The Signaling Role of Leaders in Global Games

Panagiotis Kyriazis and Edmund Lou

July 2022

Motivation

- Coordination problems naturally arise in a diverse variety of socio-economic phenomena.

Motivation

- Coordination problems naturally arise in a diverse variety of socio-economic phenomena.
- In many of such scenarios, there can exist "leaders" whose visibility gives them a special role.

Motivation

- Coordination problems naturally arise in a diverse variety of socio-economic phenomena.
- In many of such scenarios, there can exist "leaders" whose visibility gives them a special role.
- Examples: large investors like Soros, vanguards in revolutions.

Motivation

- Coordination problems naturally arise in a diverse variety of socio-economic phenomena.
- In many of such scenarios, there can exist "leaders" whose visibility gives them a special role.
- Examples: large investors like Soros, vanguards in revolutions.
- We study the signaling role of leaders in global games and its effect on equilibrium and rationalizable behavior.

This Paper

- A leader and a set of followers face a coordination problem and have binary, irreversible actions.

This Paper

- A leader and a set of followers face a coordination problem and have binary, irreversible actions.
- The leader moves first and her action is observable by the followers

This Paper

- A leader and a set of followers face a coordination problem and have binary, irreversible actions.
- The leader moves first and her action is observable by the followers – **signaling** by the leader

This Paper

- A leader and a set of followers face a coordination problem and have binary, irreversible actions.
- The leader moves first and her action is observable by the followers
  – **signaling** by the leader
  – **learning** by the followers.

This Paper

- A leader and a set of followers face a coordination problem and have binary, irreversible actions.
- The leader moves first and her action is observable by the followers
  – **signaling** by the leader
  – **learning** by the followers.
- **Result:** Conditions that guarantee the uniqueness of rationalizable (and hence, equilibrium) play.

This Paper

- A leader and a set of followers face a coordination problem and have binary, irreversible actions.
- The leader moves first and her action is observable by the followers
  – **signaling** by the leader
  – **learning** by the followers.
- **Result:** Conditions that guarantee the uniqueness of rationalizable (and hence, equilibrium) play.
- **Result:** (In)Efficiency of outcomes.

Related Literature

- **Simultaneous-Move Global Games**: Carlsson and Van Damme (1993), Morris and Shin (2003), Frankel, Morris and Pauzner (2003).
- **Dynamic Global Games, Efficiency and Learning**: Xue (2003), Angeletos, Hellwig and Pavan (2007), Chassang (2010), Dasgupta (2007), Kováč and Steiner (2012).
- Signaling and Policy Traps: Angeletos, Hellwig and Pavan (2006).
- **Extensive-Form Bayesian Games with Strategic Complementarities**: Echenique (2002), Van Zandt and Vives (2007).
- **Applications**: Corsetti et al. (2004), Bueno de Mesquita (2010), Shadmehr and Bernhardt (2019).

The Model - Complete Information

- A leader and n followers have to decide whether to invest in a project or not.

The Model - Complete Information

- A leader and n followers have to decide whether to invest in a project or not.
- Payoffs: $u_i(\theta, A) = \theta + A - 1$ where θ is the state of nature and A is the fraction of players that invest.

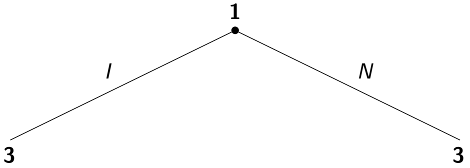
The Model - Complete Information

- A leader and n followers have to decide whether to invest in a project or not.
- Payoffs: $u_i(\theta, A) = \theta + A - 1$ where θ is the state of nature and A is the fraction of players that invest.
- Leader moves first and then followers play a simultaneous move game.

The Model - Complete Information

- A leader and n followers have to decide whether to invest in a project or not.
- Payoffs: $u_i(\theta, A) = \theta + A - 1$ where θ is the state of nature and A is the fraction of players that invest.
- Leader moves first and then followers play a simultaneous move game.
- Two-stage game of strategic complementarities across and within stages.

The Game



		I	N
2	I	θ , θ , θ	$\theta - \frac{1}{3}$, $\theta - \frac{1}{3}$, 0
	N	$\theta - \frac{1}{3}$, 0 , $\theta - \frac{1}{3}$	$\theta - \frac{2}{3}$, 0 , 0

	I	N
I	0 , $\theta - \frac{1}{3}$, $\theta - \frac{1}{3}$	0 , $\theta - \frac{2}{3}$, 0
N	0 , 0 , $\theta - \frac{2}{3}$	0 , 0 , 0

Equilibrium

Under complete information it is easy to verify that:

- If $\theta < 0$ then the game has a unique SPE given by ($\mathcal{N}$, $\mathcal{N}$.$\mathcal{N}$,$\mathcal{N}$.$\mathcal{N}$),

Equilibrium

Under complete information it is easy to verify that:

- If $\theta < 0$ then the game has a unique SPE given by ($\mathcal{N}$, $\mathcal{N}.\mathcal{N}$,$\mathcal{N}.\mathcal{N}$),
- If $\theta > 2/3$ then the game has a unique SPE given by ($\mathcal{I}$,$\mathcal{I}.\mathcal{I}$, $\mathcal{I}.\mathcal{I}$).

Equilibrium

Under complete information it is easy to verify that:

- If $\theta < 0$ then the game has a unique SPE given by ($\mathcal{N}$, $\mathcal{N}.\mathcal{N}$,$\mathcal{N}.\mathcal{N}$),
- If $\theta > 2/3$ then the game has a unique SPE given by ($\mathcal{I}$,$\mathcal{I}.\mathcal{I}$, $\mathcal{I}.\mathcal{I}$).
- If $\theta \in (1/3, 2/3]$ then there two outcome equivalent SPNE, namely ($\mathcal{I}$, $\mathcal{I}.\mathcal{N}$, $\mathcal{I}.\mathcal{N}$) and ($\mathcal{I}$, $\mathcal{I}.\mathcal{I}$, $\mathcal{I}.\mathcal{I}$).

Equilibrium

Under complete information it is easy to verify that:

- If $\theta < 0$ then the game has a unique SPE given by ($\mathcal{N}$, $\mathcal{N}$.$\mathcal{N}$,$\mathcal{N}$.$\mathcal{N}$),
- If $\theta > 2/3$ then the game has a unique SPE given by ($\mathcal{I}$,$\mathcal{I}$.$\mathcal{I}$, $\mathcal{I}$.$\mathcal{I}$).
- If $\theta \in (1/3, 2/3]$ then there two outcome equivalent SPNE, namely ($\mathcal{I}$, $\mathcal{I}$.$\mathcal{N}$, $\mathcal{I}$.$\mathcal{N}$) and ($\mathcal{I}$, $\mathcal{I}$.$\mathcal{I}$, $\mathcal{I}$.$\mathcal{I}$).
- If $\theta \in [0, 1/3]$ then there are two SPE, namely ($\mathcal{I}$, $\mathcal{I}$.$\mathcal{N}$, I.$\mathcal{N}$) and ($\mathcal{N}$, $\mathcal{N}$.$\mathcal{N}$, $\mathcal{N}$.$\mathcal{N}$).

Equilibrium

Under complete information it is easy to verify that:

- If $\theta < 0$ then the game has a unique SPE given by ($\mathcal{N}$, $\mathcal{N}.\mathcal{N}$,$\mathcal{N}.\mathcal{N}$),
- If $\theta > 2/3$ then the game has a unique SPE given by ($\mathcal{I}$,$\mathcal{I}.\mathcal{I}$, $\mathcal{I}.\mathcal{I}$).
- If $\theta \in (1/3, 2/3]$ then there two outcome equivalent SPNE, namely ($\mathcal{I}$, $\mathcal{I}.\mathcal{N}$, $\mathcal{I}.\mathcal{N}$) and ($\mathcal{I}$, $\mathcal{I}.\mathcal{I}$, $\mathcal{I}.\mathcal{I}$).
- If $\theta \in [0, 1/3]$ then there are two SPE, namely ($\mathcal{I}$, $\mathcal{I}.\mathcal{N}$, I.$\mathcal{N}$) and ($\mathcal{N}$, $\mathcal{N}.\mathcal{N}$, $\mathcal{N}.\mathcal{N}$).

When there *n* followers, two SPEs for $\theta \in (0, 1/(n+1))$

Incomplete Information-Signaling

- Introduce uncertainty in the global game way.

Incomplete Information-Signaling

- Introduce uncertainty in the global game way.
- We assume that θ is drawn from an improper prior on the real line.
- At the beginning of the game the leader observes the true state θ

Incomplete Information-Signaling

- Introduce uncertainty in the global game way.
- We assume that θ is drawn from an improper prior on the real line.
- At the beginning of the game the leader observes the true state θ
- Followers observe signals $x_j = \theta + \epsilon_i$ where ϵ_i's are i.i.d and independent of θ and $\epsilon_j \sim N(0, \sigma^2)$.

Incomplete Information-Signaling

- Introduce uncertainty in the global game way.
- We assume that θ is drawn from an improper prior on the real line.
- At the beginning of the game the leader observes the true state θ
- Followers observe signals $x_j = \theta + \epsilon_i$ where ϵ_i's are i.i.d and independent of θ and $\epsilon_j \sim N(0, \sigma^2)$.

Timing:

1. Nature draws θ.
2. Leader observes θ and each follower observes signal x_j.
3. Leader chooses $a_L \in \{\mathcal{I}, \mathcal{N}\}$.
4. Followers observe the realized history (i.e. the choice of leader) and choose $a_j \in \{\mathcal{I}, \mathcal{N}\}$, $j = 1, \ldots, n$.
5. Playoffs realize.

Strategies

- Leader: $s_L : \mathbb{R} \to \{\mathcal{I}, \mathcal{N}\}$.

Strategies

- Leader: $s_L : \mathbb{R} \to \{\mathcal{I}, \mathcal{N}\}$.
- Followers: $s_j : \mathbb{R} \times \mathcal{H} \to \{\mathcal{I}, \mathcal{N}\}$, $j = 1, \ldots, n$, where $\mathcal{H} = \{\mathcal{I}, \mathcal{N}\}$.

Strategies

- Leader: $s_L : \mathbb{R} \to \{\mathcal{I}, \mathcal{N}\}$.
- Followers: $s_j : \mathbb{R} \times \mathcal{H} \to \{\mathcal{I}, \mathcal{N}\}$, $j = 1, \ldots, n$, where $\mathcal{H} = \{\mathcal{I}, \mathcal{N}\}$.

Definition (Monotone Strategies)

We say that players play a monotone strategy if there exist thresholds $\widehat{\theta}_L$, $\hat{x}_j^h$ and $\hat{x}_j^h$ such that:

$$s_L = \begin{cases} \mathcal{I} & \text{if } \theta \geq \widehat{\theta}_L \\ \mathcal{N} & \text{if } \theta < \widehat{\theta}_L \end{cases}$$

and for $j = 1, \ldots,$ and $h \in \{I, N\}$,

$$s_j^h = \begin{cases} I & \text{if } x_j \geq \hat{x}_j^h \\ N & \text{if } x_j < \hat{x}_j^h \end{cases}$$

Analysis

- Consider type x_j for follower j: observing $h = \mathcal{I}$ is equivalent to knowing that the event $\{\theta > \widehat{\theta}_L\}$ has happened.
- Thus, type x_j's posterior belief about θ has a truncated Gaussian distribution:

$$\psi^{\mathcal{I}}(\theta; x_j, \widehat{\theta}_L) = \frac{\frac{1}{\sigma_F}\phi\left(\frac{\theta - x_j}{\sigma_F}\right)}{1 - \Phi\left(\frac{\widehat{\theta}_L - x_j}{\sigma_F}\right)}\mathbf{1}(\theta > \widehat{\theta}_L),$$

Analysis

- Consider type x_j for follower j: observing $h = \mathcal{I}$ is equivalent to knowing that the event $\{\theta > \widehat{\theta_L}\}$ has happened.
- Thus, type x_j's posterior belief about θ has a truncated Gaussian distribution:

$$\psi^{\mathcal{I}}(\theta; x_j, \widehat{\theta}_L) = \frac{\frac{1}{\sigma_F}\phi\left(\frac{\theta - x_j}{\sigma_F}\right)}{1 - \Phi\left(\frac{\widehat{\theta}_L - x_j}{\sigma_F}\right)}\mathbf{1}(\theta > \widehat{\theta}_L),$$

- Similarly for history $h = \mathcal{N}$,

$$\psi^{\mathcal{N}}(\theta; x_j, \widehat{\theta}_L) = \frac{\frac{1}{\sigma_F}\phi\left(\frac{\theta - x_j}{\sigma_F}\right)}{\Phi\left(\frac{\widehat{\theta}_L - x_j}{\sigma_F}\right)}\mathbf{1}(\theta \leq \widehat{\theta}_L)$$

Analysis

- Under history h, the expected payoff to investing of type x_j of follower j is given by:

$$\pi_F^h(x_j;\widehat{\theta}_L,\widehat{x}_h) = \int_{-\infty}^{\infty}\left(\theta - \frac{n-1}{n+1}\Phi\left(\frac{\widehat{x}_h-\theta}{\sigma_F}\right)\right)\mathrm{d}\Psi^h(\theta;x_j,\widehat{\theta}_L) - \frac{\mathbf{1}(h=\mathcal{N})}{n+1}$$

Analysis

- Under history h, the expected payoff to investing of type x_j of follower j is given by:

$$\pi_F^h(x_j;\widehat{\theta}_L,\widehat{x}_h)=\int_{-\infty}^{\infty}\left(\theta-\frac{n-1}{n+1}\Phi\left(\frac{\widehat{x}_h-\theta}{\sigma_F}\right)\right)\mathrm{d}\Psi^h(\theta;x_j,\widehat{\theta}_L)-\frac{\mathbf{1}(h=\mathcal{N})}{n+1}$$

Lemma

Type x_j's payoff to investing under history h has the following properties:
(1) $\pi_F^h(x_j;\widehat{\theta}_L,\widehat{x}_h)$ is strictly increasing in x_j and $\widehat{\theta}_L$.
(2) $\pi_F^h(x_j;\widehat{\theta}_L,\widehat{x}_h)$ is strictly decreasing in $\widehat{x}_h$.

- Thus, all followers will best respond by using a monotone strategy.

Analysis

- The payoff to investing for type θ of the leader is

$$\pi_L(\theta; \widehat{x}_{\mathcal{I}}) = \theta - \frac{n}{n+1}\Phi\left(\frac{\widehat{x}_{\mathcal{I}} - \theta}{\sigma_F}\right).$$

- Note that the behavior of followers matters to the leader only when $a_L = \mathcal{I}$ (otherwise the safe action $a_L = \mathcal{N}$ gives a payoff of zero).
- It is straightforward to see that $\pi_L(\theta; \widehat{x}_{\mathcal{I}})$ is strictly increasing in θ and crosses zero only once from below.

Analysis

- The payoff to investing for type θ of the leader is

$$\pi_L(\theta; \widehat{x}_{\mathcal{I}}) = \theta - \frac{n}{n+1}\Phi\left(\frac{\widehat{x}_{\mathcal{I}} - \theta}{\sigma_F}\right).$$

- Note that the behavior of followers matters to the leader only when $a_L = \mathcal{I}$ (otherwise the safe action $a_L = \mathcal{N}$ gives a payoff of zero).
- It is straightforward to see that $\pi_L(\theta; \widehat{x}_{\mathcal{I}})$ is strictly increasing in θ and crosses zero only once from below.
- Therefore, the leader's best response must be a monotone strategy with a non-negative threshold because $\pi_L(\theta, \widehat{x}_{\mathcal{I}}) < 0$, regardless of $\widehat{x}_{\mathcal{I}}$, for all $\theta < 0$.

Equilibrium

A monotone equilibrium with thresholds $(\theta_L^*, x_{\mathcal{I}}^*, x_{\mathcal{N}}^*)$ obtains if the threshold types are indifferent between investing and not investing. Thus, it must solve the following system of equations:

$$\pi_L(\theta_L^*; x_{\mathcal{I}}^*) = 0; \tag{E-1}$$

$$\pi_F^{\mathcal{I}}(x_{\mathcal{I}}^*; \theta_L^*, x_{\mathcal{I}}^*) = 0; \tag{E-2}$$

$$\pi_F^{\mathcal{N}}(x_{\mathcal{N}}^*; \theta_L^*, x_{\mathcal{N}}^*) = 0. \tag{E-3}$$

Equilibrium

A monotone equilibrium with thresholds $(\theta_L^*, x_\mathcal{I}^*, x_\mathcal{N}^*)$ obtains if the threshold types are indifferent between investing and not investing. Thus, it must solve the following system of equations:

$$\pi_L(\theta_L^*; x_\mathcal{I}^*) = 0; \tag{E-1}$$

$$\pi_F^\mathcal{I}(x_\mathcal{I}^*; \theta_L^*, x_\mathcal{I}^*) = 0; \tag{E-2}$$

$$\pi_F^\mathcal{N}(x_\mathcal{N}^*; \theta_L^*, x_\mathcal{N}^*) = 0. \tag{E-3}$$

Proposition (Herding equilibrium)

There exists a unique monotone equilibrium that simultaneously solves (E-1), (E-2) and (E-3). In this equilibrium, any positive type of the leader invests, and all followers invest when $h = \mathcal{I}$ and do not invest when $h = \mathcal{N}$ (i.e., with thresholds $\theta_L^* = 0$, $x_\mathcal{I}^* = -\infty$, and $x_\mathcal{N}^* = \infty$).

Rationalizable Behavior

- We now turn to the question of which actions are rationalizable.

Rationalizable Behavior

- We now turn to the question of which actions are rationalizable.
- We resort to Δ-rationalizability of Battigalli and Siniscalchi (2003), which extends Pearce's notion of extensive-form rationalizability to games with incomplete information.

Rationalizable Behavior

- We now turn to the question of which actions are rationalizable.
- We resort to Δ-rationalizability of Battigalli and Siniscalchi (2003), which extends Pearce's notion of extensive-form rationalizability to games with incomplete information.
- The "Δ" indicates a specific set of restrictions on beliefs that is required to be satisfied at each round of the iterative procedure, which, in our case, it is the signal structure commonly known to all players.

Rationalizable Behavior

Definition (Δ-rationalizability)

Consider the following procedure.
(Round 0) Let $R_L^0 = \Theta \times \{\mathcal{I}, \mathcal{N}\}$ and $R_{F,j}^0 = X_j \times S_j$ for each $j \in F$, where S_j is the set of strategies $s_j(h)$ that maps each history into an action.
(Round $k \geq 1$) Let $R_F^k = \prod_{j \in F} R_{F,j}^k$ and $R_{F,-j}^k = \prod_{\ell \neq j} R_{F,\ell}^k$. Then

(i) $(\theta, a_L) \in R_L^k$ if and only if $(\theta, a_L) \in R_L^{k-1}$ and there exists a belief μ_L over R_F^0 such that $\mu_L(R_F^{k-1}) = 1$ and a_L is a best response with respect to μ_L for type θ of the leader.

(ii) For every follower $j \in F$, $(x_j, s_j) \in R_L^{k-1}$ if and only if $(x_j, s_j) \in R_j^{k-1}$ and for each history h there exists a belief $\mu_j(\cdot|h)$ over $R_L^0 \times R_{F,-j}^0$ such that $\mu_j(R_L^k \times R_{F,-j}^{k-1}|h) = 1$ and $s_j(h)$ is a best response with respect to $\mu_j(\cdot|h)$ for type x_j.

Finally, let $R_L^\infty = \bigcap_{k=0}^\infty R_L^k$ and $R_{F,j}^\infty = \bigcap_{k=0}^\infty R_{F,j}^\infty$. Then an action a_L is Δ-rationalizable for type θ of the leader if $(\theta, a_L) \in R_L^\infty$. Analogously, a strategy s_j is Δ-rationalizable for type x_j of follower j if $(x_j, s_j) \in R_{F,j}^\infty$.

Rationalizable Behavior

- We now illustrate how Δ-rationalizability proceeds in the case $n = 2$.
- First note that the payoff for the leader satisfies the standard two-sided "limit dominance" property of global games with the *dominance regions* being $(-\infty, 0)$ and $(2/3, \infty)$.

Rationalizable Behavior

- We now illustrate how Δ-rationalizability proceeds in the case $n = 2$.
- First note that the payoff for the leader satisfies the standard two-sided “limit dominance” property of global games with the *dominance regions* being $(-\infty, 0)$ and $(2/3, \infty)$.
- So the leader will delete, in Round 1, all type-action pairs $(\theta, \mathcal{I})$ for $\theta < \underline{\theta}_L^1 = 0$ and $(\theta, \mathcal{N})$ for $\theta > \overline{\theta}_L^1 = 2/3$.

Rationalizable Behavior

- We now illustrate how Δ-rationalizability proceeds in the case $n = 2$.
- First note that the payoff for the leader satisfies the standard two-sided "limit dominance" property of global games with the *dominance regions* being $(-\infty, 0)$ and $(2/3, \infty)$.
- So the leader will delete, in Round 1, all type-action pairs $(\theta, \mathcal{I})$ for $\theta < \underline{\theta}_L^1 = 0$ and $(\theta, \mathcal{N})$ for $\theta > \overline{\theta}_L^1 = 2/3$.
- Investing is never a best response for type x_j if $\mathbb{E}[\theta \,|\, x_j, \theta \geq \underline{\theta}_L^1] < 0$ and not investing never a best response for type x_j if $\mathbb{E}[\theta \,|\, x_j, \theta \geq \underline{\theta}_L^1] > 1/3$.

Rationalizable Behavior

- We now illustrate how Δ-rationalizability proceeds in the case $n = 2$.
- First note that the payoff for the leader satisfies the standard two-sided "limit dominance" property of global games with the *dominance regions* being $(-\infty, 0)$ and $(2/3, \infty)$.
- So the leader will delete, in Round 1, all type-action pairs $(\theta, \mathcal{I})$ for $\theta < \underline{\theta}_L^1 = 0$ and $(\theta, \mathcal{N})$ for $\theta > \overline{\theta}_L^1 = 2/3$.
- Investing is never a best response for type x_j if $\mathbb{E}[\theta \,|\, x_j, \theta \geq \underline{\theta}_L^1] < 0$ and not investing never a best response for type x_j if $\mathbb{E}[\theta \,|\, x_j, \theta \geq \underline{\theta}_L^1] > 1/3$.
- But since

$$\mathbb{E}[\theta \,|\, x_j, \theta \geq \underline{\theta}_L^1] = x_j + \sigma_F \frac{\phi\left(\frac{\underline{\theta}_L^1 - x_j}{\sigma_F}\right)}{1 - \Phi\left(\frac{\underline{\theta}_L^1 - x_j}{\sigma_F}\right)}$$

is strictly increasing in x_j and positive, investing is not dominated for any type of follower j.

Rationalizable Behavior

- We now illustrate how Δ-rationalizability proceeds in the case $n = 2$.
- First note that the payoff for the leader satisfies the standard two-sided "limit dominance" property of global games with the *dominance regions* being $(-\infty, 0)$ and $(2/3, \infty)$.
- So the leader will delete, in Round 1, all type-action pairs $(\theta, \mathcal{I})$ for $\theta < \underline{\theta}_L^1 = 0$ and $(\theta, \mathcal{N})$ for $\theta > \overline{\theta}_L^1 = 2/3$.
- Investing is never a best response for type x_j if $\mathbb{E}[\theta \,|\, x_j, \theta \geq \underline{\theta}_L^1] < 0$ and not investing never a best response for type x_j if $\mathbb{E}[\theta \,|\, x_j, \theta \geq \underline{\theta}_L^1] > 1/3$.
- But since

$$\mathbb{E}[\theta \,|\, x_j, \theta \geq \underline{\theta}_L^1] = x_j + \sigma_F \frac{\phi\left(\frac{\underline{\theta}_L^1 - x_j}{\sigma_F}\right)}{1 - \Phi\left(\frac{\underline{\theta}_L^1 - x_j}{\sigma_F}\right)}$$

  is strictly increasing in x_j and positive, investing is not dominated for any type of follower j.
- Let $\underline{x}_{\mathcal{I}}^1 = -\infty$ and $\overline{x}_{\mathcal{I}}^1$ be the unique value of x_j solving $\mathbb{E}[\theta \,|\, x_j, \theta \geq 0] = 1/3$.

Rationalizable Behavior

- Thus, in Round 1, follower j will delete all type-strategy pairs (x_j, s_j) such that $x_j > \overline{x}^1_{\mathcal{I}}$ and $s_j(\mathcal{I}) = \mathcal{N}$.

Rationalizable Behavior

- Thus, in Round 1, follower j will delete all type-strategy pairs (x_j, s_j) such that $x_j > \overline{x}^1_{\mathcal{I}}$ and $s_j(\mathcal{I}) = \mathcal{N}$.
- Likewise, in the no-investment subgame there is no type x_j viewing investing as a dominant action. We set $\underline{x}^1_{\mathcal{N}}$ to be the unique solution to $\mathbb{E}[\theta \,|\, x_j, \theta \leq \overline{\theta}_L] = 0$ and $\overline{x}^1_{\mathcal{N}} = \infty$.

Rationalizable Behavior

- In Round 2, with the knowledge of $\underline{x}_{\mathcal{I}}^1$ and $\overline{x}_{\mathcal{I}}^1$ the payoff to investing for type θ of the leader is bounded above by θ and below by

$$\theta - \frac{1}{3}\Phi\left(\frac{\overline{x}_{\mathcal{I}}^1 - \theta}{\sigma_F}\right).$$

Rationalizable Behavior

- In Round 2, with the knowledge of $\underline{x}_{\mathcal{I}}^1$ and $\overline{x}_{\mathcal{I}}^1$ the payoff to investing for type θ of the leader is bounded above by θ and below by

$$\theta - \frac{1}{3}\Phi\left(\frac{\overline{x}_{\mathcal{I}}^1 - \theta}{\sigma_F}\right).$$

- The upper bound is justified by a belief that all types between $\underline{x}_{\mathcal{I}}^1$ and $\overline{x}_{\mathcal{I}}^1$ would invest and the lower bound is supported by believing that no type below $\overline{x}_{\mathcal{I}}^1$ would invest.

Rationalizable Behavior

- In Round 2, with the knowledge of $\underline{x}_{\mathcal{I}}^{1}$ and $\overline{x}_{\mathcal{I}}^{1}$ the payoff to investing for type θ of the leader is bounded above by θ and below by

$$\theta - \frac{1}{3}\Phi\left(\frac{\overline{x}_{\mathcal{I}}^{1} - \theta}{\sigma_F}\right).$$

- The upper bound is justified by a belief that all types between $\underline{x}_{\mathcal{I}}^{1}$ and $\overline{x}_{\mathcal{I}}^{1}$ would invest and the lower bound is supported by believing that no type below $\overline{x}_{\mathcal{I}}^{1}$ would invest.
- The best response in the best-case scenario is $a_L = \mathcal{I}$ if $\theta > \underline{\theta}_L^2 = 0$ and $a_L = \mathcal{N}$ if $\theta < \underline{\theta}_L^2$.

Rationalizable Behavior

- In Round 2, with the knowledge of $\underline{x}_{\mathcal{I}}^1$ and $\overline{x}_{\mathcal{I}}^1$ the payoff to investing for type θ of the leader is bounded above by θ and below by

$$\theta - \frac{1}{3}\Phi\left(\frac{\overline{x}_{\mathcal{I}}^1 - \theta}{\sigma_F}\right).$$

- The upper bound is justified by a belief that all types between $\underline{x}_{\mathcal{I}}^1$ and $\overline{x}_{\mathcal{I}}^1$ would invest and the lower bound is supported by believing that no type below $\overline{x}_{\mathcal{I}}^1$ would invest.
- The best response in the best-case scenario is $a_L = \mathcal{I}$ if $\theta > \underline{\theta}_L^2 = 0$ and $a_L = \mathcal{N}$ if $\theta < \underline{\theta}_L^2$.
- The payoff lower bound is strictly increasing in θ; therefore type θ best responds in this case by investing if $\theta > \overline{\theta}_L^2$ and not investing if $\theta < \overline{\theta}_L^2$ where $\overline{\theta}_L^2$ solves $\theta - (1/3)\Phi((\overline{x}_{\mathcal{I}}^1 - \theta)/\sigma_F) = 0$ and $\overline{\theta}_L^2 < \overline{\theta}_L^1$.

Rationalizable Behavior

- In Round 2, with the knowledge of $\underline{x}_{\mathcal{I}}^1$ and $\overline{x}_{\mathcal{I}}^1$ the payoff to investing for type θ of the leader is bounded above by θ and below by

$$\theta - \frac{1}{3}\Phi\left(\frac{\overline{x}_{\mathcal{I}}^1 - \theta}{\sigma_F}\right).$$

- The upper bound is justified by a belief that all types between $\underline{x}_{\mathcal{I}}^1$ and $\overline{x}_{\mathcal{I}}^1$ would invest and the lower bound is supported by believing that no type below $\overline{x}_{\mathcal{I}}^1$ would invest.
- The best response in the best-case scenario is $a_L = \mathcal{I}$ if $\theta > \underline{\theta}_L^2 = 0$ and $a_L = \mathcal{N}$ if $\theta < \underline{\theta}_L^2$.
- The payoff lower bound is strictly increasing in θ; therefore type θ best responds in this case by investing if $\theta > \overline{\theta}_L^2$ and not investing if $\theta < \overline{\theta}_L^2$ where $\overline{\theta}_L^2$ solves $\theta - (1/3)\Phi((\overline{x}_{\mathcal{I}}^1 - \theta)/\sigma_F) = 0$ and $\overline{\theta}_L^2 < \overline{\theta}_L^1$.
- In sum, the leader will delete all $(\theta, \mathcal{I})$ such that $\theta < \underline{\theta}_L^2$ and all $(\theta, \mathcal{N})$ such that $\theta > \overline{\theta}_L^2$.

Rationalizable Behavior

- Now type x_j of follower j reasons that if he finds himself in the investment subgame then the type of the leader must be at least as great as $\underline{\theta}_L^2$.

Rationalizable Behavior

- Now type x_j of follower j reasons that if he finds himself in the investment subgame then the type of the leader must be at least as great as $\underline{\theta}_L^2$.
- He also knows that other followers will invest for sure if their type is greater than $\overline{x}_{\mathcal{I}}^1$.

Rationalizable Behavior

- Now type x_j of follower j reasons that if he finds himself in the investment subgame then the type of the leader must be at least as great as $\underline{\theta}_L^2$.
- He also knows that other followers will invest for sure if their type is greater than $\overline{x}_{\mathcal{I}}^1$.
- It follows that x_j's payoff to investing is lower bounded by

$$\int_{-\infty}^{\infty} \left(\theta - \frac{1}{3}\Phi\left(\frac{\overline{x}_{\mathcal{I}}^1 - \theta}{\sigma_F}\right)\right) \mathrm{d}\Psi^{\mathcal{I}}(\theta; x_j, \underline{\theta}_L^2)$$

if all types below $\overline{x}_{\mathcal{I}}^1$ do not invest, and is upper bounded by

$$\int_{-\infty}^{\infty} \theta \, \mathrm{d}\Psi^{\mathcal{I}}(\theta; x_j, \underline{\theta}_L^2)$$

if all type below $\overline{x}_{\mathcal{I}}^1$ invest.

Rationalizable Behavior

- Now type x_j of follower j reasons that if he finds himself in the investment subgame then the type of the leader must be at least as great as $\underline{\theta}_L^2$.
- He also knows that other followers will invest for sure if their type is greater than $\overline{x}_{\mathcal{I}}^1$.
- It follows that x_j's payoff to investing is lower bounded by

$$\int_{-\infty}^{\infty} \left(\theta - \frac{1}{3} \Phi \left(\frac{\overline{x}_{\mathcal{I}}^1 - \theta}{\sigma_F} \right) \right) \mathrm{d}\Psi^{\mathcal{I}}(\theta; x_j, \underline{\theta}_L^2)$$

  if all types below $\overline{x}_{\mathcal{I}}^1$ do not invest, and is upper bounded by

$$\int_{-\infty}^{\infty} \theta \, \mathrm{d}\Psi^{\mathcal{I}}(\theta; x_j, \underline{\theta}_L^2)$$

  if all type below $\overline{x}_{\mathcal{I}}^1$ invest.
- Both bounds are strictly increasing in x_j.and the upper bound is positive.

Rationalizable Behavior

- Now type x_j of follower j reasons that if he finds himself in the investment subgame then the type of the leader must be at least as great as $\underline{\theta}_L^2$.
- He also knows that other followers will invest for sure if their type is greater than $\overline{x}_{\mathcal{I}}^1$.
- It follows that x_j's payoff to investing is lower bounded by

$$\int_{-\infty}^{\infty} \left(\theta - \frac{1}{3} \Phi \left(\frac{\overline{x}_{\mathcal{I}}^1 - \theta}{\sigma_F} \right) \right) \mathrm{d}\Psi^{\mathcal{I}}(\theta; x_j, \underline{\theta}_L^2)$$

  if all types below $\overline{x}_{\mathcal{I}}^1$ do not invest, and is upper bounded by

$$\int_{-\infty}^{\infty} \theta \, \mathrm{d}\Psi^{\mathcal{I}}(\theta; x_j, \underline{\theta}_L^2)$$

  if all type below $\overline{x}_{\mathcal{I}}^1$ invest.
- Both bounds are strictly increasing in x_j.and the upper bound is positive.
- This implies that x_j's best response in the best-case scenario is always investing (i.e., $\underline{x}_{\mathcal{I}}^2 = -\infty$).

Rationalizable Behavior

- Let $\overline{x}_{\mathcal{I}}^{2}$ be the value of x_j that makes the lower bound equal to zero.

Rationalizable Behavior

- Let $\overline{x}^2_{\mathcal{I}}$ be the value of x_j that makes the lower bound equal to zero.
- Then the worst-case best response for x_j is investing if $x_j > \overline{x}^2_{\mathcal{I}}$ and not investing if $x_j < \overline{x}^2_{\mathcal{I}}$.

Rationalizable Behavior

- Let $\overline{x}_{\mathcal{I}}^2$ be the value of x_j that makes the lower bound equal to zero.
- Then the worst-case best response for x_j is investing if $x_j > \overline{x}_{\mathcal{I}}^2$ and not investing if $x_j < \overline{x}_{\mathcal{I}}^2$.
- Note that $\overline{x}_{\mathcal{I}}^2 > \overline{x}_{\mathcal{I}}^1$ and we can get $\underline{x}_{\mathcal{N}}^2$ and $\overline{x}_{\mathcal{N}}^2$ in the same vein.

Rationalizable Behavior

- Let $\overline{x}_{\mathcal{I}}^2$ be the value of x_j that makes the lower bound equal to zero.
- Then the worst-case best response for x_j is investing if $x_j > \overline{x}_{\mathcal{I}}^2$ and not investing if $x_j < \overline{x}_{\mathcal{I}}^2$.
- Note that $\overline{x}_{\mathcal{I}}^2 > \overline{x}_{\mathcal{I}}^1$ and we can get $\underline{x}_{\mathcal{N}}^2$ and $\overline{x}_{\mathcal{N}}^2$ in the same vein.
- Thus, follower j will delete all type-strategy pairs (x_j, s_j) such that $x_j > \overline{x}_{\mathcal{I}}^2$ and $s_j(\mathcal{I}) = \mathcal{N}$ and such that $x_j < \underline{x}_{\mathcal{N}}^2$ and $s_j(\mathcal{N}) = \mathcal{I}$.

Rationalizable Behavior

- Let $\overline{x}_{\mathcal{I}}^2$ be the value of x_j that makes the lower bound equal to zero.
- Then the worst-case best response for x_j is investing if $x_j > \overline{x}_{\mathcal{I}}^2$ and not investing if $x_j < \overline{x}_{\mathcal{I}}^2$.
- Note that $\overline{x}_{\mathcal{I}}^2 > \overline{x}_{\mathcal{I}}^1$ and we can get $\underline{x}_{\mathcal{N}}^2$ and $\overline{x}_{\mathcal{N}}^2$ in the same vein.
- Thus, follower j will delete all type-strategy pairs (x_j, s_j) such that $x_j > \overline{x}_{\mathcal{I}}^2$ and $s_j(\mathcal{I}) = \mathcal{N}$ and such that $x_j < \underline{x}_{\mathcal{N}}^2$ and $s_j(\mathcal{N}) = \mathcal{I}$.
- Repeating the above procedure yields six sequences: $(\underline{\theta}_L^k, \overline{\theta}_L^k, \underline{x}_{\mathcal{I}}^k, \overline{x}_{\mathcal{I}}^k, \underline{x}_{\mathcal{N}}^k, \overline{x}_{\mathcal{N}}^k)_{k=1}^{\infty}$ where $\underline{\theta}_L^k = 0$, $\underline{x}_{\mathcal{I}}^k = -\infty$, and $\overline{x}_{\mathcal{N}}^k = \infty$ for all k.

Rationalizable Behavior

- Let $\overline{x}_{\mathcal{I}}^2$ be the value of x_j that makes the lower bound equal to zero.
- Then the worst-case best response for x_j is investing if $x_j > \overline{x}_{\mathcal{I}}^2$ and not investing if $x_j < \overline{x}_{\mathcal{I}}^2$.
- Note that $\overline{x}_{\mathcal{I}}^2 > \overline{x}_{\mathcal{I}}^1$ and we can get $\underline{x}_{\mathcal{N}}^2$ and $\overline{x}_{\mathcal{N}}^2$ in the same vein.
- Thus, follower j will delete all type-strategy pairs (x_j, s_j) such that $x_j > \overline{x}_{\mathcal{I}}^2$ and $s_j(\mathcal{I}) = \mathcal{N}$ and such that $x_j < \underline{x}_{\mathcal{N}}^2$ and $s_j(\mathcal{N}) = \mathcal{I}$.
- Repeating the above procedure yields six sequences: $(\underline{\theta}_L^k, \overline{\theta}_L^k, \underline{x}_{\mathcal{I}}^k, \overline{x}_{\mathcal{I}}^k, \underline{x}_{\mathcal{N}}^k, \overline{x}_{\mathcal{N}}^k)_{k=1}^{\infty}$ where $\underline{\theta}_L^k = 0$, $\underline{x}_{\mathcal{I}}^k = -\infty$, and $\overline{x}_{\mathcal{N}}^k = \infty$ for all k.
- Since $\overline{\theta}_L^k$ and $\overline{x}_{\mathcal{I}}^k$ are decreasing and $\underline{x}_{\mathcal{N}}^k$ is increasing, a unique Δ-rationalizable strategy profile obtains when $\overline{\theta}_L^k$ converges to zero, and $\overline{x}_{\mathcal{I}}^k$ and $\underline{x}_{\mathcal{N}}^k$ diverge to $-\infty$ and ∞, respectively.

Unique Rationalizable Behavior

Proposition

There exists a unique $\widehat{\sigma}_F$ *such that the unique monotone equilibrium is uniquely* Δ*-rationalizable if and only if* $\sigma_F > \widehat{\sigma}_F$. *Moreover,* $\widehat{\sigma}_F$ *is strictly increasing in* n.

Rationalizable Behavior

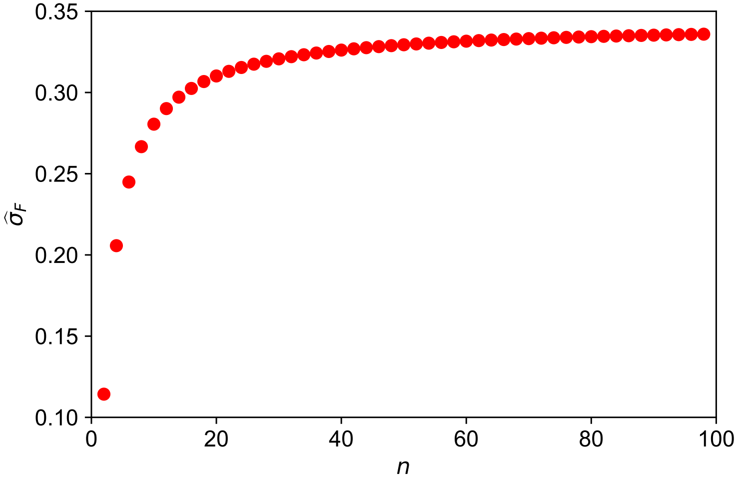


Figure: $\widehat{\sigma}_F$ increases with n.

Intuition-Conditinal Rank Beliefs

- Define

$$R^{\mathcal{I}}(x;\widehat{\theta}_L) = \Pr(x_\ell < x_j \,|\, x_j = x, \theta > \widehat{\theta}_L) = \frac{1}{2}\Phi\left(\frac{x - \widehat{\theta}_L}{\sigma_F}\right)$$

  to be follower j's *conditional rank belief function* under history $h = \mathcal{I}$; that is, the probability follower j assigns to the event that another follower's type x_ℓ is lower than his own ($x_j = x$) given that the leader's type θ is greater than a threshold $\widehat{\theta}_L$.

Intuition-Conditinal Rank Beliefs

- Define

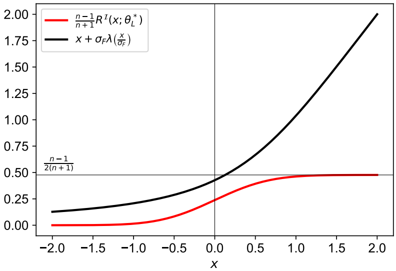
$$R^{\mathcal{I}}(x; \widehat{\theta}_L) = \Pr(x_\ell < x_j \mid x_j = x, \theta > \widehat{\theta}_L) = \frac{1}{2}\Phi\left(\frac{x - \widehat{\theta}_L}{\sigma_F}\right)$$

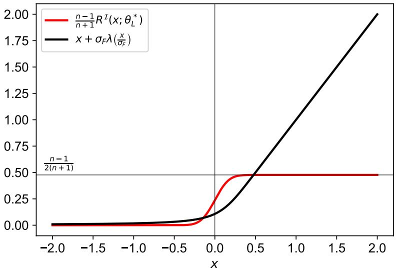
  to be follower j's *conditional rank belief function* under history $h = \mathcal{I}$; that is, the probability follower j assigns to the event that another follower's type x_ℓ is lower than his own ($x_j = x$) given that the leader's type θ is greater than a threshold $\widehat{\theta}_L$.

- Follower's payoff to investing in the investment subgame can be written as

$$\pi_F^{\mathcal{I}}(x; \widehat{\theta}_L, x) = \underbrace{x + \sigma_F \lambda\left(\frac{x - \widehat{\theta}_L}{\sigma_F}\right)}_{\text{expected gross return}} - \underbrace{\frac{n-1}{n+1} R^{\mathcal{I}}(x; \widehat{\theta}_L)}_{\text{expected loss}},$$

Intuition


$\frac{n-1}{n+1}R^{\mathcal{I}}(x;\theta_L^*)$
$x+\sigma_F\lambda(\frac{x}{\sigma_F})$
$\frac{n-1}{2(n+1)}$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x


$\frac{n-1}{n+1}R^{\mathcal{I}}(x;\theta_L^*)$
$x+\sigma_F\lambda(\frac{x}{\sigma_F})$
$\frac{n-1}{2(n+1)}$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x

Intuition

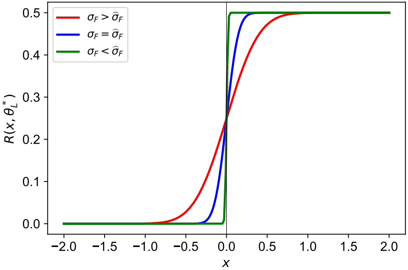


Figure: Conditional Rank Belief

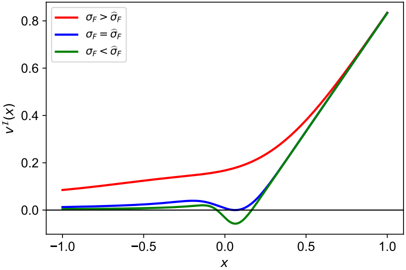


Figure: Expected Payoff from Investing

Stackelberg Selection and Efficiency

- When the unique rationalizable strategy profile obtains, our result can also be interpreted in terms of equilibrium selection.

Corollary (Unique Stackelberg selection)

If $\sigma_F > \widehat{\sigma}_F$, then the signaling game uniquely selects a fully efficient SPE of the complete information game.

Extension: Noisy Observation By the Leader

- Suppose, now, that instead of perfectly learning the state θ, the leader observes a noisy signal $x_L = \theta + \sigma_L \varepsilon_L$ with ε_L being a standard Gaussian noise independent of θ and ε_j for any $j \in F$.

Extension: Noisy Observation By the Leader

- Suppose, now, that instead of perfectly learning the state θ, the leader observes a noisy signal $x_L = \theta + \sigma_L \varepsilon_L$ with ε_L being a standard Gaussian noise independent of θ and ε_j for any $j \in F$.
- This generalization is important from a global games perspective, since perturbations that remove common knowledge of the fact that the leader is perfectly informed may affect the results of the previous analysis.
- Now the strategy of the leader will depend on her signal rather than the true state.

Extension: Noisy Observation By the Leader

- Suppose, now, that instead of perfectly learning the state θ, the leader observes a noisy signal $x_L = \theta + \sigma_L \varepsilon_L$ with ε_L being a standard Gaussian noise independent of θ and ε_j for any $j \in F$.
- This generalization is important from a global games perspective, since perturbations that remove common knowledge of the fact that the leader is perfectly informed may affect the results of the previous analysis.
- Now the strategy of the leader will depend on her signal rather than the true state.
- Thus, the equilibrium will be given by thresholds x_L^*, $x_{\mathcal{I}}^*$, and $x_{\mathcal{N}}^*$

Extension: Noisy Observation By the Leader-Monotone Equilibrium

Proposition

There exists a unique monotone equilibrium. Moreover, this equilibrium converges to that of the signaling game when $\sigma_L \to 0^+$ *(while keep* σ_F *fixed), or when* $\sigma_L \to 0^+$ *,* $\sigma_F \to 0^+$ *and* $\frac{\sigma_L}{\sigma_F} \to 0^+$*.*

Extension: Noisy Observation By the Leader-Rationalizable Behavior

Proposition

The unique monotone equilibrium is uniquely Δ*-rationalizable if* σ_L *and* σ_F *satisfy a condition on the* (σ_L, σ_F)*-space*

Extension: Noisy Observation By the Leader-Rationalizable Behavior

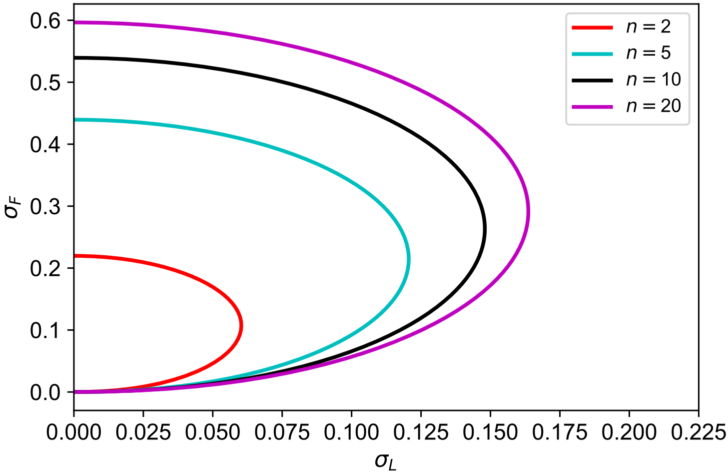


Figure: Curve $\widehat{\sigma}_L(\gamma)$ in the (σ_L, σ_F)-space.

Noisy Observation by the Leader

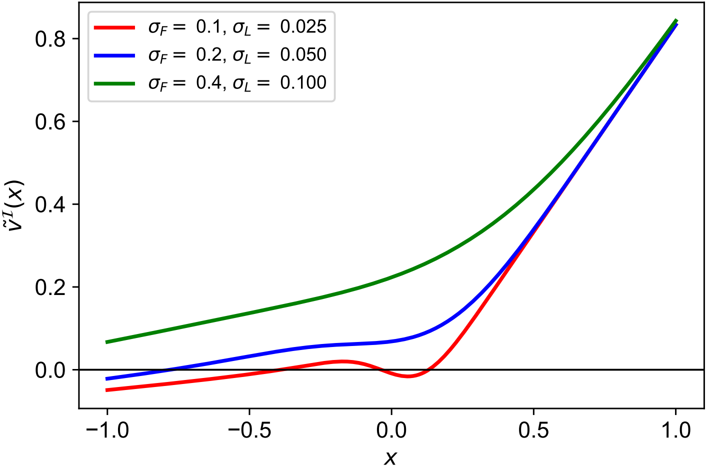


Figure: Function $\tilde{v}^{\mathcal{I}}(x)$ for $n = 2$

Concluding Remarks

- Our results challenge the robustness of the uniqueness results of the global games framework when one moves away from the static benchmark and the exogenous information structure.

Concluding Remarks

- Our results challenge the robustness of the uniqueness results of the global games framework when one moves away from the static benchmark and the exogenous information structure.
- We derive conditions that guarantee the uniqueness of equilibrium behavior and characterize the (in)efficiency of outcomes that may arise.

Concluding Remarks

- Our results challenge the robustness of the uniqueness results of the global games framework when one moves away from the static benchmark and the exogenous information structure.
- We derive conditions that guarantee the uniqueness of equilibrium behavior and characterize the (in)efficiency of outcomes that may arise.
- Implications for applied work may be signifficant if one wishes to use rationalizability as the solution concept.

Concluding Remarks

- Our results challenge the robustness of the uniqueness results of the global games framework when one moves away from the static benchmark and the exogenous information structure.
- We derive conditions that guarantee the uniqueness of equilibrium behavior and characterize the (in)efficiency of outcomes that may arise.
- Implications for applied work may be signifficant if one wishes to use rationalizability as the solution concept.
- In this case, a leader may or may not be able to discipline the game depending on weather certain conditions on the noise of the signals are satisfied, even if she is arbitrarily better informed than the followers.

Thank you!