Debt-Stabilizing Properties of GDP-Linked Securities: A Macro-Finance Perspective

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The views expressed are solely those of the authors and do not necessarily reflect the views of the Banque de France.
The Next New Thing in Finance – Bonds Linked Directly to the Economy,

Introduction
Many Proponents, Few Implementations

- Numerous economists / think tanks call for the introduction of GDP-LBs.
- Could provide the government with an automatic stabilizer to its finance:

  Debt service linked to GDP ⇒ debt-to-GDP volatility ↓ ⇒ sovereign defaults ↓

GDP-L instruments are still the exception

- No case of sovereign issuance where investors take on *symmetrical* risks.
- GDP-linked “warrants” have been issued as part of debt restructuring agreements.
Introduction

Standard analysis

• Debt accumulation process:

\[ D_t = D_{t-1} \exp(rate_t) - BS_t \]
\[ \Rightarrow d_t = d_{t-1} \exp(rate_t - \pi_t - y_t) - bs_t, \]

where BS = budget surplus and \( d \) = debt-to-GDP ratio \((D/(Y \times P))\).

• \( rate_t \) depends on the bonds issued on date \( t-1 \) by the government:

<table>
<thead>
<tr>
<th>Type of bond</th>
<th>( rate_t )</th>
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<tbody>
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With GDP-LBs, \( d_t \) remains constant \((\equiv d_0)\) if \( bs_t \) is set at \( d_0(\exp(r^*) - 1)\).

• Limitations:
  (i) \( i, r \) and \( r^* \) are cst, (ii) only short-term bonds, (iii) unclear pricing assump.
Why should one expect GDP risk premiums to be positive?

Panel (a) – GDP Surprises and Stock Returns

Panel (b) – Realized-over-Expected Real Payoff

- Allow for time-varying rates and bonds of any maturity.
- Standard macro-finance asset-pricing model.
- What are the debt-stabilizing properties of GDP-LBs?
This paper

The model:
- Closed-form (approximated) bond pricing formula.
Main findings:

(a) Countercyclical GDP-LB yields: embed a RP $\sim$ 40 bps on avg.

$[GDP-LB \text{ RP} = \text{avg excess return over inflation-linked bond of same maturity}]$

(b) GDP-LBs make it easier to forecast debt-to-GDP ratios in the SR/MR.

(c) Unclear ability of GDP-LBs to avoid high debt-to-GDP ratios in the LR.

(d) Debt-stabilizing budget surplus: More predictable but higher on avg.
Model
Consumption, GDP and inflation dynamics

• Infinity of identical investors whose consumption $C_t$ follows:

$$c_t \equiv \log C_t = c_{t-1} + g_c + \nu_t,$$

(1)

where $\nu_t$ is an i.i.d. zero-mean shock (c.d.f. is $f$).

• Dynamics of real GDP ($Y_t$) and price index ($P_t$):

$$\Delta y_t = \log \frac{Y_t}{Y_{t-1}} = g_y + \rho_y \nu_t + \varepsilon^y_t,$$

(2)

$$\pi_t = \log \frac{P_t}{P_{t-1}} = \bar{\pi}(1 - \psi) + \psi \pi_{t-1} + \rho_{\pi} \nu_t + \varepsilon^\pi_t,$$

(3)

with $\varepsilon^y_t \sim i.i.d. \mathcal{N}(0, \sigma^2_y)$ and $\varepsilon^\pi_t \sim i.i.d. \mathcal{N}(0, \sigma^2_{\pi})$.

• Dividends:

$$\log Div_t = \log Div_{t-1} + \bar{div} + \rho_d \nu_t + \varepsilon^d_t.$$

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Model

Agents' preferences

- Utility over consumption: ([Campbell and Cochrane, 1999] and [Wachter, 2006])

\[
\mathbb{E}_t \left( \sum_{h=0}^{+\infty} \delta^h \frac{(C_{t+h} - X_{t+h})^{1-\gamma} - 1}{1-\gamma} \right),
\]

where \( X_t \) is the external “habit” stock, defined through surplus consumption \( S_t \):

\[
S_t \equiv \frac{\bar{C}_t - X_t}{\bar{C}_t}
\]

where \( \bar{C} \): aggr. consumption.

- External habit: Small individual investors take \( X_t \) as given (unit mass ⇒ \( \bar{C} \equiv C \)).
- In this context, s.d.f. between dates \( t \) and \( t+1 \):

\[
M_{t,t+1} = \delta \left( \frac{S_{t+1}}{S_t} \frac{C_{t+1}}{C_t} \right)^{-\gamma}.
\]

- \( S_t \) (and therefore \( X_t \) by eq.5) deterministically depends on consumption:

\[
s_{t+1} = \log S_{t+1} = (1 - \phi)\bar{s} + \phi s_t + \lambda(s_t)(c_{t+1} - \mathbb{E}_t(c_{t+1}))
\]

where \( \lambda(s_t) \leq 0 \).
Model
Bond pricing and yields definitions

• Model is discretised and solved. 
• $\Pi :=$ matrix of transition probabilities (across discretised states). 
• Explicit formula for $Q$, the risk-neutral equivalent of $\Pi$. 
$\Rightarrow$ Explicit (discretised) pricing formula.
• Nominal and real interest-rates are given by: 

$$i_{t,h} = F^n_h(s_t) + a_h + b'_h\pi_t$$
$$r_{t,h} = F^r_h(s_t),$$

Approximate recursive solutions for $F^n_h$, $F^r_h$, $a_h$ and $b_h$. 

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We define the maturity-\( h \) GDP-LB “yields” as:

\[
r_{t,h}^* = -\frac{1}{h} \log P_{t,h}^* ,
\]

\( P_{t,h}^* \) is the price of a bond whose nominal payoff (on date \( t + h \)) is \( \propto Y_{t+h} P_{t+h} \).

For \( r_{t,h}^* \) to be comparable to a real yield, the real expected payoff of the GDP-LB has to be 1 (at issuance) \( \Rightarrow \) nominal payoff of

\[
\frac{Y_{t+h} P_{t+h}}{E_t(Y_{t+h}) P_t}.
\]

If no (real) GDP uncertainty, GDP-LB \( \equiv \) ILB, and \( r_{t,h}^* = r_{t,h} \).

Note: Definition of \( r^* \) is purely conventional, not an assumption.

We derive \( F_{h}^*(s_t) \) s.t. \( r_{t,h}^* = F_{h}^*(s_t) \).

GDP-LB pricing
Model, estimation approach and data

Overview

- Estimation approach:
  - Some parameters are calibrated (preference parameters)
  - Some parameters are estimated to fit moments.

- Feasible because
  - tractable formulas for the moments of bonds and stock returns.


- Moments based on:
  - GDP growth, consumption growth, inflation, nominal and real interest rates
  - Proxies of interest-rate conditional variances (based on realized volatilities)
  - Equity price and dividend data
Yield curves

![Graph showing yield curves with labels for various rates and maturity in years.]
Debt-management implications

Debt dynamics

- Debt accumulation process:

\[ D_t = D_{t-1} \exp(rate_t) - BS_t \]

\[ \Rightarrow d_t = d_{t-1} \exp(rate_t - \pi_t - y_t) - bs_t, \]

where BS = budget surplus and \( d = \) debt-to-GDP ratio \((D/(Y \times P))\).

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- With GDP-LBs, \( d_t \) remains constant (\( \equiv d_0 \)) if \( bs_t \) is set at \( d_0(\exp[r^*_t] - 1) \).

- Limitations:
  (i) \( i, r \) and \( r^* \) are time-varying and (ii) bonds of different maturities are issued.
Debt-management implications

Debt dynamics

debt securities accounting

When (i) and (ii) are taken into account, we obtain the (model-free) equation:

\[ d_t = d_{t-1} - bs_t + \]

\[ \sum_{h=1}^{H} \sum_{k=1}^{h} iss^n_{t-k,h} \left[ e^{k(i_{t-k,h} - y_{t-k,t} - \pi_{t-k,t})} - e^{(k-1)(i_{t-k,h} - y_{t-k,t} - 1 - \pi_{t-k,t} - 1)} \right] + \]

\[ \sum_{h=1}^{H} \sum_{k=1}^{h} iss^r_{t-k,h} \left[ e^{k(r_{t-k,h} - y_{t-k,t})} - e^{(k-1)(r_{t-k,h} - y_{t-k,t} - 1)} \right] + \]

\[ \sum_{h=1}^{H} \sum_{k=1}^{h} iss^*_{t-k,h} \left[ e^{k(r^*_{t-k,h} - y^*_{t-k,h})} - e^{(k-1)(r^*_{t-k,h} - y^*_{t-k,h})} \right], \]

(8)

where

- \( iss^n_{t,h} = ISS^n_{t,h} / (Y_t \times P_t) \)
  
  \( ISS^n_{t,h} \): # of nominal bonds of maturity \( h \) issued at time \( t \) (proceeds),

- \( H \) is the maximal bond maturity,

- \( y_{t-k,t} = \frac{1}{k} \log \left( \frac{Y_t}{Y_{t-k}} \right) \), \( y^e_{t,h} = \frac{1}{h} \log \mathbb{E}_t \left( \frac{Y_{t+h}}{Y_t} \right) \) and \( \pi_{t-k,t} = \frac{1}{k} \log \left( \frac{P_t}{P_{t-k}} \right) \).
Debt-management implications
Debt dynamics

- Eq. (8) reflects that:
  - If only nominal bonds have been issued in the past:
    Inflation or real-GDP growth surprises have a negative effect on $d_t$.
  - If only ILBs have been issued in the past:
    A GDP growth surprise has a negative effect on $d_t$.
  - If only GDP-LBs have been issued in the past:
    $d_t$ is immune to date-$t$ inflation or to real-GDP surprises.

- However: $d_{t+K}$ is affected by:
  - future GDP-LB rates $r_{t,h}^*, r_{t+1,h}^*, \ldots, r_{t+K-1,h}^*$
  - future values of nominal and real rates.
Debt-management implications

Debt dynamics

- More generally, $d_t$’s dynamics depend on:
  (a) debt strategy (choice of issuances), (b) $b_s$ and (c) dynamics of interest rates.

- (a) and (b) are determined by the government.

- Natural simulation exercise:
  - Posit a simple process for $b_s$,
  - Consider different issuance strategies.
    (same types of bonds issued in the same proportion over time)
  - Strategies defined across two dimensions:
    (i) Type of bonds (nominal, IL, GDP-L) and (ii) maturities.

- Not a normative approach. No maximized criteria.
  [Bohn, 1990], [Angeletos, 2002], [Buera and Nicolini, 2004].

Smoothing performance and implications of using different debt instruments.
Densities of future debt-to-GDP ratios
Initial debt-to-GDP ratio of 100%, $b_{t-1} \equiv -1\%$

Maturity of bonds: 1 year, Horizon: 2 years

Maturity of bonds: 1 year, Horizon: 20 years

Maturity of bonds: 10 years, Horizon: 2 years

Maturity of bonds: 10 years, Horizon: 20 years
Densities of future debt services (as fractions of GDP)
Initial debt-to-GDP ratio of 100%, $b_s \equiv -1\%$
Debt-management implications
Debt dynamics

- We consider the dual problem: how should $bs_t$ evolve so as to keep $d_t$ constant?

- Examples:
  - **Strategy A:**
    $\mathbb{E}(bs_t)$ is high, but small (predictable) changes in $bs_t$ over time.
  - **Strategy B:**
    $\mathbb{E}(bs_t)$ is lower, but process $(bs_t)$ is volatile.
Debt-stabilizing budget surplus

Dispersion

<table>
<thead>
<tr>
<th></th>
<th>Nominal (N)</th>
<th>Inflation-linked (IL)</th>
<th>GDP-linked (GDP-L)</th>
<th>Maturity: 1 year</th>
<th>Maturity: 5 years</th>
<th>Maturity: 10 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average of debt-stabilizing budget surplus (% of GDP)</td>
<td>-3.5</td>
<td>-3.0</td>
<td>-2.5</td>
<td>-2.0</td>
<td>-1.5</td>
<td>-1.0</td>
</tr>
<tr>
<td>Std. dev. of budget surplus (% of GDP)</td>
<td>1.8</td>
<td>2.0</td>
<td>2.2</td>
<td>2.4</td>
<td>2.6</td>
<td></td>
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</tbody>
</table>
Debt-stabilizing budget surplus
Smoothness
Debt-stabilizing budget surplus
Tail of the distribution

Risk measure: 90th percentile of DSBS

Risk measure: 95th percentile of DSBS
Potential benefits of diversification
International risk-sharing arguments

- Various studies suggest that diversification could reduce GDP-RP
  [Athanasoulis et al., 1999] and [Kamstra and Shiller, 2009].

- GDP-RP could be muted if international business cycles are desynchronised.

- We propose a simple analysis to gauge the potential for GDP risk diversification.
  - Pairwise correlations between GDP surprises for 8 large economies.
  - Low correlations between Brazil, China, India and adv. western economies.
    (scope for diversification)
  - By contrast, strong correlations between adv. western economies.
    (moderate diversification opportunities)

Diversification across adv. western economies unlikely to markedly reduce GDP-RP.
### Correlations between 2-year-ahead GDP surprises

**Panel (a) – 1980-2018**

<table>
<thead>
<tr>
<th></th>
<th>U.S.</th>
<th>U.K.</th>
<th>Germany</th>
<th>France</th>
<th>Canada</th>
<th>Japan</th>
<th>Brazil</th>
<th>China</th>
<th>India</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S.</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>U.K.</td>
<td>0.63</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Germ.</td>
<td>0.21</td>
<td>0.19</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>France</td>
<td>0.52</td>
<td>0.41</td>
<td>0.22</td>
<td>1.00</td>
<td></td>
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<tr>
<td>Canada</td>
<td>0.75</td>
<td>0.66</td>
<td>−0.02</td>
<td>0.60</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Japan</td>
<td>0.55</td>
<td>0.67</td>
<td>0.67</td>
<td>0.38</td>
<td>0.29</td>
<td>1.00</td>
<td></td>
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</tr>
<tr>
<td>Brazil</td>
<td>−0.27</td>
<td>−0.17</td>
<td>0.51</td>
<td>−0.44</td>
<td>−0.45</td>
<td>0.15</td>
<td>1.00</td>
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</tr>
<tr>
<td>China</td>
<td>−0.13</td>
<td>−0.07</td>
<td>−0.12</td>
<td>−0.56</td>
<td>−0.16</td>
<td>−0.27</td>
<td>0.48</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>India</td>
<td>−0.08</td>
<td>0.09</td>
<td>0.11</td>
<td>−0.13</td>
<td>−0.16</td>
<td>−0.01</td>
<td>0.34</td>
<td>0.46</td>
<td>1.00</td>
</tr>
</tbody>
</table>

**Panel (b) – 2000-2018**

<table>
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<th>China</th>
<th>India</th>
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<tr>
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<tr>
<td>U.K.</td>
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<td>1.00</td>
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<tr>
<td>Germ.</td>
<td>0.66</td>
<td>0.64</td>
<td>1.00</td>
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<tr>
<td>France</td>
<td>0.82</td>
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<tr>
<td>Japan</td>
<td>0.91</td>
<td>0.92</td>
<td>0.66</td>
<td>0.72</td>
<td>0.68</td>
<td>1.00</td>
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<td></td>
</tr>
<tr>
<td>Brazil</td>
<td>−0.15</td>
<td>−0.10</td>
<td>−0.21</td>
<td>−0.22</td>
<td>0.14</td>
<td>−0.29</td>
<td>1.00</td>
<td></td>
<td></td>
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<tr>
<td>China</td>
<td>0.02</td>
<td>−0.00</td>
<td>0.06</td>
<td>0.23</td>
<td>0.21</td>
<td>−0.25</td>
<td>0.64</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>India</td>
<td>−0.05</td>
<td>−0.22</td>
<td>0.01</td>
<td>−0.04</td>
<td>0.07</td>
<td>−0.30</td>
<td>0.52</td>
<td>0.81</td>
<td>1.00</td>
</tr>
</tbody>
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Conclusion

• Extension of [Wachter, 2006]'s habit-based model ⇒ pricing of GDP-LB
  ⇒ Extract the term structure of GDP-LB premiums.

• Generalization of the debt accumulation process
  ⇒ Examine the debt-stabilizing properties of GDP-LBs.

• Results:
  (a) Countercyclical GDP-LB yields: embed a RP ∼ 40 bps on avg.
  (b) GDP-LBs make it easier to forecast debt-to-GDP ratios in the SR/MR.
  (c) Unclear ability of GDP-LBs to avoid high debt-to-GDP ratios in the LR.
  (d) Debt-stabilizing budget surplus: More predictable but higher on avg.

Our findings call into question the view that GDP-LBs tame debt.
Risk Premia and Term Premia in General Equilibrium.

Fiscal Policy with Noncontingent Debt and the Optimal Maturity Structure.


GDP-Linked Bonds and Sovereign Default.

The Case for Growth-Indexed Bonds in Advanced Economies Today.
Policy Briefs PB16-2, Peterson Institute for International Economics.

Tax Smoothing with Financial Instruments.

The Case for GDP-Indexed Bonds.
Optimal Maturity of Government Debt without State Contingent Bonds.

Consumption-Based Asset Pricing.


IMF Working Papers 08/109, International Monetary Fund.

Pricing Growth-Indexed Bonds.

Tips from TIPS: The Informational Content of Treasury Inflation-Protected Security Prices.


– Appendix –
### Model parametrisation

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<th>Value</th>
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<tr>
<td>Rate of preference for present</td>
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</tr>
<tr>
<td>Risk aversion parameter</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>Consumption growth</td>
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</tr>
<tr>
<td>( g_c \times 10^2 )</td>
<td>0.550</td>
</tr>
<tr>
<td>( \sigma_\varepsilon \times 10^3 )</td>
<td>5.420</td>
</tr>
<tr>
<td>( p \times 10^2 )</td>
<td>2.117</td>
</tr>
<tr>
<td>( \eta \times 10^2 )</td>
<td>2.496</td>
</tr>
<tr>
<td>GDP growth shocks</td>
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</tr>
<tr>
<td>( \rho_y \times 10^3 )</td>
<td>1.000</td>
</tr>
<tr>
<td>( \sigma_y \times 10^3 )</td>
<td>0.015</td>
</tr>
<tr>
<td>Inflation dynamics</td>
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</tr>
<tr>
<td>( \bar{\pi} \times 10^2 )</td>
<td>0.697</td>
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<tr>
<td>( \psi )</td>
<td>0.981</td>
</tr>
<tr>
<td>( \rho_{\pi} )</td>
<td>0.039</td>
</tr>
<tr>
<td>( \sigma_{\pi} \times 10^3 )</td>
<td>0.661</td>
</tr>
<tr>
<td>Dynamics of consumption ratio</td>
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<tr>
<td>( \phi \times 10^2 )</td>
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<tr>
<td>( b \times 10^2 )</td>
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<tr>
<td>Growth rate of dividends</td>
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<tr>
<td>( \text{div} \times 10^2 )</td>
<td>0.550</td>
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<tr>
<td>( \rho_d )</td>
<td>2.000</td>
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<tr>
<td>( \sigma_d \times 10^3 )</td>
<td>0.031</td>
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## Fit of moments

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<th>Data</th>
<th>Model</th>
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<tr>
<td>Mean of GDP growth rate $\Delta y_t$</td>
<td>$10^2$</td>
<td>0.65</td>
</tr>
<tr>
<td>Mean of consumption growth rate $\Delta c_t$</td>
<td>$10^2$</td>
<td>0.47</td>
</tr>
<tr>
<td>Mean of inflation $\pi_t$</td>
<td>$10^2$</td>
<td>0.54</td>
</tr>
<tr>
<td>Std. dev. of GDP growth rate $\Delta y_t$</td>
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<tr>
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<td>0.48</td>
</tr>
<tr>
<td>Std. dev. of inflation $\pi_t$</td>
<td>$10^2$</td>
<td>0.24</td>
</tr>
<tr>
<td>Mean of short-term nom. rate</td>
<td>$10^2$</td>
<td>2.32</td>
</tr>
<tr>
<td>Mean of 10-year nom. rate</td>
<td>$10^2$</td>
<td>5.43</td>
</tr>
<tr>
<td>Mean of 30-year nom. rate</td>
<td>$10^2$</td>
<td>5.77</td>
</tr>
<tr>
<td>Std. dev. of short-term nom. rate</td>
<td>$10^2$</td>
<td>2.17</td>
</tr>
<tr>
<td>Std. dev. of 10-year nom. rate</td>
<td>$10^2$</td>
<td>2.33</td>
</tr>
<tr>
<td>Std. dev. of 30-year nom. rate</td>
<td>$10^2$</td>
<td>1.95</td>
</tr>
<tr>
<td>Auto-correl. of short-term nom. rate</td>
<td></td>
<td>0.99</td>
</tr>
<tr>
<td>Auto-correl. of 10-year nom. rate</td>
<td></td>
<td>0.99</td>
</tr>
<tr>
<td>Auto-correl. of 30-year nom. rate</td>
<td></td>
<td>0.98</td>
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<tr>
<td>Mean of slope of the nom. yd curve (3m-30yrs)</td>
<td>$10^2$</td>
<td>2.45</td>
</tr>
<tr>
<td>Std. dev. of slope of the nom. yd curve (3m-30yrs)</td>
<td>$10^2$</td>
<td>1.43</td>
</tr>
</tbody>
</table>
Fit of moments (2/2)

<table>
<thead>
<tr>
<th>Moment description</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean of short-term real rate</td>
<td>$10^2$</td>
<td>0.14</td>
</tr>
<tr>
<td>Mean of 2-year real rate</td>
<td>$10^2$</td>
<td>0.08</td>
</tr>
<tr>
<td>Mean of 10-year real rate</td>
<td>$10^2$</td>
<td>1.67</td>
</tr>
<tr>
<td>Std. dev. of short-term real rate</td>
<td>$10^2$</td>
<td>1.95</td>
</tr>
<tr>
<td>Std. dev. of 2-year real rate</td>
<td>$10^2$</td>
<td>1.36</td>
</tr>
<tr>
<td>Std. dev. of 10-year real rate</td>
<td>$10^2$</td>
<td>1.29</td>
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<tr>
<td>Auto-correl. of short-term real rate</td>
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<td>0.81</td>
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<tr>
<td>Auto-correl. of 2-year real rate</td>
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<td>0.86</td>
</tr>
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<td>Auto-correl. of 10-year real rate</td>
<td></td>
<td>0.98</td>
</tr>
<tr>
<td>Mean of slope of the real yd curve (3m-10yrs)</td>
<td>$10^2$</td>
<td>2.01</td>
</tr>
<tr>
<td>Std. dev. of slope of the real yd curve (3m-10yrs)</td>
<td>$10^2$</td>
<td>1.25</td>
</tr>
<tr>
<td>Mean of condi. var. of the short-term nom. rate</td>
<td>$10^5$</td>
<td>1.39</td>
</tr>
<tr>
<td>Mean of condi. var. of the 30-year nom. rate</td>
<td>$10^5$</td>
<td>2.22</td>
</tr>
<tr>
<td>Mean of condi. var. of the 10-year real rate</td>
<td>$10^5$</td>
<td>1.33</td>
</tr>
<tr>
<td>Average expected excess return (annualized)</td>
<td>$10^2$</td>
<td>7.78</td>
</tr>
<tr>
<td>Average cond. volat. of stock return (annualized)</td>
<td>$10^2$</td>
<td>15.69</td>
</tr>
<tr>
<td>Average P/D</td>
<td></td>
<td>40.93</td>
</tr>
<tr>
<td>Std. dev. of P/D</td>
<td></td>
<td>17.32</td>
</tr>
</tbody>
</table>
Consumption shocks ($\nu_t$) p.d.f.

- $\nu_t$ is drawn from a mixture of Gaussian distributions. Specifically:
  \[
  \nu_t = B_t W_{1,t} + (1 - B_t) W_{2,t},
  \]
  where $B_t \sim B(p)$, $W_{1,t} \sim \mathcal{N}(-\eta(1-p), \sigma_{\nu}^2)$ and $W_{2,t} \sim \mathcal{N}(\eta p, \sigma_{\nu}^2)$.

- Alternatively, we have:
  \[
  \nu_t = -(B_t - p) \eta + \varepsilon_{\nu}' , \quad \varepsilon_{\nu}' \sim i.i.d. \mathcal{N}(0, \sigma_{\nu}^2).
  \]

- $p$, $\sigma_{\nu}$ and $\eta$ are calibrated so as to match the first three moments of $\log(C_t/C_{t-4})$. 
Specification of $\lambda(s_t)$

- $S_t$ (and therefore $X_t$ by eq. 5) deterministically depends on consumption:

$$s_{t+1} = \log S_{t+1} = (1 - \phi)\bar{s} + \phi s_t + \lambda(s_t)(c_{t+1} - \mathbb{E}_t(c_{t+1}))$$

$$\lambda(s_t) = \begin{cases} 
\frac{1}{\exp(\bar{s})} \sqrt{1 - 2(s_t - \bar{s})} - 1 & \text{if } s_t \leq s_{\max} \\
0 & \text{otherwise.} 
\end{cases}$$

where $s_{\max} = \bar{s} + \frac{1}{2}(1 - \exp(\bar{s})^2)$, which ensures that $\lambda(s_t) \geq 0$, and with with

$$\bar{s} = \log \left( \sqrt{\text{Var}(\nu)} \frac{\gamma}{1 - \phi - b/\gamma} \right).$$

- This specification notably implies that habit is predetermined at the steady state. Using the notation $x_t = \log(X_t)$, eq. (9) indeed implies that $\frac{dx_{t+1}}{dc_{t+1}} = 1 - \frac{\lambda(s_t)}{\exp(-s_t) - 1}$.

$\Rightarrow$ Habit is predetermined at the steady state if $\lambda(\bar{s}) = \frac{1}{\exp(\bar{s})} - 1$, which is mechanically the case if $\lambda(s_t)$ is specified as in eq. (9).
At each date $t$, $\tilde{s}_t$ is the discretised value of $s_t$: $s_t \approx \tilde{s}_t = \mu' z_t$. 

The dynamics of $\tilde{s}_t$ (i.e. of $z_t$) is defined by the $p_{ij}$’s, where $p_{ij} = \mathbb{P}(z_{t+1} = e_j | z_t = e_i)$. 

$p_{ij}$ defined as the proba. that $s_{t+1} \in \left[ \frac{1}{2} (\mu_{j-1} + \mu_j), \frac{1}{2} (\mu_j + \mu_{j+1}) \right]$ given that $s_t = \mu_i$. 

That is (using $\mu_0 = -\infty$ and $\mu_{N+1} = +\infty$ and eq. 9):

$$p_{ij} = \mathbb{P}(\tilde{s}_{t+1} = \mu_j | \tilde{s}_t = \mu_i) = \mathbb{P}(z_{t+1} = e_j | z_t = e_i)$$

$$= \mathbb{P}\left(s_{t+1} \in \left[ \frac{1}{2} (\mu_{j-1} + \mu_j), \frac{1}{2} (\mu_j + \mu_{j+1}) \right] | s_t = \mu_i \right)$$

$$= f\left( \frac{1}{\lambda_i} \left[ \frac{1}{2} (\mu_j + \mu_{j+1}) - (1 - \phi)\bar{s} - \phi \mu_i \right] \right)$$

$$- f\left( \frac{1}{\lambda_i} \left[ \frac{1}{2} (\mu_{j-1} + \mu_j) - (1 - \phi)\bar{s} - \phi \mu_i \right] \right).$$
• $\Pi = \{p_{ij}\}_{i,j \in [1,N]^2}$ is the matrix of transition probabilities:

$$
\begin{align*}
\begin{array}{ccccccc}
\mu_1 & \mu_2 & \cdots & \mu_j & \cdots & \mu_N \\
\uparrow & \uparrow & \cdots & \uparrow & \cdots & \uparrow \\
\end{array}
\end{align*}
$$

$$
\begin{array}{cccc}
\mu_1 & \mu_2 & \cdots & \mu_j & \cdots & \mu_N \\
\uparrow & \uparrow & \cdots & \uparrow & \cdots & \uparrow \\
\end{array}
\begin{pmatrix}
p_{11} & p_{12} & \cdots & \cdots & p_{1N} \\
p_{21} & p_{22} & \cdots & \cdots & p_{2N} \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
\vdots & \vdots & \cdots & p_{ij} & \cdots \\
p_{N1} & p_{N2} & \cdots & \cdots & p_{NN} \\
\end{pmatrix}.
\end{align*}
$$

• We also introduce vector $\lambda$, whose $i^{th}$ component is $\lambda_i \equiv \lambda(\mu_i)$. 
• The (discretised) s.d.f. $M_{t,t+1}$ (see eq.6) is given by

$$\exp \left( \log(\delta) - \gamma \mathbb{E}(\Delta c) - \gamma \left[ \mu'(z_{t+1} - z_t) + \frac{1}{\lambda' z_t} \left( \mu' z_{t+1} - (1 - \phi) \bar{s} - \phi \mu' z_t \right) \right] \right).$$

• The risk-neutral dynamics of $z_t$ is defined through:

$$\mathbb{E}_t^Q(z_{t+1}) = \mathbb{E}_t \left( \frac{M_{t,t+1}}{\mathbb{E}_t M_{t,t+1}} z_{t+1} \right)$$

$$= \left( \exp \left( -\gamma \left\{ 1 + \frac{1}{\lambda} \right\} \mu' \right) \mathbb{E}_t \left( \exp \left( -\gamma \left\{ 1 + \frac{1}{\lambda} \right\} \mu' \right) \odot \Pi \right) 11' \right)^z.$$ 

$Q$, is the risk-neutral equivalent of $\Pi = \{p_{i,j}\}_{i,j \in [1,N]}$, the matrix of (physical) transition probabilities (see previous slide).
Specific vectorial notations

We use specific vectorial notations. If $M = \{m_{i,j}\}_{(i,j)\in [1,n] \times [1,p]}$:

- $\frac{1}{M} \equiv \left\{ \frac{1}{m_{i,j}} \right\}_{(i,j)\in [1,n] \times [1,p]}$
- $\exp M \equiv \{\exp m_{i,j}\}_{(i,j)\in [1,n] \times [1,p]}$
- $\log M \equiv \{\log m_{i,j}\}_{(i,j)\in [1,n] \times [1,p]}$
- $\sqrt{M} \equiv \{\sqrt{m_{i,j}}\}_{(i,j)\in [1,n] \times [1,p]}$
- If $\alpha$ is a scalar, $M + \alpha \equiv \{\alpha + m_{i,j}\}_{(i,j)\in [1,n] \times [1,p]}$

Let’s denote by $\mu_r$ the vector containing the discretised values of the real short-term rate, i.e. $r_t = \mu'_r z_t$. We have:

$$
\mu_r = -\log(\delta) + \gamma g_c - \gamma (1 - \phi)\bar{s} \frac{1}{\lambda} - \gamma \mu - \gamma \phi \frac{\mu}{\lambda} - \log \left[ \left( \prod \circ \exp \left[ -\gamma \mu \left( 1 + \frac{1}{\lambda} \right) \right] \right) \mathbf{1} \right].
$$
GDP-LB Pricing

Date-\( t \) price of a maturity-\( h \) bond indexed on nominal GDP:

\[
P_{t,h}^* = (H^h 1)' z_t,
\]

with

\[
H = \frac{1}{\mathbb{E} \exp(\rho_y \nu)} \left( \exp \left( \left( -\mu_r - \frac{(1 - \phi) \rho_y \bar{S}}{\lambda} - \rho_y \phi \frac{\mu}{\lambda} \right) 1' + \rho_y \frac{\mu'}{\lambda} \right) \right) \odot Q.
\]

ILB pricing

Yield of a zero-coupon Inflation-Linked bond of maturity \( h \) is given by

\[
r_{t,h} = -\frac{1}{h} \log P_{t,h}^r,
\]

where \( P_{t,h}^r \), the date-\( t \) price of a maturity-\( h \) zero-coupon ILB is given by:

\[
P_{t,h}^r = \left( \{ Q \times \text{Diag}[\exp(-\mu_r)] \}^{h-1} \exp(-\mu_r) \right)' z_t.
\]
Nominal bonds

Let's denote by $P_{t,h}^n$ the date-$t$ price of a zero-coupon nominal bond of maturity $h$. We have

$$P_{t,h}^n = \exp(-b_h \pi_t) F_{h}^{n'} z_t,$$

where $F_h^n$ and $b_h$ are computed recursively using:

$$
\begin{align*}
F_h^n &= -(b_{h-1} + 1) \bar{\pi} (1 - \psi) + \frac{(b_{h-1} + 1)^2 \sigma^2}{2} \\
&+ \left( \exp \left[ \left( -\mu_r + \rho \pi (b_{h-1} + 1) \frac{(1 - \phi) \bar{s}}{\lambda} + \rho \pi (b_{h-1} + 1) \phi \frac{\mu}{\lambda} \right) 1' - \rho \pi (b_{h-1} + 1) \frac{1}{\lambda} \mu' \right] \circ Q \right) F_{h-1}^n \\
b_h &= (b_{h-1} + 1) \psi,
\end{align*}
$$

with the initial conditions $b_0 = 0$ and $F_0^n = 1$.

Nominal-bond yields-to-maturity are further obtained as:

$$i_{t,h} = \frac{b_h}{h} \pi_t - \frac{1}{h} \log(F_h^n)' z_t.$$
Accounting concept of debt

- Eq. (8) is implicitly based on an accounting concept of debt.
- “Nominal valuation of debt securities”: the debt outstanding reflects
  - the sum of funds originally advanced,
  - less any repayments,
  - plus any accrued interest.

[Handbook of Securities Statistics (2015), BIS/ECB/IMF]

- Accrued interest: “debtor approach”, i.e. interest defined from the perspective of the issuer of debt securities:
  Accrued interest based on market interest rate at the time of issuance; indexations based on observations.
  See next slide, with $ky_{t-k,t} = \log \left( \frac{Y_t}{Y_{t-k}} \right)$, $hy_{t,h} = \log \mathbb{E}_t \left( \frac{Y_{t+h}}{Y_t} \right)$ and $k\pi_{t-k,t} = \log \left( \frac{P_t}{\bar{P}_{t-k}} \right)$.  

Zero coupon accounting

Nominal ZC bond:

$$100 \cdot 100 \exp(i_{t,h}) \cdot 100 \exp(ki_{t,h}) \cdot 100 \exp(hi_{t,h})$$

$$t \quad t+1 \quad \ldots \quad t+k \quad \ldots \quad t+h$$

Inflation-Linked ZC bond:

$$100 \cdot 100 \times \exp(r_{t,h} + \pi_{t,t+1}) \cdot 100 \times \exp(kr_{t,h} + k\pi_{t,t+k}) \cdot 100 \times \exp(hr_{t,h} + h\pi_{t,t+h})$$

$$t \quad t+1 \quad \ldots \quad t+k \quad \ldots \quad t+h$$

GDP-Linked ZC bond:

$$100 \cdot 100 \times \exp(r_{t,h} + \pi_{t,t+1}) \times \exp(y_{t,t+1} - y_{t,h}^e) \cdot 100 \times \exp(kr_{t,h} + k\pi_{t,t+k}) \times \exp(k(y_{t,t+k} - y_{t,h}^e)) \cdot 100 \times \exp(hr_{t,h} + h\pi_{t,t+h}) \times \exp(h(y_{t,t+h} - y_{t,h}^e))$$

$$t \quad t+1 \quad \ldots \quad t+k \quad \ldots \quad t+h$$
Debt-management implications

Debt dynamics

- We also have:

\[
\sum_{h=1}^{H} iss_{t,h}^n + iss_{t,h}^r + iss_{t,h}^* = -bs_t +
\]

\(=: iss_t\), debt issued on date \(t\)

\[
\sum_{h=1}^{H} iss_{t-h,h}^n e^{h[t_{t-h,h}-\gamma_{t-h,t}-\pi_{t-h,t}]} + iss_{t-h,h}^r e^{h[r_{t-h,h}-\gamma_{t-h,t}]} + iss_{t-h,h}^* e^{h[r^*_{t-h,h}-\gamma^*_{t-h,h}]}.
\]

Debt maturing on date \(t\)

- Using eqs (8) and (10), one can simulate \(\{d_t\}\) given

(i) paths for macroeconomic variables and interest rates,

(ii) how \(bs_t\) is determined and

(iii) an issuance strategy (how \(iss_t\) is divided into \(iss_{t,h}^n\), \(iss_{t,h}^r\) and \(iss_{t,h}^*\)).
Fiscal fatigue

• Simple case: only one-period bonds issued; \( d_t \) is the single state variable; no \( \pi_t \).
• \( bs_t \) increases to reduce debt when it is high, but “fiscal fatigue” at some point (\( \gamma \)):
  \[
  bs_t = \min(\alpha + \beta d_{t-1}, \gamma)
  \]
• Debt dynamics:
  \[
  d_t = (1 - def_t) d_{t-1} \exp(rate_t - \Delta y_t) - bs_t + \varepsilon_t.
  \]
• Govies: Risk-neutral investors get 100% if no default and 0 if default, hence:
  \[
  \exp(-rate_t) = \exp(-r) \times (1 - p(d_t)),
  \]
  where \( p(d_t) \) is the probability of default and \( r \) is the real risk-free rate.
• \( \exists \bar{d} \) (debt limit) such that \( def_t \iff d_t \geq \bar{d} \). Function \( p(.) \) must satisfy:
  \[
  p(d) = \mathbb{P}\left( \frac{1}{1 - p(d)} d \exp(r - \Delta y_t) - \min(\alpha + \beta d, \gamma) + \varepsilon_t > \bar{d} \right)
  \]
  (fixed-point problem).
Limitations of GDP-LB studies

(i) Simulations rely on constant yields

- Several papers rely on Monte-Carlo simulations to illustrate the budget smoothing properties of GDP-LBs.

- Though these exercises take into account the indexation of the coupons, they abstract from the state-contingency of bond prices (i.e. \( i, r \) and \( r^* \) are cst).

- Examples of potential problems:
  - If part of the inflation fluctuations is predictable, \( i_{t-1} \) is correlated to \( \pi_t \), reducing the variability of \( i_{t-1} - \pi_t \).
  - If GDP yields tend to be higher in low-growth contexts (say), the government then has to issue relatively more GDP-LBs to meet a given funding requirement (+ non-linearities if persistent bad state).

\[ \Rightarrow \] Difficult to appropriately analyse the smoothing properties of these bonds without resorting to a non-trivial asset-pricing dynamic model.
Limitations of GDP-LB studies

(ii) A single (short-term) maturity is considered

- Studies investigating the debt smoothing properties of GDP-LBs consider a single (short-term) maturity.
- However, Treasuries issue large amounts of medium- to long-term bonds.
- As soon as \( i, r \) and \( r^* \) are not constant, the fact that short-term GDP-LBs are better than standard short-term bonds to stabilize the debt-to-GDP ratio does not necessarily imply that the same holds true for long-term bonds.
Limitations of GDP-LB studies

(iii) Unclear GDP-LB pricing

- Several studies use the (market) CAPM to assess the GDP-LB risk premiums.

(*) \( \beta \)'s are obtained by regressing GDP growth on market index returns. Risk premiums are computed as \( \beta \times \) market excess return.

- This approach assumes that the projection of the s.d.f. on stock returns is sufficient to price all GDP risk.

In other words, approach (*) implicitly supposes that the partial correlation between \( \Delta y_t \) & \( \Delta c_t \), controlling for \( s_t \), is zero. This may not be the case [Roll, 1977].

<table>
<thead>
<tr>
<th>Study</th>
<th>R.P.</th>
<th>Approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Barr et al., 2014]</td>
<td>35 bps</td>
<td>CCAPM</td>
</tr>
<tr>
<td>[Kamstra and Shiller, 2009]</td>
<td>150 bps</td>
<td>CAPM</td>
</tr>
<tr>
<td>[Borensztein and Mauro, 2004]</td>
<td>40 bps</td>
<td>CAPM</td>
</tr>
<tr>
<td>[Blanchard et al., 2016]</td>
<td>100 bps</td>
<td>NA</td>
</tr>
<tr>
<td>[Pisani-Ferry et al., 2013]</td>
<td>150 bps</td>
<td>NA</td>
</tr>
<tr>
<td>[Fratzscher et al., 2014]</td>
<td>0 bp</td>
<td>Risk-neutral investors</td>
</tr>
<tr>
<td>[Chamon and Mauro, 2006]</td>
<td>0 bp</td>
<td>Risk-neutral investors</td>
</tr>
</tbody>
</table>

Note: These studies consider short-term debt instruments only.
Yields p.d.f.

Maturity: 2 years

\[ \begin{array}{cccccc}
0.00 & 0.02 & 0.04 & 0.06 & 0.08 \\
\end{array} \]

GDP-LB yields
ILB yields

Maturity: 10 years

\[ \begin{array}{cccccc}
0.00 & 0.02 & 0.04 & 0.06 & 0.08 \\
\end{array} \]

GDP-LB yields
ILB yields

Note: Vertical lines indicate means.
Sensitivity of yields to surplus

![Graph showing sensitivity of yields to surplus](image)

- **Consumption surplus ($S_t$)**
- **Annualized yield to maturity**
- **GDP-LB (1 year)**
- **GDP-LB (10 years)**
- **ILB (1 year)**
- **ILB (10 years)**
- **p.d.f. of consumption surplus ($S_t$)**
Simulated paths of debt-to-GDP ratios
Initial debt-to-GDP ratio of 100%, $b_{st} \equiv -1\%$
Additional issues associated with GDP-LBs

- Novelty premiums.
  [Chamon et al., 2008]: Argentine GDP warrants, 500 bps.
  [D'Amico et al., 2018]: TIPS, 100 bps (“shadow real yields”, ATSM).

- Adverse selection: First issuers could be suspected to hide weak fundamentals.

- Significant issuances could lead to a reduction in the supply of (safe) conventional assets (that markets need).

- Significant issuances could transfer excessive risk to the private sector (increase in business- or financial-cycles volatility + worsening sovereign-bank nexus).

- Risk of moral hazard.
  - Lower incentives for the government to implement growth policies
  - Higher incentives for issuers to manipulate data in their favour.
The sources of risk premiums: Payoff exposures

- Two-year investment.
- Three possible types of (zero-coupon) bonds: Inflation-Linked (IL), nominal, GDP-L.
- Expected real payoffs:

<table>
<thead>
<tr>
<th>Type of bond</th>
<th>Expected real payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflation-linked bond with (real) face value of 100</td>
<td>100</td>
</tr>
<tr>
<td>Nominal bond with (nominal) face value of 100</td>
<td>$\mathbb{E} \left[ \frac{100}{(1 + \pi_{t,t+2})^2} \right]$</td>
</tr>
<tr>
<td>GDP-LB with “face value” of 100</td>
<td>$\mathbb{E} \left[ 100 \times (1 + y_{t,t+2})^2 \right]$</td>
</tr>
</tbody>
</table>

$\pi_{t,t+2}$ (resp. $y_{t,t+2}$): annualized growth rate of price index (GDP) between $t$ and $t + 2$.

- Next slide: ratio between realized and expected payoffs of 1-year bonds with unit face value (expectations measured using US SPF).
Choosing the model

- Model has to be flexible enough to capture moments of observed data (term structures of nominal and real yields, GDP growth, inflation).
- Model must feature a sufficient amount of structure for it to generate realistic GDP-LB prices.
  - GDP-LB share important features with stocks: Higher (lower) payoffs in expansions (recessions).
  - Dividends often modelled as levered consumption, i.e. $Div_t = C_t^\lambda$ as in [Abel, 1999, Campbell and Cochrane, 1999, Campbell, 2003].
    - Calibration includes stock-based data moments.
- Chosen model and estimation approach:
  - Extended version of [Wachter, 2006].
    - Key ingredient: consumption habits.
  - Closed-form (approximated) bond pricing formula.