The Signaling Role of Leaders in Global Games

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Motivation

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- In many of such scenarios, there can exist ”leaders” whose visibility gives them a special role.
- Examples: large investors like Soros, vanguards in revolutions.
- We study the signaling role of leaders in global games and its effect on equilibrium and rationalizable behavior.
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This Paper

- A leader and a set of followers face a coordination problem and have binary, irreversible actions.
- The leader moves first and her action is observable by the followers – signaling by the leader.
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- The leader moves first and her action is observable by the followers
  - **signaling** by the leader
  - **learning** by the followers.
A leader and a set of followers face a coordination problem and have binary, irreversible actions.

The leader moves first and her action is observable by the followers
– **signaling** by the leader
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**Result:** Conditions that guarantee the uniqueness of rationalizable (and hence, equilibrium) play.
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The leader moves first and her action is observable by the followers
- **signaling** by the leader
- **learning** by the followers.

**Result:** Conditions that guarantee the uniqueness of rationalizable (and hence, equilibrium) play.

**Result:** (In)Efficiency of outcomes.
Related Literature


- **Signaling and Policy Traps**: Angeletos, Hellwig and Pavan (2006).


- **Applications**: Corsetti et al. (2004), Bueno de Mesquita (2010), Shadmehr and Bernhardt (2019).
The Model - Complete Information

- A leader and $n$ followers have to decide whether to invest in a project or not.
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- Payoffs: $u_i(\theta, A) = \theta + A - 1$ where $\theta$ is the state of nature and $A$ is the fraction of players that invest.
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- Leader moves first and then followers play a simultaneous move game.
- Two-stage game of strategic complementarities across and within stages.
The Game

\[
\begin{array}{c|c|c|c|c}
2 & I & N & 3 & 1 \\
\hline
N & \theta - \frac{1}{3}, 0, \theta - \frac{1}{3} & \theta - \frac{2}{3}, 0, 0 & & \\
\hline
I & \theta, \theta, \theta & \theta - \frac{1}{3}, \theta - \frac{1}{3}, 0 & & \\
\hline
N & 0, \theta - \frac{1}{3}, \theta - \frac{1}{3} & 0, \theta - \frac{2}{3}, 0 & & \\
\hline
I & 0, 0, \theta - \frac{2}{3} & 0, 0, 0 & & \\
\hline
N & 0, 0, \theta - \frac{2}{3} & 0, 0, 0 & & \\
\end{array}
\]
Equilibrium

Under complete information it is easy to verify that:

- If \( \theta < 0 \) then the game has a unique SPE given by \((\mathcal{N}, \mathcal{N}, \mathcal{N}, \mathcal{N})\),

- If \( \theta > \frac{2}{3} \) then the game has a unique SPE given by \((\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I})\).

- If \( \theta \in \left(\frac{1}{3}, \frac{2}{3}\right) \) then there are two outcome equivalent SPNE, namely \((\mathcal{I}, \mathcal{I}, \mathcal{N}, \mathcal{I})\) and \((\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I})\).

- If \( \theta \in \left[0, \frac{1}{3}\right) \) then there are two SPE, namely \((\mathcal{I}, \mathcal{I}, \mathcal{N}, \mathcal{N})\) and \((\mathcal{N}, \mathcal{N}, \mathcal{N}, \mathcal{N})\).
Under complete information it is easy to verify that:

- If $\theta < 0$ then the game has a unique SPE given by $(\mathcal{N}, \mathcal{N} \cdot \mathcal{N}, \mathcal{N} \cdot \mathcal{N})$,
- If $\theta > 2/3$ then the game has a unique SPE given by $(\mathcal{I}, \mathcal{I} \cdot \mathcal{I}, \mathcal{I} \cdot \mathcal{I})$.
Equilibrium

Under complete information it is easy to verify that:

- If $\theta < 0$ then the game has a unique SPE given by $(N, N.N.N)$.
- If $\theta > 2/3$ then the game has a unique SPE given by $(I.I.I, I.I)$.
- If $\theta \in (1/3, 2/3]$ then there two outcome equivalent SPNE, namely $(I, I.N, I.N)$ and $(I, I.I, I.I)$.
Equilibrium

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- If $\theta \in [0, 1/3]$ then there are two SPE, namely $(\mathcal{I}, \mathcal{I}.\mathcal{N}, I.\mathcal{N})$ and $(\mathcal{N}, \mathcal{N}.\mathcal{N}, \mathcal{N}.\mathcal{N})$. 
Equilibrium

Under complete information it is easy to verify that:

- If $\theta < 0$ then the game has a unique SPE given by $(N, N.N,N.N)$,
- If $\theta > 2/3$ then the game has a unique SPE given by $(I,I.I, I.I)$.
- If $\theta \in (1/3, 2/3]$ then there two outcome equivalent SPNE, namely $(I, I.N, I.N)$ and $(I, I.I, I.I)$.
- If $\theta \in [0, 1/3]$ then there are two SPE, namely $(I, I.N, I.N)$ and $(N, N.N, N.N)$.

When there $n$ followers, two SPEs for $\theta \in (0, 1/(n + 1))$
Incomplete Information-Signaling

- Introduce uncertainty in the global game way.
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- At the beginning of the game the leader observes the true state $\theta$.
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- Followers observe signals $x_j = \theta + \epsilon_i$ where $\epsilon_i$’s are i.i.d and independent of $\theta$ and $\epsilon_j \sim N(0, \sigma^2)$.
Incomplete Information-Signaling

- Introduce uncertainty in the global game way.
- We assume that \( \theta \) is drawn from an improper prior on the real line.
- At the beginning of the game the leader observes the true state \( \theta \)
- Followers observe signals \( x_j = \theta + \epsilon_i \) where \( \epsilon_i \)'s are i.i.d and independent of \( \theta \) and \( \epsilon_j \sim N(0, \sigma^2) \).

Timing:
1. Nature draws \( \theta \).
2. Leader observes \( \theta \) and each follower observes signal \( x_j \).
3. Leader chooses \( a_L \in \{I, N\} \).
4. Followers observe the realized history (i.e. the choice of leader) and choose \( a_j \in \{I, N\}, j = 1, \ldots, n \).
5. Playoffs realize.
Strategies

- Leader: \( s_L : \mathbb{R} \rightarrow \{I, N\} \).
Strategies

- **Leader**: $s_L : \mathbb{R} \to \{I, N\}$.
- **Followers**: $s_j : \mathbb{R} \times \mathcal{H} \to \{I, N\}$, $j = 1, \ldots, n$, where $\mathcal{H} = \{I, N\}$.

**Definition (Monotone Strategies)**

We say that players play a monotone strategy if there exist thresholds $b_\theta^L$ and $\hat{x}_{h j}$ such that:

For $j = 1, \ldots, n$, and $h \in \{I, N\}$,

$$s_h j = I \text{ if } x_j \geq \hat{x}_{h j}$$

$$s_h j = N \text{ if } x_j < \hat{x}_{h j}$$
Strategies

- Leader: \( s_L : \mathbb{R} \rightarrow \{I, N\} \).
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**Definition (Monotone Strategies)**

We say that players play a monotone strategy if there exist thresholds \( \hat{\theta}_L \), \( \hat{x}_j^I \) and \( \hat{x}_j^N \) such that:

\[
s_L = \begin{cases} 
I & \text{if } \theta \geq \hat{\theta}_L \\
N & \text{if } \theta < \hat{\theta}_L 
\end{cases}
\]

and for \( j = 1, \ldots, n \), and \( h \in \{I, N\} \),

\[
s_j^h = \begin{cases} 
I & \text{if } x_j \geq \hat{x}_j^h \\
N & \text{if } x_j < \hat{x}_j^h 
\end{cases}
\]
Analysis

- Consider type $x_j$ for follower $j$: observing $h = \mathcal{I}$ is equivalent to knowing that the event $\{\theta > \hat{\theta}_L\}$ has happened.
- Thus, type $x_j$’s posterior belief about $\theta$ has a truncated Gaussian distribution:

$$
\psi^{\mathcal{I}}(\theta; x_j, \hat{\theta}_L) = \frac{1}{\sigma_F} \phi \left( \frac{\theta - x_j}{\sigma_F} \right) \frac{1}{1 - \Phi \left( \frac{\hat{\theta}_L - x_j}{\sigma_F} \right)} \mathbf{1}(\theta > \hat{\theta}_L),
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$$

Similarly for history $h = \mathcal{N}$,

$$
\psi^\mathcal{N}(\theta; x_j, \hat{\theta}_L) = \frac{1}{\sigma_F} \phi \left( \frac{\theta - x_j}{\sigma_F} \right) \frac{1}{\Phi \left( \frac{\hat{\theta}_L - x_j}{\sigma_F} \right)} 1(\theta \leq \hat{\theta}_L)
$$
Analysis

- Under history $h$, the expected payoff to investing of type $x_j$ of follower $j$ is given by:

$$\pi^h_F(x_j; \hat{\theta}_L, \hat{x}_h) = \int_{-\infty}^{\infty} \left( \theta - \frac{n-1}{n+1} \Phi \left( \frac{\hat{x}_h - \theta}{\sigma_F} \right) \right) d\psi^h(\theta; x_j, \hat{\theta}_L) - \frac{1(h=N)}{n+1}$$
Analysis

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$$

Lemma

Type $x_j$’s payoff to investing under history $h$ has the following properties:

1. $\pi^h_F(x_j; \hat{\theta}_L, \hat{x}_h)$ is strictly increasing in $x_j$ and $\hat{\theta}_L$.
2. $\pi^h_F(x_j; \hat{\theta}_L, \hat{x}_h)$ is strictly decreasing in $\hat{x}_h$.

- Thus, all followers will best respond by using a monotone strategy.
Analysis

- The payoff to investing for type $\theta$ of the leader is

$$
\pi_L(\theta; \hat{x}_I) = \theta - \frac{n}{n + 1} \Phi \left( \frac{\hat{x}_I - \theta}{\sigma_F} \right).
$$

- Note that the behavior of followers matters to the leader only when $a_L = I$ (otherwise the safe action $a_L = N$ gives a payoff of zero).
- It is straightforward to see that $\pi_L(\theta; \hat{x}_I)$ is strictly increasing in $\theta$ and crosses zero only once from below.
Analysis

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- Therefore, the leader’s best response must be a monotone strategy with a non-negative threshold because $\pi_L(\theta, \hat{x}_I) < 0$, regardless of $\hat{x}_I$, for all $\theta < 0$. 

Equilibrium

A monotone equilibrium with thresholds \((\theta^*_L, x^*_I, x^*_N)\) obtains if the threshold types are indifferent between investing and not investing. Thus, it must solve the following system of equations:

\[
\begin{align*}
\pi_L(\theta^*_L; x^*_I) &= 0; \quad (E-1) \\
\pi^I_F(x^*_I; \theta^*_L, x^*_I) &= 0; \quad (E-2) \\
\pi^N_F(x^*_N; \theta^*_L, x^*_N) &= 0. \quad (E-3)
\end{align*}
\]
Equilibrium

A monotone equilibrium with thresholds \((\theta_L^*, x_I^*, x_N^*)\) obtains if the threshold types are indifferent between investing and not investing. Thus, it must solve the following system of equations:

\[
\pi_L(\theta_L^*; x_I^*) = 0; \quad (E-1)
\]
\[
\pi_F^T(x_I^*; \theta_L^*, x_I^*) = 0; \quad (E-2)
\]
\[
\pi_F^N(x_N^*; \theta_L^*, x_N^*) = 0. \quad (E-3)
\]

Proposition (Herding equilibrium)

There exists a unique monotone equilibrium that simultaneously solves (E-1), (E-2) and (E-3). In this equilibrium, any positive type of the leader invests, and all followers invest when \(h = I\) and do not invest when \(h = N\) (i.e., with thresholds \(\theta_L^* = 0, x_I^* = -\infty, \) and \(x_N^* = \infty\)).
We now turn to the question of which actions are rationalizable.
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Rationalizable Behavior

- We now turn to the question of which actions are rationalizable.
- We resort to \( \Delta \)-rationalizability of Battigalli and Siniscalchi (2003), which extends Pearce’s notion of extensive-form rationalizability to games with incomplete information.
- The “\( \Delta \)” indicates a specific set of restrictions on beliefs that is required to be satisfied at each round of the iterative procedure, which, in our case, it is the signal structure commonly known to all players.
Rationalizable Behavior

Definition (∆-rationalizability)

Consider the following procedure.

(Round 0) Let $R_L^0 = \Theta \times \{I, N\}$ and $R_{F,j}^0 = X_j \times S_j$ for each $j \in F$, where $S_j$ is the set of strategies $s_j(h)$ that maps each history into an action.

(Round $k \geq 1$) Let $R_F^k = \prod_{j \in F} R_{F,j}^k$ and $R_{F,-j}^k = \prod_{\ell \neq j} R_{F,\ell}^k$. Then

(i) $(\theta, a_L) \in R_L^k$ if and only if $(\theta, a_L) \in R_L^{k-1}$ and there exists a belief $\mu_L$ over $R_F^0$ such that $\mu_L(R_F^{k-1}) = 1$ and $a_L$ is a best response with respect to $\mu_L$ for type $\theta$ of the leader.

(ii) For every follower $j \in F$, $(x_j, s_j) \in R_L^{k-1}$ if and only if $(x_j, s_j) \in R_j^{k-1}$ and for each history $h$ there exists a belief $\mu_j(\cdot|h)$ over $R_L^0 \times R_{F,-j}^0$ such that $\mu_j(R_L^k \times R_{F,-j}^{k-1}|h) = 1$ and $s_j(h)$ is a best response with respect to $\mu_j(\cdot|h)$ for type $x_j$.

Finally, let $R_L^\infty = \bigcap_{k=0}^{\infty} R_L^k$ and $R_{F,j}^\infty = \bigcap_{k=0}^{\infty} R_{F,j}^k$. Then an action $a_L$ is $\Delta$-rationalizable for type $\theta$ of the leader if $(\theta, a_L) \in R_L^\infty$. Analogously, a strategy $s_j$ is $\Delta$-rationalizable for type $x_j$ of follower $j$ if $(x_j, s_j) \in R_{F,j}^\infty$. 
Rationalizable Behavior

- We now illustrate how $\Delta$-rationalizability proceeds in the case $n = 2$.
- First note that the payoff for the leader satisfies the standard two-sided “limit dominance” property of global games with the dominance regions being $(-\infty, 0)$ and $(2/3, \infty)$.
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- First note that the payoff for the leader satisfies the standard two-sided “limit dominance” property of global games with the dominance regions being $(-\infty, 0)$ and $(2/3, \infty)$.
- So the leader will delete, in Round 1, all type-action pairs $(\theta, I)$ for $\theta < \theta^1_L = 0$ and $(\theta, N)$ for $\theta > \overline{\theta}^1_L = 2/3$.
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- Investing is never a best response for type $x_j$ if $\mathbb{E}[\theta | x_j, \theta \geq \theta^1_L] < 0$ and not investing never a best response for type $x_j$ if $\mathbb{E}[\theta | x_j, \theta \geq \theta^1_L] > 1/3$. 
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- Investing is never a best response for type \( x_j \) if \( \mathbb{E}[\theta | x_j, \theta \geq \theta^1_L] < 0 \) and not investing never a best response for type \( x_j \) if \( \mathbb{E}[\theta | x_j, \theta \geq \theta^1_L] > 1/3 \).
- But since

\[
\mathbb{E}[\theta | x_j, \theta \geq \theta^1_L] = x_j + \sigma_F \frac{\phi \left( \frac{\theta^1_L - x_j}{\sigma_F} \right)}{1 - \Phi \left( \frac{\theta^1_L - x_j}{\sigma_F} \right)}
\]

is strictly increasing in \( x_j \) and positive, investing is not dominated for any type of follower \( j \).
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is strictly increasing in $x_j$ and positive, investing is not dominated for any type of follower $j$.
- Let $x^1_I = -\infty$ and $\bar{x}^1_I$ be the unique value of $x_j$ solving $E[\theta | x_j, \theta \geq 0] = 1/3$. 


Thus, in Round 1, follower $j$ will delete all type-strategy pairs $(x_j, s_j)$ such that $x_j > x_j^1$ and $s_j(I) = N$. 
Thus, in Round 1, follower $j$ will delete all type-strategy pairs $(x_j, s_j)$ such that $x_j > x^1_I$ and $s_j(I) = \mathcal{N}$.

Likewise, in the no-investment subgame there is no type $x_j$ viewing investing as a dominant action. We set $x^1_{\mathcal{N}}$ to be the unique solution to $\mathbb{E}[\theta | x_j, \theta \leq \bar{\theta}_L] = 0$ and $x^1_{\mathcal{N}} = \infty$. 
Rationalizable Behavior

- In Round 2, with the knowledge of $x_{1L}$ and $x_{1I}$ the payoff to investing for type $\theta$ of the leader is bounded above by $\theta$ and below by

$$\theta - \frac{1}{3} \Phi \left( \frac{x_{1I} - \theta}{\sigma_F} \right).$$
Rationalizable Behavior

- In Round 2, with the knowledge of $x^1_{L}$ and $\bar{x}^1_{L}$ the payoff to investing for type $\theta$ of the leader is bounded above by $\theta$ and below by

$$\theta - \frac{1}{3} \Phi \left( \frac{x^1_{L} - \theta}{\sigma_F} \right).$$

- The upper bound is justified by a belief that all types between $x^1_{L}$ and $\bar{x}^1_{L}$ would invest and the lower bound is supported by believing that no type below $\bar{x}^1_{L}$ would invest.
Rationalizable Behavior

- In Round 2, with the knowledge of $x_{1I}^1$ and $\bar{x}_{1I}^1$ the payoff to investing for type $\theta$ of the leader is bounded above by $\theta$ and below by

$$\theta - \frac{1}{3} \Phi \left( \frac{\bar{x}_{1I}^1 - \theta}{\sigma_F} \right).$$

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- The best response in the best-case scenario is $a_L = I$ if $\theta > \theta_L^2 = 0$ and $a_L = N$ if $\theta < \theta_L^2$. 
Rationalizable Behavior

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$$\theta - \frac{1}{3}\Phi\left(\frac{\bar{x}^1_I - \theta}{\sigma_F}\right).$$

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- The best response in the best-case scenario is $a_L = I$ if $\theta > \theta^2_L = 0$ and $a_L = N$ if $\theta < \theta^2_L$.

- The payoff lower bound is strictly increasing in $\theta$; therefore type $\theta$ best responds in this case by investing if $\theta > \bar{\theta}_L^2$ and not investing if $\theta < \bar{\theta}_L^2$ where $\bar{\theta}_L^2$ solves $\theta - (1/3)\Phi((\bar{x}_I^1 - \theta)/\sigma_F) = 0$ and $\bar{\theta}_L^2 < \bar{\theta}^1_L$. 
In Round 2, with the knowledge of $x_I^1$ and $\bar{x}_I^1$ the payoff to investing for type $\theta$ of the leader is bounded above by $\theta$ and below by

$$\theta - \frac{1}{3} \Phi \left( \frac{x_I^1 - \theta}{\sigma_F} \right).$$

The upper bound is justified by a belief that all types between $x_I^1$ and $\bar{x}_I^1$ would invest and the lower bound is supported by believing that no type below $\bar{x}_I^1$ would invest.

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The payoff lower bound is strictly increasing in $\theta$; therefore type $\theta$ best responds in this case by investing if $\theta > \bar{\theta}_L^2$ and not investing if $\theta < \bar{\theta}_L^2$ where $\bar{\theta}_L^2$ solves $\theta - (1/3)\Phi((\bar{x}_I^1 - \theta)/\sigma_F) = 0$ and $\bar{\theta}_L^2 < \theta^1_L$.

In sum, the leader will delete all $(\theta, I)$ such that $\theta < \theta^2_L$ and all $(\theta, N)$ such that $\theta > \bar{\theta}_L^2$. 
Rationalizable Behavior

- Now type $x_j$ of follower $j$ reasons that if he finds himself in the investment subgame then the type of the leader must be at least as great as $\theta_L^2$. 

He also knows that other followers will invest for sure if their type is greater than $x_1$. It follows that $x_j$'s payoff to investing is lower bounded by 

$$Z_{-\infty}^{\infty} \theta - \frac{1}{3} \Phi(x_1) - \theta \sigma_F \Psi_I(\theta; x_j, \theta_L^2)$$

if all types below $x_1$ do not invest, and is upper bounded by 

$$Z_{-\infty}^{\infty} \theta \Psi_I(\theta; x_j, \theta_L^2)$$

if all type below $x_1$ invest. Both bounds are strictly increasing in $x_j$ and the upper bound is positive. This implies that $x_j$'s best response in the best-case scenario is always investing (i.e., $x_2^I = -\infty$).
Rationalizable Behavior

- Now type $x_j$ of follower $j$ reasons that if he finds himself in the investment subgame then the type of the leader must be at least as great as $\theta^2_L$.
- He also knows that other followers will invest for sure if their type is greater than $\bar{x}_1$. 

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Rationalizable Behavior

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- He also knows that other followers will invest for sure if their type is greater than $\bar{x}_L^1$.
- It follows that $x_j$’s payoff to investing is lower bounded by
  \[
  \int_{-\infty}^{\infty} \left( \theta - \frac{1}{3} \Phi \left( \frac{\bar{x}_L^1 - \theta}{\sigma_F} \right) \right) d\psi^I(\theta; x_j, \theta^2_L)
  \]
  if all types below $\bar{x}_L^1$ do not invest, and is upper bounded by
  \[
  \int_{-\infty}^{\infty} \theta d\psi^I(\theta; x_j, \theta^2_L)
  \]
  if all type below $\bar{x}_L^1$ invest.
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- Now type $x_j$ of follower $j$ reasons that if he finds himself in the investment subgame then the type of the leader must be at least as great as $\theta_2^L$.
- He also knows that other followers will invest for sure if their type is greater than $\bar{x}^1_I$.
- It follows that $x_j$’s payoff to investing is lower bounded by
  \[ \int_{-\infty}^{\infty} \left( \theta - \frac{1}{3} \Phi \left( \frac{\bar{x}^1_I - \theta}{\sigma_F} \right) \right) \, d\psi^I(\theta; x_j, \theta_2^L) \]
  if all types below $\bar{x}^1_I$ do not invest, and is upper bounded by
  \[ \int_{-\infty}^{\infty} \theta \, d\psi^I(\theta; x_j, \theta_2^L) \]
  if all type below $\bar{x}^1_I$ invest.
- Both bounds are strictly increasing in $x_j$ and the upper bound is positive.
Rationalizable Behavior

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- Both bounds are strictly increasing in $x_j$ and the upper bound is positive.
- This implies that $x_j$’s best response in the best-case scenario is always investing (i.e., $x_j^2 = -\infty$).
Rationalizable Behavior

- Let $x_j^2$ be the value of $x_j$ that makes the lower bound equal to zero.
Rationalizable Behavior

- Let $x^2_I$ be the value of $x_j$ that makes the lower bound equal to zero.
- Then the worst-case best response for $x_j$ is investing if $x_j > x^2_I$ and not investing if $x_j < x^2_I$. 
Rationalizable Behavior

- Let $\bar{x}_I^2$ be the value of $x_j$ that makes the lower bound equal to zero.
- Then the worst-case best response for $x_j$ is investing if $x_j > \bar{x}_I^2$ and not investing if $x_j < \bar{x}_I^2$.
- Note that $\bar{x}_I^2 > \bar{x}_I^1$ and we can get $\bar{x}_N^2$ and $\bar{x}_N^2$ in the same vein.
Rationalizable Behavior

- Let $\bar{x}_{\mathcal{I}}^2$ be the value of $x_j$ that makes the lower bound equal to zero.
- Then the worst-case best response for $x_j$ is investing if $x_j > \bar{x}_{\mathcal{I}}^2$ and not investing if $x_j < \bar{x}_{\mathcal{I}}^2$.
- Note that $\bar{x}_{\mathcal{I}}^2 > \bar{x}_{\mathcal{I}}^1$ and we can get $x_{\bar{\mathcal{N}}}$ and $\bar{x}_{\mathcal{N}}^2$ in the same vein.
- Thus, follower $j$ will delete all type-strategy pairs $(x_j, s_j)$ such that $x_j > \bar{x}_{\mathcal{I}}^2$ and $s_j(\mathcal{I}) = \mathcal{N}$ and such that $x_j < \bar{x}_{\mathcal{N}}^2$ and $s_j(\mathcal{N}) = \mathcal{I}$. 
Rationalizable Behavior

- Let $\overline{x}_I^2$ be the value of $x_j$ that makes the lower bound equal to zero.
- Then the worst-case best response for $x_j$ is investing if $x_j > \overline{x}_I^2$ and not investing if $x_j < \overline{x}_I^2$.
- Note that $\overline{x}_I^2 > \overline{x}_I^1$ and we can get $\underline{x}_N^2$ and $\overline{x}_N^2$ in the same vein.
- Thus, follower $j$ will delete all type-strategy pairs $(x_j, s_j)$ such that $x_j > \overline{x}_I^2$ and $s_j(I) = N$ and such that $x_j < \underline{x}_N^2$ and $s_j(N) = I$.
- Repeating the above procedure yields six sequences:
  $$(\theta_L^k, \overline{\theta}_L^k, \underline{x}_I^k, \overline{x}_I^k, \underline{x}_N^k, \overline{x}_N^k)_{k=1}^{\infty}$$
  where $\theta_L^k = 0$, $\underline{x}_I^k = -\infty$, and $\overline{x}_N^k = \infty$ for all $k$. 
Rationalizable Behavior

- Let $\bar{x}_I^2$ be the value of $x_j$ that makes the lower bound equal to zero.
- Then the worst-case best response for $x_j$ is investing if $x_j > \bar{x}_I^2$ and not investing if $x_j < \bar{x}_I^2$.
- Note that $\bar{x}_I^2 > \bar{x}_I^1$ and we can get $\underline{x}_N^2$ and $\bar{x}_N^2$ in the same vein.
- Thus, follower $j$ will delete all type-strategy pairs $(x_j, s_j)$ such that $x_j > \bar{x}_I^2$ and $s_j(I) = N$ and such that $x_j < \underline{x}_N^2$ and $s_j(N) = I$.
- Repeating the above procedure yields six sequences:
  $$(\theta_L^k, \bar{\theta}_L^k, x_I^k, \underline{x}_I^k, x_N^k, \bar{x}_N^k)_{k=1}^{\infty}$$
  where $\theta_L^k = 0$, $x_I^k = -\infty$, and $\bar{x}_N^k = \infty$ for all $k$.
- Since $\bar{\theta}_L^k$ and $\underline{x}_I^k$ are decreasing and $\underline{x}_N^k$ is increasing, a unique $\Delta$-rationalizable strategy profile obtains when $\bar{\theta}_L^k$ converges to zero, and $\underline{x}_I^k$ and $\underline{x}_N^k$ diverge to $-\infty$ and $\infty$, respectively.
Proposition

There exists a unique $\hat{\sigma}_F$ such that the unique monotone equilibrium is uniquely $\Delta$-rationalizable if and only if $\sigma_F > \hat{\sigma}_F$. Moreover, $\hat{\sigma}_F$ is strictly increasing in $n$. 
Rationalizable Behavior

Figure: $\hat{\sigma}_F$ increases with $n$. 
Intuition-Conditional Rank Beliefs

Define

$$R^\mathcal{I}(x; \hat{\theta}_L) = \Pr(x_\ell < x_j \mid x_j = x, \theta > \hat{\theta}_L) = \frac{1}{2} \Phi \left( \frac{x - \hat{\theta}_L}{\sigma_F} \right)$$

to be follower $j$’s *conditional rank belief function* under history $h = \mathcal{I}$; that is, the probability follower $j$ assigns to the event that another follower’s type $x_\ell$ is lower than his own ($x_j = x$) given that the leader’s type $\theta$ is greater than a threshold $\hat{\theta}_L$. 
Intuition-Conditional Rank Beliefs

- Define

\[ R^I(x; \hat{\theta}_L) = \Pr(x_\ell < x_j \mid x_j = x, \theta > \hat{\theta}_L) = \frac{1}{2} \Phi \left( \frac{x - \hat{\theta}_L}{\sigma_F} \right) \]

- to be follower \( j \)'s conditional rank belief function under history \( h = I \); that is, the probability follower \( j \) assigns to the event that another follower's type \( x_\ell \) is lower than his own \( (x_j = x) \) given that the leader's type \( \theta \) is greater than a threshold \( \hat{\theta}_L \).

- Follower's payoff to investing in the investment subgame can be written as

\[ \pi^I_F(x; \hat{\theta}_L, x) = x + \sigma_F \lambda \left( \frac{x - \hat{\theta}_L}{\sigma_F} \right) - \frac{n - 1}{n + 1} R^I(x; \hat{\theta}_L), \]

\[ \underbrace{\text{expected gross return}}_{\text{expected loss}} \]
Intuition
Intuition

Figure: Conditional Rank Belief

Figure: Expected Payoff from Investing
When the unique rationalizable strategy profile obtains, our result can also be interpreted in terms of equilibrium selection.

**Corollary (Unique Stackelberg selection)**

If $\sigma_F > \hat{\sigma}_F$, then the signaling game uniquely selects a fully efficient SPE of the complete information game.
Suppose, now, that instead of perfectly learning the state $\theta$, the leader observes a noisy signal $x_L = \theta + \sigma_L \varepsilon_L$ with $\varepsilon_L$ being a standard Gaussian noise independent of $\theta$ and $\varepsilon_j$ for any $j \in F$. This generalization is important from a global games perspective, since perturbations that remove common knowledge of the fact that the leader is perfectly informed may affect the results of the previous analysis. Now the strategy of the leader will depend on her signal rather than the true state. Thus, the equilibrium will be given by thresholds $x^*_L$, $x^*_I$, and $x^*_N$. 
Extension: Noisy Observation By the Leader

- Suppose, now, that instead of perfectly learning the state $\theta$, the leader observes a noisy signal $x_L = \theta + \sigma_L \varepsilon_L$ with $\varepsilon_L$ being a standard Gaussian noise independent of $\theta$ and $\varepsilon_j$ for any $j \in F$.

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- Now the strategy of the leader will depend on her signal rather than the true state.

- Thus, the equilibrium will be given by thresholds $x_L^*$, $x_I^*$, and $x_N^*$. 
Extension: Noisy Observation By the Leader-Monotone Equilibrium

**Proposition**

There exists a unique monotone equilibrium. Moreover, this equilibrium converges to that of the signaling game when $\sigma_L \to 0^+$ (while keep $\sigma_F$ fixed), or when $\sigma_L \to 0^+$, $\sigma_F \to 0^+$ and $\frac{\sigma_L}{\sigma_F} \to 0^+$. 
Extension: Noisy Observation By the Leader-Rationalizable Behavior

**Proposition**

*The unique monotone equilibrium is uniquely $\Delta$-rationalizable if* $\sigma_L$ *and* $\sigma_F$ *satisfy a condition on the* $(\sigma_L, \sigma_F)$-*space*
Extension: Noisy Observation By the Leader-Rationalizable Behavior

![Diagram](image)

**Figure:** Curve $\hat{\sigma}_L(\gamma)$ in the $(\sigma_L, \sigma_F)$-space.
Noisy Observation by the Leader

Figure: Function $\tilde{v}(x)$ for $n = 2$
Concluding Remarks

- Our results challenge the robustness of the uniqueness results of the global games framework when one moves away from the static benchmark and the exogenous information structure.

Implications for applied work may be significant if one wishes to use rationalizability as the solution concept. In this case, a leader may or may not be able to discipline the game depending on whether certain conditions on the noise of the signals are satisfied, even if she is arbitrarily better informed than the followers.
Concluding Remarks

- Our results challenge the robustness of the uniqueness results of the global games framework when one moves away from the static benchmark and the exogenous information structure.
- We derive conditions that guarantee the uniqueness of equilibrium behavior and characterize the (in)efficiency of outcomes that may arise.

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Thank you!