^s_t)\right] \\
&- \frac{b_n\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s)^2(a^s_1)^3(\kappa^s + rD^s_t)}{2-(r+\kappa^s)a^s_1}\frac{\partial a^i_{n1}}{\partial u_n}(\kappa^i_n\bar{D}^i_n + rD^i_{nt})
\end{aligned} \tag{A.11}$$

and

$$\mathcal{Z}_{u2} \equiv \left[\frac{\partial}{\partial u_n}\left[\frac{\eta_n(\sigma^i_n)^2(a^i_{n1})^3(\kappa^i_n\bar{D}^i_n + rD^i_{nt})^2}{2-(r+\kappa^i_n)a^i_{n1}}\right]a^i_{n1} - \frac{\eta_n(\sigma^i_n)^2(a^i_{n1})^3(\kappa^i_n\bar{D}^i_n + rD^i_{nt})^2}{2-(r+\kappa^i_n)a^i_{n1}}\frac{\partial a^i_{n1}}{\partial u_n}\right]$$

$$= (a_{n1}^i)^2 \frac{\partial}{\partial u_n} \left[\frac{\eta_n (\sigma_n^i)^2 (a_{n1}^i)^2 (\kappa_n^i \bar{D}_n^i + r D_{nt}^i)^2}{2 - (r + \kappa_n^i) a_{n1}^i} \right].$$

Since (3.15) implies that a_{n1}^i increases in u_n, $\mathcal{Z}_{u2}$ is positive. To show that $\mathcal{Z}_{u1}$ is non-negative, we follow the same argument as when showing that $\mathcal{Z}_{\eta 1}$ is positive in the case $\mu_2 \lambda + \hat{u} > 1$. The counterpart of (A.9) is

$$\frac{\partial}{\partial u_n} \left[\frac{\eta_n (\sigma_n^i)^2 (a_{n1}^i)^3}{2 - (r + \kappa_n^i) a_{n1}^i} \right] - \frac{\left(\sum_{m=1}^N \eta_m' b_m \right) (\sigma^s a_1^s)^2}{2 - (r + \kappa^s) a_1^s} \frac{\partial a_{n1}^i}{\partial u_n} \geq 0,$$

and the counterpart of (A.10) is

$$\frac{3\eta_n (\sigma_n^i a_{n1}^i)^2}{2 - (r + \kappa_n^i) a_{n1}^i} - \frac{\left(\sum_{m=1}^N \eta_m' b_m \right) (\sigma^s a_1^s)^2}{2 - (r + \kappa^s) a_1^s} \geq 0. \tag{A.12}$$

Since a_{n1}^i increases in u_n, (5.4) ensures that (A.12) holds for all $u \in [u_{n'}, u_n]$, which in turn ensures that Z_{u1} is non-negative for all $u \in [u_{n'}, u_n]$.

We next consider the case where μ_2 increases and μ_1 decreases by a corresponding fraction $\phi \in (0, 1]$. We begin with a lemma.

Lemma A.1. *The following inequality holds*

$$\frac{\partial (\mu_2 \lambda)}{\partial \mu_2} + \phi \frac{\mu_2 \lambda + \max_m \frac{u_m}{\eta_m} - 1}{\mu_1} \equiv \Delta \geq 0 \tag{A.13}$$

and is strict when $\phi < 1$ *or* $\mathcal{I} \subsetneq \{1, .., N\}$ *or* $\#\{\frac{u_m}{\eta_m} : m \in \{1, .., N\}\} > 1$.

Proof of Lemma A.1. We proceed by contradiction, assuming that $\Delta \leq 0$ and that this inequality is strict except when $\phi < 1$ or $\mathcal{I} \subsetneq \{1, .., N\}$ or $\#\{\frac{u_m}{\eta_m} : m \in \{1, .., N\}\} > 1$. Differentiating with respect to μ_2 and using $\frac{\partial \mu_1}{\partial \mu_2} = -\phi$, we find

$$\frac{\partial}{\partial \mu_2} \left(\frac{\eta_n - \mu_2 \lambda \eta_n' - u_n}{\mu_1} \right) = -\frac{\partial (\mu_2 \lambda)}{\partial \mu_2} \frac{\eta_n'}{\mu_1} + \phi \frac{\eta_n - \mu_2 \lambda \eta_n' - u_n}{\mu_1^2}$$
$$= \begin{cases} -\left(\frac{\partial (\mu_2 \lambda)}{\partial \mu_2} + \phi \frac{\mu_2 \lambda + \frac{u_n}{\eta_n} - 1}{\mu_1} \right) \frac{\eta_n}{\mu_1} & \text{for} \quad n \in \mathcal{I}, \\ \phi \frac{\eta_n - u_n}{\mu_1^2} & \text{for} \quad n \notin \mathcal{I}. \end{cases} \tag{A.14}$$

Consider first the case where $\mathcal{I} = \{1, .., N\}$ and $\#\{\frac{u_m}{\eta_m} : m \in \{1, .., N\}\} = 1$. Equation (A.14) and our assumption on the sign of Δ imply that the derivative of $\frac{\eta_n - \mu_2 \lambda \eta_n' - u_n}{\mu_1}$ with respect to μ_2 is positive when $\phi = 1$ and is non-negative when $\phi < 1$. Since the function $\Phi(z)$ is increasing, (3.14) and (3.15) imply that the derivative with respect to μ_2 of the left-hand side of (A.5) is positive when $\phi = 1$ and is non-negative when $\phi < 1$. Likewise, since the function $\Psi(z)$ is decreasing, (3.14) and (3.15) imply that the derivative with respect to μ_2 of the term in square brackets multiplying λ in the left-hand side of (A.5) is negative when $\phi = 1$ and is non-positive when $\phi < 1$. Therefore, (A.5) implies

$$\frac{\partial \lambda}{\partial \mu_2} \geq 0 \tag{A.15}$$

and that this inequality is strict when $\phi = 1$. Equation (3.16) also implies

$$\lambda(\mu_1 + \mu_2) \geq 1 - \max_m \frac{u_m}{\eta_m}. \tag{A.16}$$

Combining (A.15) and (A.16), we find

$$\begin{aligned} \Delta = \frac{\partial(\mu_2 \lambda)}{\partial \mu_2} + \phi \frac{\mu_2 \lambda + \max_m \frac{u_m}{\eta_m} - 1}{\mu_1} &= \lambda + \mu_2 \frac{\partial \lambda}{\partial \mu_2} + \phi \frac{\mu_2 \lambda + \max_m \frac{u_m}{\eta_m} - 1}{\mu_1} \\ &= \mu_2 \frac{\partial \lambda}{\partial \mu_2} + (1 - \phi)\lambda, \end{aligned}$$

which is positive when $\phi = 1$ because of (A.15) and when $\phi < 1$ because $\lambda > 0$. This contradicts our assumption on the sign of Δ. Consider next the case where $\mathcal{I} \subsetneq \{1, .., N\}$ or $\#\{\frac{u_m}{\eta_m} : m \in \{1, .., N\}\} > 1$. Equations (A.14), $\eta_n > u_n$ and our assumption on the sign of Δ imply that the derivative of $\frac{\eta_n - \mu_2 \lambda \eta_n' - u_n}{\mu_1}$ with respect to μ_2 is non-negative for all n and positive for some n. The same argument as in the case where $\mathcal{I} = \{1, .., N\}$ and $\#\{\frac{u_m}{\eta_m} : m \in \{1, .., N\}\} = 1$ then implies that (A.15) holds as a strict inequality and this contradicts our assumption on the sign of Δ. □

Using (3.13)-(3.15) and $\frac{\partial \mu_1}{\partial \mu_2} = -\phi$, we find

$$\frac{1}{S_{nt}} \frac{\partial S_{nt}}{\partial \mu_2} = \frac{\partial(\mu_2 \lambda)}{\partial \mu_2} \frac{1}{S_{nt}} \frac{\partial S_{nt}}{\partial(\mu_2 \lambda)} + \phi \frac{1}{S_{nt}} \frac{\partial S_{nt}}{\partial \mu_1}$$

$$= \frac{\rho}{\mu_1} \frac{\frac{b_n\left(\sum_{m=1}^N \hat{\Delta}_m b_m\right)(\sigma^s)^2(a_1^s)^3(\kappa^s + rD_t^s)}{2-(r+\kappa^s)a_1^s} + \frac{\hat{\Delta}_n(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)}{2-(r+\kappa_n^i)a_{n1}^i}}{\left[\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)\right]} \equiv \frac{\rho}{\mu_1}\hat{\mathcal{Z}},$$

where

$$\hat{\Delta}_n \equiv \frac{\partial(\mu_2\lambda)}{\partial\mu_2}\eta_n' + \phi\frac{\mu_2\lambda\eta_n' + u_n - \eta_n}{\mu_1}.$$

Using the definition of $\hat{\Delta}_n$, we find

$$\begin{aligned}
\frac{\sum_{m=1}^N \hat{\Delta}_m b_m}{\sum_{m=1}^N \eta_m' b_m} &= \frac{\sum_{m=1}^N \left(\frac{\partial(\mu_2\lambda)}{\partial\mu_2}\eta_m' + \phi\frac{\mu_2\lambda\eta_m' + u_m - \eta_m}{\mu_1}\right) b_m}{\sum_{m=1}^N \eta_m' b_m} \\
&= \frac{\sum_{m\in\mathcal{I}} \left(\frac{\partial(\mu_2\lambda)}{\partial\mu_2}\eta_m + \phi\frac{\mu_2\lambda\eta_m + u_m - \eta_m}{\mu_1}\right) b_m + \sum_{m\notin\mathcal{I}}\left(\phi\frac{u_m - \eta_m}{\mu_1}\right) b_m}{\sum_{m\in\mathcal{I}} \eta_m b_m} \\
&\leq \frac{\sum_{m\in\mathcal{I}} \left(\frac{\partial(\mu_2\lambda)}{\partial\mu_2}\eta_m + \phi\frac{\mu_2\lambda\eta_m + u_m - \eta_m}{\mu_1}\right) b_m}{\sum_{m\in\mathcal{I}} \eta_m b_m} \\
&\leq \frac{\sum_{m\in\mathcal{I}} \left(\frac{\partial(\mu_2\lambda)}{\partial\mu_2}\eta_m + \phi\frac{\mu_2\lambda\eta_m + \left(\max_{m'}\frac{u_{m'}}{\eta_{m'}}\right)\eta_m - \eta_m}{\mu_1}\right) b_m}{\sum_{m\in\mathcal{I}} \eta_m b_m} \\
&= \frac{\partial(\mu_2\lambda)}{\partial\mu_2} + \phi\frac{\mu_2\lambda + \max_m\frac{u_m}{\eta_m} - 1}{\mu_1} = \Delta,
\end{aligned} \tag{A.17}$$

and for $n \in \mathcal{I} \cap \text{argmax}_m \frac{u_m}{\eta_m}$,

$$\begin{aligned}
\hat{\Delta}_n &= \frac{\partial(\mu_2\lambda)}{\partial\mu_2}\eta_n + \phi\frac{\mu_2\lambda\eta_n + u_n - \eta_n}{\mu_1} \\
&= \left(\frac{\partial(\mu_2\lambda)}{\partial\mu_2} + \phi\frac{\mu_2\lambda + \frac{u_n}{\eta_n} - 1}{\mu_1}\right)\eta_n \\
&= \left(\frac{\partial(\mu_2\lambda)}{\partial\mu_2} + \phi\frac{\mu_2\lambda + \max_m\frac{u_m}{\eta_m} - 1}{\mu_1}\right)\eta_n = \Delta\eta_n.
\end{aligned} \tag{A.18}$$

Since $\phi < 1$ or $\mathcal{I} \subsetneq \{1,..,N\}$ or $\#\{\frac{u_m}{\eta_m} : m \in \{1,..,N\}\} > 1$, Lemma A.1 implies $\Delta > 0$.

The result in Case (ii) will follow if we show that $\hat{\mathcal{Z}}$ increases in η_n holding $\sum_{m=1}^N \eta_m' b_m$, $\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta_m' - u_m}{\mu_1} b_m$ and $\frac{u_n}{\eta_n} \equiv \hat{u}$ constant, and setting $\eta_n' = \eta_n$. The derivative of $\hat{\mathcal{Z}}$ with respect to η_n has the same sign as $\hat{\mathcal{Z}}_{\eta 1} + \hat{\mathcal{Z}}_{\eta 2}$, where

$$\hat{\mathcal{Z}}_{\eta 1} \equiv \frac{\partial}{\partial \eta_n}\left[\frac{\hat{\Delta}_n(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)}{2-(r+\kappa_n^i)a_{n1}^i}\right]\left[\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s)\right] \\ - \frac{\left(\sum_{m=1}^N \hat{\Delta}_m b_m\right)(\sigma^s)^2(a_1^s)^3(\kappa^s + rD_t^s)}{2-(r+\kappa^s)a_1^s}\frac{\partial a_{n1}^i}{\partial \eta_n}(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)$$

and

$$\hat{\mathcal{Z}}_{\eta 2} \equiv \left[\frac{\partial}{\partial \eta_n}\left[\frac{\hat{\Delta}_n(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right]a_{n1}^i - \frac{\hat{\Delta}_n(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\frac{\partial a_{n1}^i}{\partial \eta_n}\right] \\ = (a_{n1}^i)^2\frac{\partial}{\partial \eta_n}\left[\frac{\hat{\Delta}_n(\sigma_n^i a_{n1}^i)^2(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right].$$

Equation (A.18) implies $\hat{\mathcal{Z}}_{\eta 2} = \Delta\mathcal{Z}_{\eta 2}$. Since $\mathcal{Z}_{\eta 2}$ is positive, so is $\hat{\mathcal{Z}}_{\eta 2}$. If $\mu_2\lambda + \frac{u_n}{\eta_n} > 1$, then (3.15) and $\eta_n' = \eta_n$ imply that a_{n1}^i increases in η_n. Combining with (A.17) and (A.18), we find $\hat{\mathcal{Z}}_{\eta 1} \geq \Delta\mathcal{Z}_{\eta 2}$. Since (5.3) ensures that $\mathcal{Z}_{\eta 1}$ is non-negative for all $\eta \in [\eta_{n'}, \eta_n]$, it also ensures that $\hat{\mathcal{Z}}_{\eta 1}$ is non-negative.

The result in Case (iii) will follow if we show that $\hat{\mathcal{Z}}$ increases in u_n holding $\sum_{m=1}^N \eta_m' b_m$, $\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta_m' - u_m}{\mu_1} b_m$ and Δ_n constant. Holding Δ_n constant is a sufficient condition because Δ_n increases in u_n. The derivative of $\hat{\mathcal{Z}}$ with respect to u_n has the same sign as $\hat{\mathcal{Z}}_{u1} + \hat{\mathcal{Z}}_{u2}$, where

$$\hat{\mathcal{Z}}_{u1} \equiv \frac{\partial}{\partial u_n}\left[\frac{\Delta_n(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)}{2-(r+\kappa_n^i)a_{n1}^i}\right]\left[\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s)\right] \\ - \frac{b_n\left(\sum_{m=1}^N \Delta_m b_m\right)(\sigma^s)^2(a_1^s)^3(\kappa^s + rD_t^s)}{2-(r+\kappa^s)a_1^s}\frac{\partial a_{n1}^i}{\partial u_n}(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)$$

and

$$\hat{\mathcal{Z}}_{u2} \equiv \left[\frac{\partial}{\partial u_n}\left[\frac{\Delta_n(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right]a_{n1}^i - \frac{\Delta_n(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\frac{\partial a_{n1}^i}{\partial u_n}\right] \\ = (a_{n1}^i)^2\frac{\partial}{\partial u_n}\left[\frac{\Delta_n(\sigma_n^i)^2(a_{n1}^i)^2(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right].$$

Equation (A.18) implies $\hat{\mathcal{Z}}_{u2} = \Delta\mathcal{Z}_{u2}$. Since $\mathcal{Z}_{u2}$ is positive, so is $\hat{\mathcal{Z}}_{u2}$. Since a_{n1}^i increases in

u_n, (A.17) and (A.18) imply $\hat{\mathcal{Z}}_{u1} \geq \Delta\mathcal{Z}_{u2}$. Since (5.3) ensures that Z_{u1} is non-negative for all $u \in [u_{n'}, u_n]$, it also ensures that $\hat{\mathcal{Z}}_{u1}$ is non-negative. □

Proof of Proposition 5.2. We first consider the case where μ_2 increases holding μ_1 constant. Equations (3.2), (3.10), (3.11) and (3.13) imply that conditional return volatility is

$$\sqrt{\frac{\mathbb{V}\text{ar}_t(dR_{nt})}{dt}} = \frac{\sqrt{b_n^2(\sigma^s a_1^s)^2 D_t^s + (\sigma_n^i a_{n1}^i)^2 D_{nt}^i}}{\frac{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa_n^i \bar{D}_n^i + rD_{nt}^i)}{r}}. \tag{A.19}$$

The change in volatility has the same sign as the change in variance (the square of (A.19)), which has the same sign as

$$\begin{aligned}
&\left[b_n^2(\sigma^s)^2 2a_1^s \frac{\partial a_1^s}{\partial(\mu_2\lambda)} D_t^s + (\sigma_n^i)^2 2a_{n1}^i \frac{\partial a_{n1}^i}{\partial(\mu_2\lambda)} D_{nt}^i\right]\left[\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa_n^i \bar{D}_n^i + rD_{nt}^i)\right] \\
&- 2\left[b_n^2(\sigma^s a_1^s)^2 D_t^s + (\sigma_n^i a_{n1}^i)^2 D_{nt}^i\right]\left[b_n \frac{\partial a_1^s}{\partial(\mu_2\lambda)}(\kappa^s + rD_t^s) + \frac{\partial a_{n1}^i}{\partial(\mu_2\lambda)}(\kappa_n^i \bar{D}_n^i + rD_{nt}^i)\right] \\
&= 2\left[b_n^2(\sigma^s)^2 a_1^s \frac{\partial a_1^s}{\partial(\mu_2\lambda)} D_t^s + (\sigma_n^i)^2 a_{n1}^i \frac{\partial a_{n1}^i}{\partial(\mu_2\lambda)} D_{nt}^i\right]\bar{D}_n \\
&\quad + 2\left[b_n^2(\sigma^s)^2 a_1^s D_t^s(\kappa_n^i \bar{D}_n^i + rD_{nt}^i) - (\sigma_n^i)^2 a_{n1}^i D_{nt}^i(\kappa^s + rD_t^s)\right] b_n\left[a_{n1}^i \frac{\partial a_s^i}{\partial(\mu_2\lambda)} - a_1^s \frac{\partial a_{n1}^i}{\partial(\mu_2\lambda)}\right].
\end{aligned} \tag{A.20}$$

Using (3.14) and (3.15), we find that (A.20) has the same sign as

$$\begin{aligned}
&\left[\frac{b_n^2\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^4 D_t^s}{2-(r+\kappa^s)a_1^s} + \frac{\eta_n'(\sigma_n^i a_{n1}^i)^4 D_{nt}^i}{2-(r+\kappa_n^i)a_{n1}^i}\right]\bar{D}_n \\
&+ \left[b_n(\sigma^s)^2 a_1^s D_t^s(\kappa_n^i \bar{D}_n^i + rD_{nt}^i) - (\sigma_n^i)^2 a_{n1}^i D_{nt}^i(\kappa^s + rD_t^s)\right] b_n a_1^s a_{1n}^i \\
&\times \left[\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n'(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right],
\end{aligned}$$

which simplifies to

$$\left[\frac{b_n^2\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^4}{2-(r+\kappa^s)a_1^s} + \frac{\eta_n'(\sigma_n^i a_{n1}^i)^4 \bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i}\right]\bar{D}_n$$

$$
\begin{aligned}
&+ \left[(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s - (r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i\right] b_n a_1^s a_{1n}^i \bar{D}_n^i \\
&\times \left[\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n'(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right] \equiv \mathcal{V}
\end{aligned} \tag{A.21}
$$

for $D_t^s = \bar{D}^s = 1$ and $D_{nt}^i = \bar{D}_n^i$.

Consider first Case (i). When $\eta_{n'} \approx 0$, $\frac{u_n}{\eta_n} = \frac{u_{n'}}{\eta_{n'}}$ implies $u_{n'} \approx 0$, and (3.15) implies $a_{n'1}^i = \frac{1}{r+\kappa_n^i}$. Equation (A.21) implies that volatility declines for stock n' if

$$
\begin{aligned}
&\frac{b_n^2\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^4}{2-(r+\kappa^s)a_1^s}\bar{D}_n \\
&+ \left[(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s - \frac{(r+\kappa^s)(\sigma_n^i)^2}{r+\kappa_n^i}\right]\frac{b_n a_1^s \bar{D}_n^i}{r+\kappa_n^i}\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} < 0 \\
&\Leftrightarrow \bar{D}_n < \left(\frac{(r+\kappa^s)(\sigma_n^i)^2}{(r+\kappa_n^i)^2 b_n(\sigma^s)^2 a_1^s} - 1\right)\bar{D}_n^i \equiv \bar{D}_{n,\max}.
\end{aligned} \tag{A.22}
$$

Conversely, (A.21) implies that volatility rises for stock $n \in \mathcal{I}$ if

$$
\bar{D}_n > \frac{\left(\frac{(r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i}{(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s} - 1\right)(r+\kappa_n^i)(b_n\sigma^s a_1^s)^2 a_{1n}^i \bar{D}_n^i\left[\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right]}{\frac{b_n^2\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^4}{2-(r+\kappa^s)a_1^s} + \frac{\eta_n(\sigma_n^i a_{n1}^i)^4 \bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i}} \equiv \bar{D}_{n,\min} \tag{A.23}
$$

Equations (A.22) and (A.23) imply that volatility rises for stock n and declines for stock n' if $\bar{D}_n \in (\min\{\bar{D}_{n,\min}, 0\}, \bar{D}_{n,\max})$. If instead $\bar{D}_n > \bar{D}_{n,\max}$ then volatility rises for both stocks, and if $\bar{D}_n \in [0, \min\{\bar{D}_{n,\min}, 0\})$ then volatility declines for both stocks. Since (5.4) for $n' = n$ implies

$$
\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} \le \frac{2}{3}\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} \tag{A.24}
$$

and

$$
\frac{\eta_n(\sigma_n^i a_{n1}^i)^4 \bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i} \ge \frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s \sigma_n^i a_1^s a_{n1}^i)^2 \bar{D}_n^i}{3\left[2-(r+\kappa^s)a_1^s\right]}, \tag{A.25}
$$

the lower bound $\bar{D}_{n,\min}$ satisfies

$$\bar{D}_{n,\min} \leq \frac{2\left(\frac{(r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i}{(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s} - 1\right)(r+\kappa_n^i)a_{1n}^i \bar{D}_n^i}{3 + \frac{(\sigma_n^i a_{n1}^i)^2 \bar{D}_n^i}{(b_n \sigma^s a_1^s)^2}}, \tag{A.26}$$

and the interval $(\min\{\bar{D}_{n,\min}, 0\}, \bar{D}_{n,\max})$ is non-empty if (5.5) holds.

Consider next Case (ii). We begin by showing that if $\bar{D}_n \geq \frac{1}{2}(r+\kappa^s)b_n a_1^s$ (and the other sufficient conditions in the proposition are met) then $\mathcal{V}$ is larger for stock n than for stock n', or equivalently $\mathcal{V}$ increases in u_n holding $\sum_{m=1}^N \eta'_m b_m$ and $\sum_{m=1}^N \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m$ constant. Since $(n, n') \in \mathcal{I} \times \mathcal{I}$ and a_{n1}^i increases in u_n, the derivative of $\mathcal{V}$ with respect to u_n has the same sign as $\mathcal{V}_{u1} + \mathcal{V}_{u2} + \mathcal{V}_{u3} + \mathcal{V}_{u4}$, where

$$\begin{aligned}
\mathcal{V}_{u1} &\equiv \frac{\partial}{\partial a_{n1}^i}\left[\frac{\eta_n(\sigma_n^i a_{n1}^i)^4 \bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i}\right]\bar{D}_n, \\
\mathcal{V}_{u2} &\equiv -(r+\kappa^s)(\sigma_n^i)^2 b_n a_1^s a_{1n}^i \bar{D}_n^i \left[\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right], \\
\mathcal{V}_{u3} &\equiv \left[(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s - (r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i\right] b_n a_1^s \bar{D}_n^i \left[\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right], \\
\mathcal{V}_{u4} &\equiv -\left[(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s - (r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i\right] b_n a_1^s a_{1n}^i \bar{D}_n^i \frac{\partial}{\partial a_{n1}^i}\left[\frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right].
\end{aligned}$$

Since a_{n1}^i increases in u_n, (5.6) implies that the term in the first bracket in $\mathcal{V}_{u3}$ and $\mathcal{V}_{u4}$ is negative for all $u \in [u_{n'}, u_n]$. Therefore, $\mathcal{V}_{u3} + \mathcal{V}_{u4}$ is positive if

$$a_{n1}^i \frac{\partial}{\partial a_{n1}^i}\left[\frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right] - \left[\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right] > 0. \tag{A.27}$$

Equation (A.27) holds under the sufficient condition

$$\begin{aligned}
&\frac{2\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} - \left[\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right] \geq 0 \\
\Leftrightarrow\ &\frac{3\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} \geq \frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s},
\end{aligned}$$

which holds for all $u \in [u_{n'}, u_n]$ if (5.4) holds. The sum $\mathcal{V}_{u1} + \mathcal{V}_{u2}$ is positive under the sufficient condition

$$\frac{4\eta_n(\sigma_n^i)^4(a_{n1}^i)^3\bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i}\bar{D}_n - (r+\kappa^s)(\sigma_n^i)^2 b_n a_1^s a_{1n}^i \bar{D}_n^i \left[\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right] \geq 0. \tag{A.28}$$

If (5.4) holds, then (A.28) holds for all $u \in [u_{n'}, u_n]$ if

$$\frac{4\eta_n(\sigma_n^i)^4(a_{n1}^i)^3\bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i}\bar{D}_n - 2(r+\kappa^s)(\sigma_n^i)^2 b_n a_1^s a_{1n}^i \bar{D}_n^i \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} \geq 0$$
$$\Leftrightarrow \bar{D}_n \geq \frac{1}{2}(r+\kappa^s)b_n a_1^s.$$

We next determine whether the volatilities of stocks n and n' rise or decline. Equation (A.21) implies that stock n's volatility rises if $\bar{D}_n > \bar{D}_{n,\min}$. If $\bar{D}_{n,\min} > \frac{1}{2}(r+\kappa^s)b_n a_1^s$, then $\mathcal{V}$ for stock n' and $\bar{D}_n = \bar{D}_{n,\min}$ is negative. This is because $\mathcal{V}$ for stock n and $\bar{D}_n = \bar{D}_{n,\min}$ is zero, and $\mathcal{V}$ is larger for stock n than for stock n' if $\bar{D}_n > \frac{1}{2}(r+\kappa^s)b_n a_1^s$. Therefore, the threshold $\bar{D}_{n,\max}$ below which volatility declines for stock n' exceeds $\bar{D}_{n,\min}$. Since (5.6) implies

$$\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} \geq \left(1-\frac{1}{\psi}\right)\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s}$$

and

$$\frac{\eta_n(\sigma_n^i a_{n1}^i)^4\bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i} \leq \frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s \sigma_n^i a_1^s a_{n1}^i)^2\bar{D}_n^i}{\psi\left[2-(r+\kappa^s)a_1^s\right]},$$

the lower bound $\bar{D}_{n,\min}$ satisfies

$$\bar{D}_{n,\min} \geq \frac{(\psi-1)\left(\frac{(r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i}{(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s} - 1\right)(r+\kappa_n^i)a_{1n}^i\bar{D}_n^i}{\psi + \frac{(\sigma_n^i a_{n1}^i)^2\bar{D}_n^i}{(b_n\sigma^s a_1^s)^2}}. \tag{A.29}$$

Since a_{n1}^i increases in u_n, $\bar{D}_{n,\min} > \frac{1}{2}(r+\kappa^s)b_n a_1^s$ holds if (5.7) holds. Therefore, if (5.7) holds, then volatility rises for stock n and declines for stock n' if $\bar{D}_n \in (\bar{D}_{n,\min}, \bar{D}_{n,\max})$. If instead $\bar{D}_n > \bar{D}_{n,\max}$

then volatility rises for both stocks, and if $\bar{D}_n \in [0, \bar{D}_{n,\min})$ then volatility declines for both stocks.

We next consider the case where μ_2 increases and μ_1 decreases by a corresponding fraction $\phi \in (0,1]$. Proceeding as in the case where μ_1 is held constant, and using the same notation as in the corresponding part of the proof of Proposition 5.1, we find that the change in volatility has the same sign as

$$\left[\frac{b_n^2\left(\sum_{m=1}^N \hat{\Delta}_m b_m\right)(\sigma^s a_1^s)^4}{2-(r+\kappa^s)a_1^s} + \frac{\hat{\Delta}_n(\sigma_n^i a_{n1}^i)^4 \bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i}\right]\bar{D}_n$$
$$+\left[(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s - (r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i\right] b_n a_1^s a_{1n}^i \bar{D}_n^i$$
$$\times\left[\frac{\left(\sum_{m=1}^N \hat{\Delta}_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\hat{\Delta}_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right] \equiv \hat{\mathcal{V}} \tag{A.30}$$

for $D_t^s = \bar{D}^s = 1$ and $D_{nt}^i = \bar{D}_n^i$. The result in Case (i) follows by proceeding as in the case where μ_1 is held constant. Equation (A.22) remains the same. Equation (A.23) remains the same except that $\sum_{m=1}^N \eta'_m b_m$ is replaced by $\sum_{m=1}^N \hat{\Delta}_m b_m$ and η_n is replaced by $\hat{\Delta}_n$. Equations (A.24) and (A.25) remain the same with the same replacements, as can be seen by combining (5.4) for $n' = n$, (A.17), (A.18) and $n \in \mathcal{I} \cap \operatorname{argmax}_m \frac{u_m}{\eta_m}$. Equation (A.26) remains the same. □

Proof of Proposition 5.3. Suppose that stock n is added to the index. Since a_1^s does not change, (3.13) implies

$$\frac{S_{nt}^{\text{post}} - S_{nt}^{\text{pre}}}{S_{nt}^{\text{pre}}} = \frac{\left(a_{n1}^{i,\text{post}} - a_{n1}^{i,\text{pre}}\right)(k_n^i \bar{D}_n^i + rD_{nt}^i)}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^{i,\text{pre}}(k_n^i \bar{D}_n^i + rD_{nt}^i)}, \tag{A.31}$$

where $(S_{nt}^{\text{pre}}, a_{n1}^{i,\text{pre}})$ denote the values of (S_{nt}, a_{n1}^i) before index addition and $(S_{nt}^{\text{post}}, a_{n1}^{i,\text{post}})$ denote the values after addition. The value $a_{n1}^{i,pre}$ is obtained from (3.15) by setting $\mu_2\lambda = 0$, and the value $a_{n1}^{i,\text{post}}$ is obtained for $\mu_2\lambda$. Treating a_{n1}^i as a function of x ranging from zero to $\mu_2\lambda$, we can write (A.31) as

$$\frac{S_{nt}^{\text{post}} - S_{nt}^{\text{pre}}}{S_{nt}^{\text{pre}}} = \frac{\left(\int_0^{\mu_2\lambda} \frac{\partial a_{n1}^i(x)}{\partial x} dx\right)(k_n^i \bar{D}_n^i + rD_{nt}^i)}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(0)(k_n^i \bar{D}_n^i + rD_{nt}^i)}$$

$$= \frac{\left(\int_0^{\mu_2\lambda} \frac{\eta_n(\sigma_n^i)^2(a_{n1}^i(x))^3}{2-(r+\kappa_n^i)a_{n1}^i(x)} dx\right)(k_n^i\bar{D}_n^i + rD_{nt}^i)}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(0)(k_n^i\bar{D}_n^i + rD_{nt}^i)},$$

where the second step follows from (3.15).

The result in Case (i) will follow if we show that

$$\frac{\frac{\eta_n(\sigma_n^i)^2(a_{n1}^i(x))^3}{2-(r+\kappa_n^i)a_{n1}^i(x)}}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(0)(k_n^i\bar{D}_n^i + rD_{nt}^i)} \equiv \mathcal{Y}$$

increases in η_n holding $\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta_m' - u_m}{\mu_1} b_m$ and $\frac{u_n}{\eta_n}$ constant. Since $\eta_n > u_n$, (3.15) implies that $a_{n1}^i(0)$ decreases in η_n and so does the denominator of $\mathcal{Y}$. The numerator of $\mathcal{Y}$ is non-decreasing in η_n under the same condition that is needed for the partial derivative in the first line of (A.6) to be non-negative. That condition is (5.2), which for general x becomes

$$(r+\kappa_n^i)^2 \geq 2(\sqrt{2}-1)\rho\frac{\eta_n(1-x)-u_n}{\mu_1}(\sigma_n^i)^2. \tag{A.32}$$

Equation (5.8) ensures that (A.32) holds for all $x \in [0, \mu_2\lambda]$ and thus $\mathcal{Y}$ increases in η_n.

The result in Case (ii) will follow if we show that $\mathcal{Y}$ increases in u_n holding $\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta_m' - u_m}{\mu_1} b_m$ constant. Since (3.15) implies that $a_{n1}^i(0)$ increases in u_n,

$$\frac{a_{n1}^i(0)}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(0)(k_n^i\bar{D}_n^i + rD_{nt}^i)}$$

increases in u_n. Therefore, to show that $\mathcal{Y}$ increases in u_n, it suffices to show that

$$\frac{(a_{n1}^i(x))^3}{a_{n1}^i(0)[2-(r+\kappa_n^i)a_{n1}^i(x)]} = (a_{n1}^i(x))^2 \frac{r+\kappa_n^i + \sqrt{(r+\kappa_n^i)^2 + 4\rho\frac{\eta_n - u_n}{\mu_1}(\sigma_n^i)^2}}{\sqrt{(r+\kappa_n^i)^2 + 4\rho\frac{\eta_n - x\eta_n - u_n}{\mu_1}(\sigma_n^i)^2}}$$

increases in u_n. That property follows from $a_{n1}^i(x)$ and

$$\frac{(r+\kappa_n^i)^2 + 4\rho\frac{\eta_n - u_n}{\mu_1}(\sigma_n^i)^2}{(r+\kappa_n^i)^2 + 4\rho\frac{\eta_n - x\eta_n - u_n}{\mu_1}(\sigma_n^i)^2}$$

being increasing in u_n for $x \in [0, \mu_2\lambda]$. □

B Return Moments

To compute conditional expected return, we divide the right-hand side of (3.12) by S_{nt}. Using (3.13), and dropping the subscript n from $(\kappa^i_n, \bar{D}^i_n, \sigma^i_n)$, we find

$$\frac{\mathbb{E}_t(dR_{nt})}{dt} = \rho r \frac{\left[b_n \left(\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1} b_m\right)(\sigma^s a^s_1)^2 D^s_t + \frac{\eta_n - \mu_2\lambda\eta'_n - u_n}{\mu_1}(\sigma^i_n a^i_{n1})^2 D^i_{nt}\right]}{\bar{D}_n + b_n a^s_1(\kappa^s + r D^s_t) + a^i_{n1}(\kappa^i \bar{D}^i + r D^i_{nt})}. \tag{B.1}$$

Unconditional expected return is the expectation of (B.1)

$$\frac{\mathbb{E}(dR_{nt})}{dt} = \rho r \mathbb{E}\left\{\frac{\left[b_n \left(\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1} b_m\right)(\sigma^s a^s_1)^2 D^s_t + \frac{\eta_n - \mu_2\lambda\eta'_n - u_n}{\mu_1}(\sigma^i a^i_{n1})^2 D^i_{nt}\right]}{\bar{D}_n + b_n a^s_1(\kappa^s + r D^s_t) + a^i_{n1}(\kappa^i \bar{D}^i + r D^i_{nt})}\right\}. \tag{B.2}$$

When the stationary distribution of (D^s_t, D^i_{nt}) is gamma, the expectation in (B.2) becomes

$$\begin{aligned}&\int_{D^i_{nt}=0}^{\infty}\int_{D^s_t=0}^{\infty} \frac{\left[b_n \left(\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1} b_m\right)(\sigma^s a^s_1)^2 D^s_t + \frac{\eta_n - \mu_2\lambda\eta'_n - u_n}{\mu_1}(\sigma^i a^i_{n1})^2 D^i_{nt}\right]}{\bar{D}_n + b_n a^s_1(\kappa^s + r D^s_t) + a^i_{n1}(\kappa^i \bar{D}^i + r D^i_{nt})} \\ &\times \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)}(D^s_t)^{\alpha^s-1} e^{-\beta^s D^s_t} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)}(D^i_{nt})^{\alpha^i-1} e^{-\beta^i D^i_{nt}} dD^s_t dD^i_{nt}.\end{aligned} \tag{B.3}$$

Because the functions $(D^s_t)^{\alpha^s-1}$ and $(D^i_{nt})^{\alpha^i-1}$ go to ∞ when D^s_t and D^i_{nt}, respectively, go to zero, the numerical calculation of the double integral in (B.3) becomes slow and inaccurate if the lower bounds are close to zero. We instead use a fast and accurate method by writing the double integral as a sum of four terms. We fix a small $\epsilon > 0$ and a large M. The integration domain for the first term is $(D^s_t, D^i_{nt}) \in [\epsilon, M] \times [\epsilon, M\bar{D}^i]$, and we compute that term using Matlab's double integration routine. The integration domain for the second term is $(D^s_t, D^i_{nt}) \in [0, \epsilon] \times [\epsilon, M\bar{D}^i]$, and we compute that term as

$$\begin{aligned}&\int_{D^i_{nt}=\epsilon}^{M\bar{D}^i}\int_{D^s_t=0}^{\epsilon} \frac{b_n \left(\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1} b_m\right)(\sigma^s a^s_1)^2 D^s_t}{\bar{D}_n + b_n a^s_1 \kappa^s + a^i_{n1}(\kappa^i \bar{D}^i + r D^i_{nt})} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)}(D^s_t)^{\alpha^s-1} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)}(D^i_{nt})^{\alpha^i-1} e^{-\beta^i D^i_{nt}} dD^s_t dD^i_{nt} \\ &+ \int_{D^i_{nt}=\epsilon}^{M\bar{D}^i}\int_{D^s_t=0}^{\epsilon} \frac{\frac{\eta_n - \mu_2\lambda\eta'_n - u_n}{\mu_1}(\sigma^i a^i_{n1})^2 D^i_{nt}}{\bar{D}_n + b_n a^s_1 \kappa^s + a^i_{n1}(\kappa^i \bar{D}^i + r D^i_{nt})} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)}(D^s_t)^{\alpha^s-1} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)}(D^i_{nt})^{\alpha^i-1} e^{-\beta^i D^i_{nt}} dD^s_t dD^i_{nt} \\ &= \int_{D^i_{nt}=\epsilon}^{M\bar{D}^i} \frac{b_n \left(\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1} b_m\right)(\sigma^s a^s_1)^2}{\bar{D}_n + b_n a^s_1 \kappa^s + a^i_{n1}(\kappa^i \bar{D}^i + r D^i_{nt})} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} \frac{\epsilon^{\alpha^s+1}}{\alpha^s+1} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)}(D^i_{nt})^{\alpha^i-1} e^{-\beta^i D^i_{nt}} dD^i_{nt}\end{aligned}$$

$$+ \int_{D^i_{nt}=\epsilon}^{M\bar{D}^i} \frac{\frac{\eta_n - \mu_2 \lambda \eta'_n - u_n}{\mu_1} (\sigma^i a^i_{n1})^2 D^i_{nt}}{\bar{D}_n + b_n a^s_1 \kappa^s + a^i_{n1}(\kappa^i \bar{D}^i + r D^i_{nt})} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} \frac{\epsilon^{\alpha^s}}{\alpha^s} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} (D^i_{nt})^{\alpha^i - 1} e^{-\beta^i D^i_{nt}} dD^i_{nt}.$$

Thus, we approximate $\kappa^s + rD^s_t$ by κ^s and $e^{-\beta^s D^s_t}$ by one, then compute the exact integrals of $(D^s_t)^{\alpha^s}$ and $(D^s_t)^{\alpha^s-1}$ over $[0, \epsilon]$, and then use Matlab's integration routine to integrate with respect to D^i_{nt} over $[\epsilon, M\bar{D}_n]$. The integration domain for the third term is $(D^s_t, D^i_{nt}) \in [\epsilon, M] \times [0, \epsilon]$, and we compute that term as

$$\begin{aligned}
&\int_{D^i_{nt}=0}^{\epsilon} \int_{D^s_t=\epsilon}^{M} \frac{b_n \left(\sum_{m=1}^N \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m \right) (\sigma^s a^s_1)^2 D^s_t}{\bar{D}_n + b_n a^s_1 (\kappa^s + r D^s_t) + a^i_{n1} \kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} (D^s_t)^{\alpha^s - 1} e^{-\beta^s D^s_t} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} (D^i_{nt})^{\alpha^i - 1} dD^s_t dD^i_{nt} \\
&+ \int_{D^i_{nt}=0}^{\epsilon} \int_{D^s_t=\epsilon}^{M} \frac{\frac{\eta_n - \mu_2 \lambda \eta'_n - u_n}{\mu_1} (\sigma^i a^i_{n1})^2 D^i_{nt}}{\bar{D}_n + b_n a^s_1 (\kappa^s + r D^s_t) + a^i_{n1} \kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} (D^s_t)^{\alpha^s - 1} e^{-\beta^s D^s_t} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} (D^i_{nt})^{\alpha^i - 1} dD^s_t dD^i_{nt} \\
&= \int_{D^s_t=\epsilon}^{M} \frac{b_n \left(\sum_{m=1}^N \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m \right) (\sigma^s a^s_1)^2 D^s_t}{\bar{D}_n + b_n a^s_1 (\kappa^s + r D^s_t) + a^i_{n1} \kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} (D^s_t)^{\alpha^s - 1} e^{-\beta^s D^s_t} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} \frac{\epsilon^{\alpha^i}}{\alpha^i} dD^i_{nt} \\
&\quad + \int_{D^s_t=\epsilon}^{M} \frac{\frac{\eta_n - \mu_2 \lambda \eta'_n - u_n}{\mu_1} (\sigma^i a^i_{n1})^2}{\bar{D}_n + b_n a^s_1 (\kappa^s + r D^s_t) + a^i_{n1} \kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} (D^s_t)^{\alpha^s - 1} e^{-\beta^s D^s_t} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} \frac{\epsilon^{\alpha^i + 1}}{\alpha^i + 1} dD^i_{nt}.
\end{aligned}$$

Thus, we approximate $\kappa^i \bar{D}^i + rD^i_{nt}$ by $\kappa^i \bar{D}^i$ and $e^{-\beta^i D^i_{nt}}$ by one, then compute the exact integrals of $(D^i_{nt})^{\alpha^i}$ and $(D^i_{nt})^{\alpha^i - 1}$ over $[0, \epsilon]$, and then use Matlab's integration routine to integrate with respect to D^s_t over $[\epsilon, M]$. The integration domain for the fourth term is $(D^s_t, D^i_{nt}) \in [0, \epsilon] \times [0, \epsilon]$, and we compute that term as

$$\begin{aligned}
&\int_{D^i_{nt}=0}^{\epsilon} \int_{D^s_t=0}^{\epsilon} \frac{b_n \left(\sum_{m=1}^N \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m \right) (\sigma^s a^s_1)^2 D^s_t}{\bar{D}_n + b_n a^s_1 \kappa^s + a^i_{n1} \kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} (D^s_t)^{\alpha^s - 1} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} (D^i_{nt})^{\alpha^i - 1} dD^s_t dD^i_{nt} \\
&+ \int_{D^i_{nt}=0}^{\epsilon} \int_{D^s_t=0}^{\epsilon} \frac{\frac{\eta_n - \mu_2 \lambda \eta'_n - u_n}{\mu_1} (\sigma^i a^i_{n1})^2 D^i_{nt}}{\bar{D}_n + b_n a^s_1 \kappa^s + a^i_{n1} \kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} (D^s_t)^{\alpha^s - 1} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} (D^i_{nt})^{\alpha^i - 1} dD^s_t dD^i_{nt} \\
&= \frac{b_n \left(\sum_{m=1}^N \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m \right) (\sigma^s a^s_1)^2}{\bar{D}_n + b_n a^s_1 \kappa^s + a^i_{n1} \kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} \frac{\epsilon^{\alpha^s + 1}}{\alpha^s + 1} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} \frac{\epsilon^{\alpha^i}}{\alpha^i} dD^i_{nt} \\
&\quad + \frac{\frac{\eta_n - \mu_2 \lambda \eta'_n - u_n}{\mu_1} (\sigma^i a^i_{n1})^2}{\bar{D}_n + b_n a^s_1 \kappa^s + a^i_{n1} \kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} \frac{\epsilon^{\alpha^s}}{\alpha^s} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} \frac{\epsilon^{\alpha^i + 1}}{\alpha^i + 1} dD^i_{nt}.
\end{aligned}$$

Thus, we approximate $\kappa^s + rD^s_t$ by κ^s, $\kappa^i \bar{D}^i + rD^i_{nt}$ by $\kappa^i \bar{D}^i$, and $e^{-\beta^s D^s_t}$ and $e^{-\beta^i D^i_{nt}}$ by one, and then compute the exact integrals of $(D^s_t)^{\alpha^s}$, $(D^s_t)^{\alpha^s - 1}$, $(D^i_{nt})^{\alpha^i}$ and $(D^i_{nt})^{\alpha^i - 1}$ over $[0, \epsilon]$. The sum of the four terms is independent of ϵ for ϵ ranging from 0.00001 to 0.01. For larger values of ϵ the approximations become inaccurate, and for smaller values of ϵ the Matlab integration routines become inaccurate.

Conditional return variance is the square of (A.19). Unconditional return variance is the expectation of conditional variance

$$\frac{\mathbb{V}\mathrm{ar}(dR_{nt})}{dt} = r^2\mathbb{E}\left\{\frac{b_n^2(\sigma^s a_1^s)^2 D_t^s + (\sigma^i a_{n1}^i)^2 D_{nt}^i}{[\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa^i \bar{D}^i + rD_{nt}^i)]^2}\right\}, \tag{B.4}$$

because infinitesimal dR_{nt} implies that $\mathbb{E}(dR_{nt}^2)$ and $\mathbb{E}_t(dR_{nt}^2)$ are equal to $\mathbb{V}\mathrm{ar}(dR_{nt})$ and $\mathbb{V}\mathrm{ar}_t(dR_{nt})$, respectively, plus smaller-order terms. We calculate the expectation in (B.4) by writing the double integral as a sum of four terms, as in the case of expected return.

Conditional CAPM beta is

$$\beta_{nt}^{\mathrm{CAPM}} = \frac{\frac{\mathbb{C}\mathrm{ov}(dR_{nt}, dR_{Mt})}{dt}}{\frac{\mathbb{V}\mathrm{ar}(dR_{Mt})}{dt}}, \tag{B.5}$$

where dR_{Mt} denotes the return of the index. The numerator of (B.5) is

$$\frac{\mathbb{C}\mathrm{ov}(dR_{nt}, dR_{Mt})}{dt} =$$
$$r^2\mathbb{E}\left\{\frac{b_n(\sum_{m=1}^N \eta'_m b_m)(\sigma^s a_1^s)^2 D_t^s + \eta'_n(\sigma^i a_{n1}^i)^2 D_{nt}^i}{[\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa^i \bar{D}^i + rD_{nt}^i)][\sum_{m=1}^N \eta'_m[\bar{D}_m + b_m a_1^s(\kappa^s + rD_t^s) + a_{m1}^i(\kappa^i \bar{D}^i + rD_{mt}^i)]}\right\}. \tag{B.6}$$

Computing the expectation in (B.6) requires integrating over $(D_t^s, \{D_{mt}^i\}_{m=1,..,N})$, i.e., $N+1$ random variables. To keep the integration manageable, we replace $\{D_{mt}^i\}_{m\neq n}$ by their expectations $\bar{D}^i$, thus applying the law of large numbers. We then calculate the expectation over (D_t^s, D_{nt}^i) by writing the double integral as a sum of four terms, as in the case of expected return. The denominator of (B.5) is

$$\frac{\mathbb{V}\mathrm{ar}(dR_{Mt})}{dt} = r^2\mathbb{E}\left\{\frac{(\sum_{m=1}^N \eta'_m b_m)^2(\sigma^s a_1^s)^2 D_t^s + \sum_{m=1}^N (\eta'_m)^2(\sigma^i a_{m1}^i)^2 D_{mt}^i}{[\sum_{m=1}^N \eta'_m[\bar{D}_m + b_m a_1^s(\kappa^s + rD_t^s) + a_{m1}^i(\kappa^i \bar{D}^i + rD_{mt}^i)]^2}\right\}. \tag{B.7}$$

We replace $\{D_{mt}^i\}_{m=1,..,N}$ by their expectations $\bar{D}^i$, and calculate the expectation over D_t^s by writing the integral as a sum of two terms, with integration domains $[0, \epsilon]$ and $[\epsilon, M]$. We do not distinguish between stock n and stocks $m \neq n$ because all stocks are symmetric in (B.7).

CAPM R-squared is

$$R^{2,\text{CAPM}} = \frac{\left[\frac{\mathbb{C}\text{ov}(dR_{nt}, dR_{Mt})}{dt}\right]^2}{\frac{\mathbb{V}\text{ar}(dR_{nt})}{dt}\frac{\mathbb{V}\text{ar}(dR_{Mt})}{dt}} = \left(\beta_{nt}^{\text{CAPM}}\right)^2 \frac{\frac{\mathbb{V}\text{ar}(dR_{Mt})}{dt}}{\frac{\mathbb{V}\text{ar}(dR_{nt})}{dt}},$$

and can be computed from the previous moments.