Fragility, Multiplicity, and Fire Sales with Debt, and Default

Thinking Outside the Edgeworth Box

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Fragility and Multiplicity: Macro-Finance?

- What does Debt have to do with Fragile Equilibria (small shocks cause big changes)?
- What do Debt and Default have to do with Multiple Equilibria?
- What does Debt have to do with fire sales? What is a GE fire sale?

- Want simple (obvious) answer without any institutional details.
- Want a micro, general equilibrium story, not a special macro-finance model
- Want a Theorem, with L goods and H agents, and no special assumptions.
- Explain *the* GE Fragility-Multiplicity Mechanism

In Debt Crises, Leaders often suggest that renewed confidence can restore the economy

- J.P. Morgan in 1907, and idea of Lender of Last Resort (making guarantees that don't need to be used much; see Bagehot))
- Franklin Roosevelt 1933 : "The only thing we have to fear is fear itself."
- Geithner, Secretary of the Treasury "Stress Tests" after 2008
- Bernanke, Chairman of FED, "Courage to Act" after 2008
- Mario Draghi "The ECB is prepared to do whatever it takes to restore the Euro. And believe me, it will be enough." 2012
- Greek ministers in 2016 saying needed to restore confidence

- They believe either that a little govt action can make a huge difference, moving global economy from crisis to normal, or believe in multiple equilibria. Like fragility or multiple equilibria.

Today's Story:
Debt Causes Fragility and Multiplicity

- Want simple answer without any institutional details. No collateral, no volatility changes, no asymmetric information, no common resources.
- Want a theorem.

- Do General Equilibrium with Negative Endowments (Debts)

Debt Makes Income Effects More Important.
Income Effects create Fragility, Multiplicity, and Fire Sales.

Debt entails the possibility of Default.
That's why nobody systematically studied GE with
a priori debt, i.e. endowments outside the Edgeworth Box.

Want answers that don't depend on
how people choose to default.

- Standard one period Walrasian Economy
- With one twist: allow for debt = negative endowments

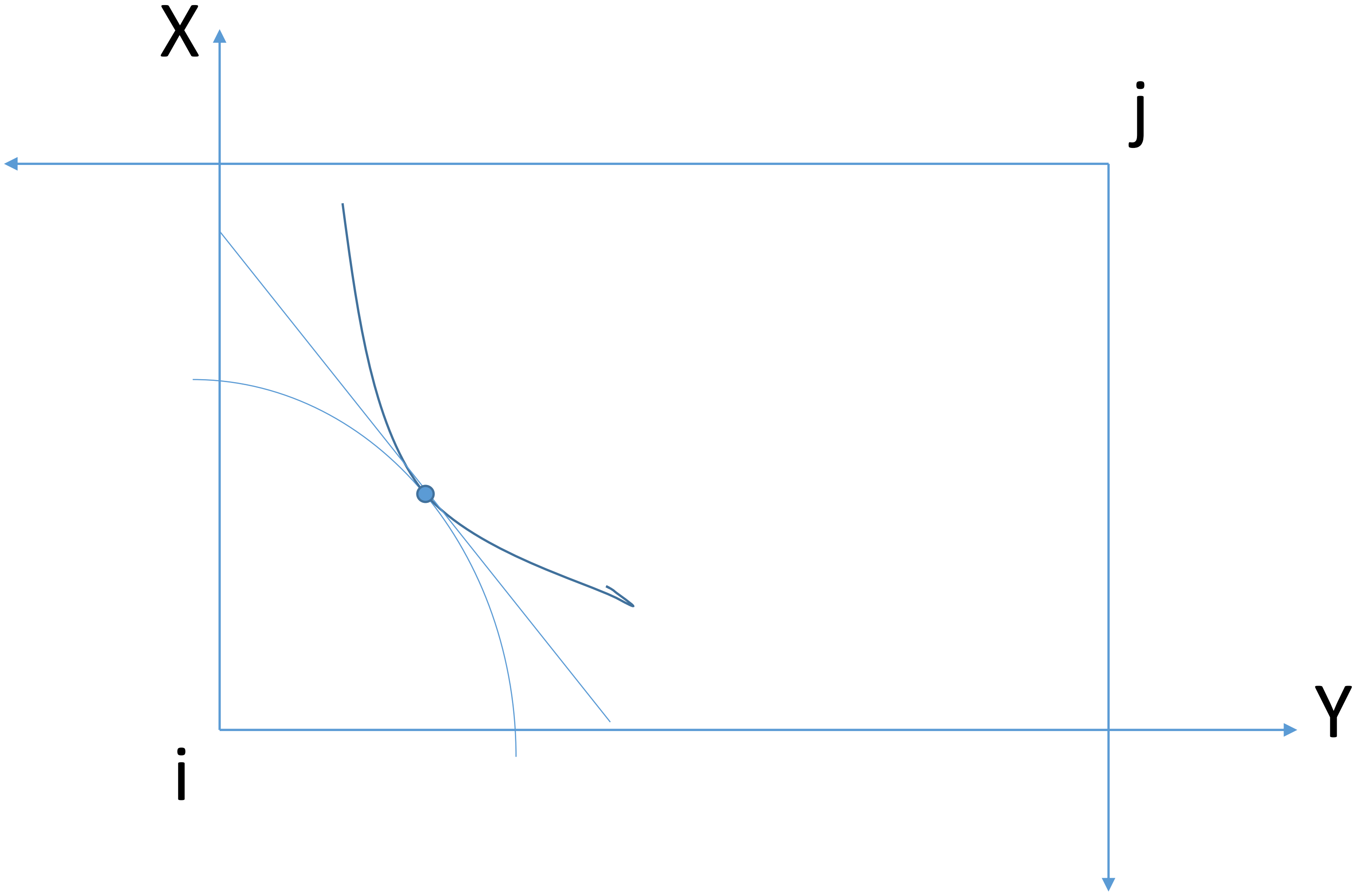
Thinking outside the Edgeworth Box

- Return to old fashioned General Equilibrium
- But with negative endowments (debts).
- Start with Special Case: 2 goods, 2 people.
- Edgeworth Box with Endowments Outside the Box.
- Finish with general L goods, H agents case.

Walrasian Debts?: Literature

- Green (1974) considered a priori debts with a specific default rule as part of temporary equilibrium. Eilenberg and Noe (2001) same model.
- Gottardi and Kajii (1999) also wrote about sunspot equilibria and multiplicity in a two period economy that is closest.
- Cass-Pavlova considered Cobb-Douglas example with a very specific real debt, which I describe later. Thorston Hens used this example to talk about sunspots.
- Balasko-Shell looked at arbitrary nominal debts, and observed that high enough price levels would make the real debt so small that it was just like a perturbation to endowments.

Edgeworth Box

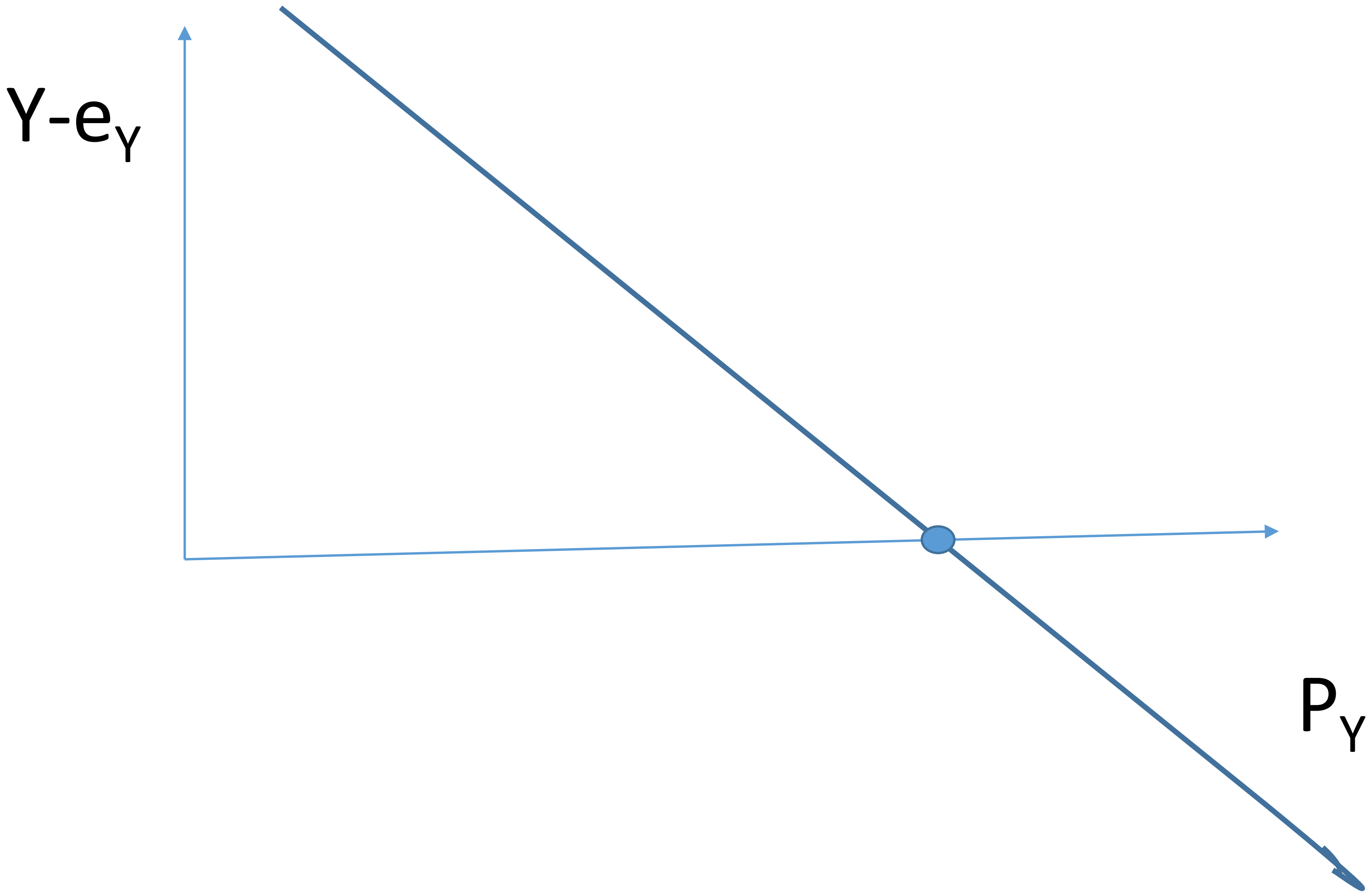


Unique Equilibrium is identical to initial endowments. No trade.

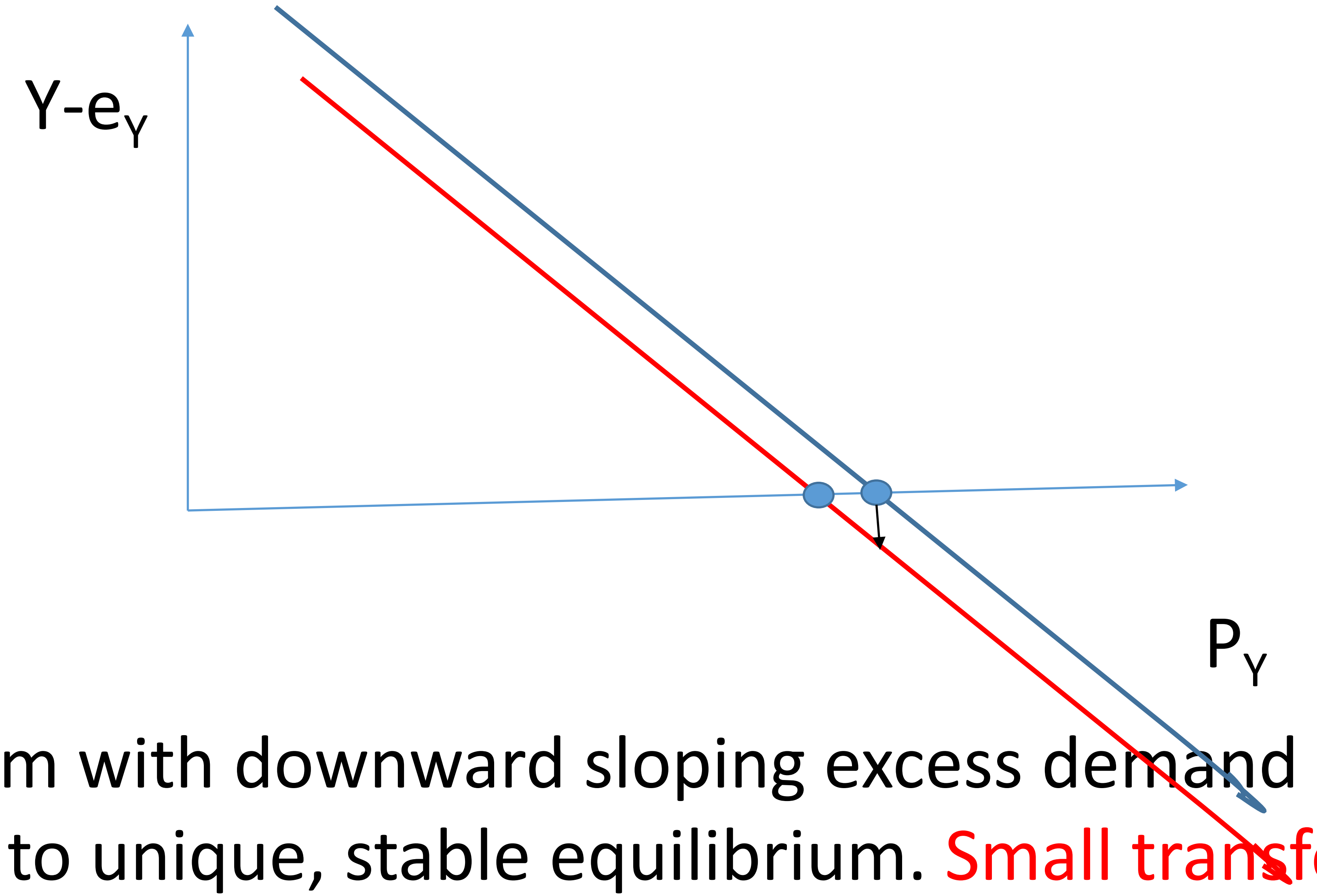
Slutsky Equation with no trade for each agent h:

$$dy^h/dp_y = dy^h/dp_y|_{subs}$$

Substitution effect is negative. Income effect is zero. Thus demand is downward sloping.

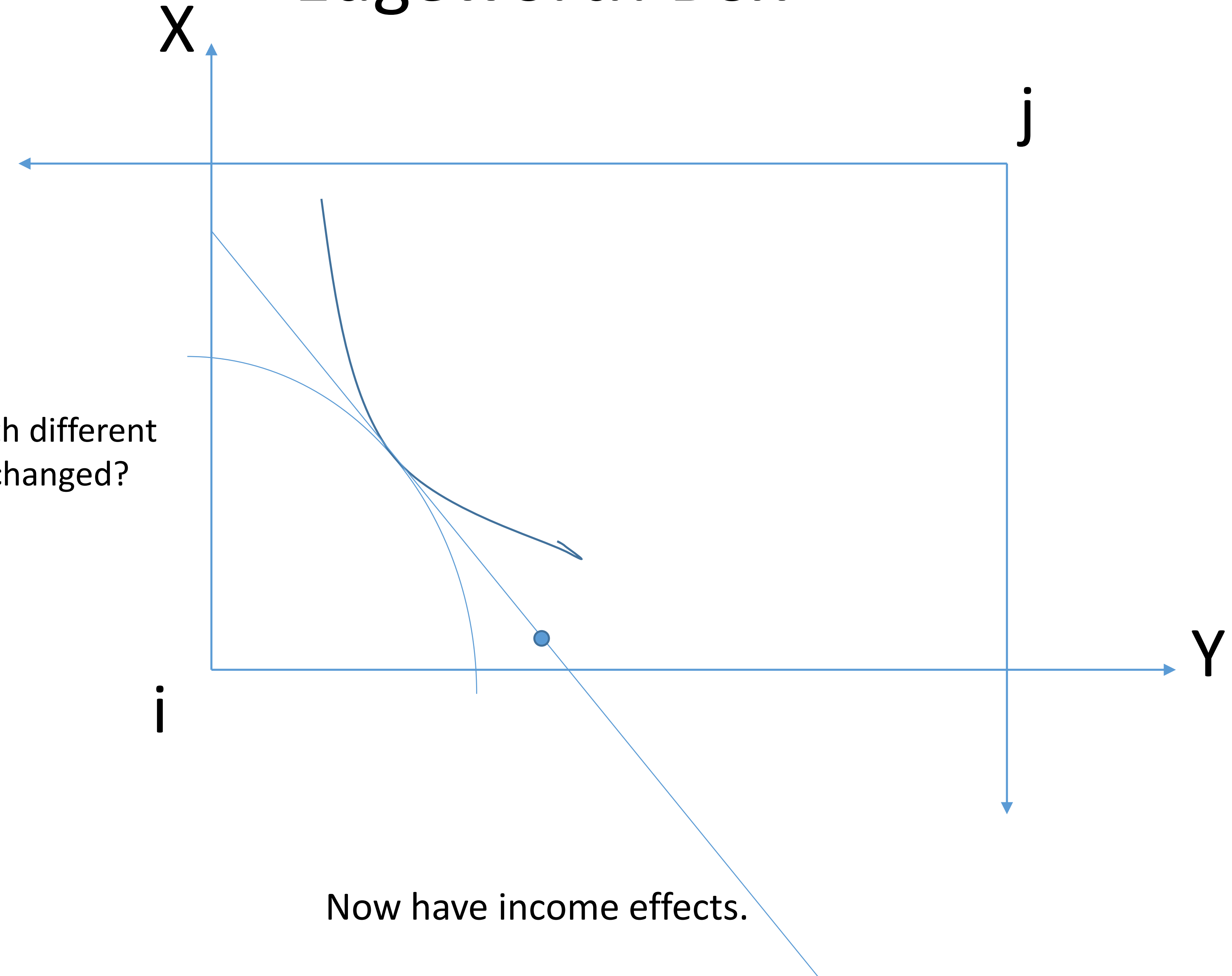


Equilibrium with downward sloping excess demand gives rise to unique, stable (regular) equilibrium.



Equilibrium with downward sloping excess demand gives rise to unique, stable equilibrium. Small transfer from i to j or change in endowments gives small change in price.

Edgeworth Box



Same equilibrium with different Endowments. What changed?

Now have income effects.

Slutsky Equation for each agent h:

$$dy^h/dp_y = dy^h/dp_y|_{subs} + MPC^h_y(e^h_y - y^h)$$

Substitution effect is negative.

Income effect reinforces substitution effect for buyer j of Y.
(For normal goods.)

Income effect counteracts substitution effect for seller i of Y.
(For normal goods.)

MPC = marginal propensity to consume out of income.

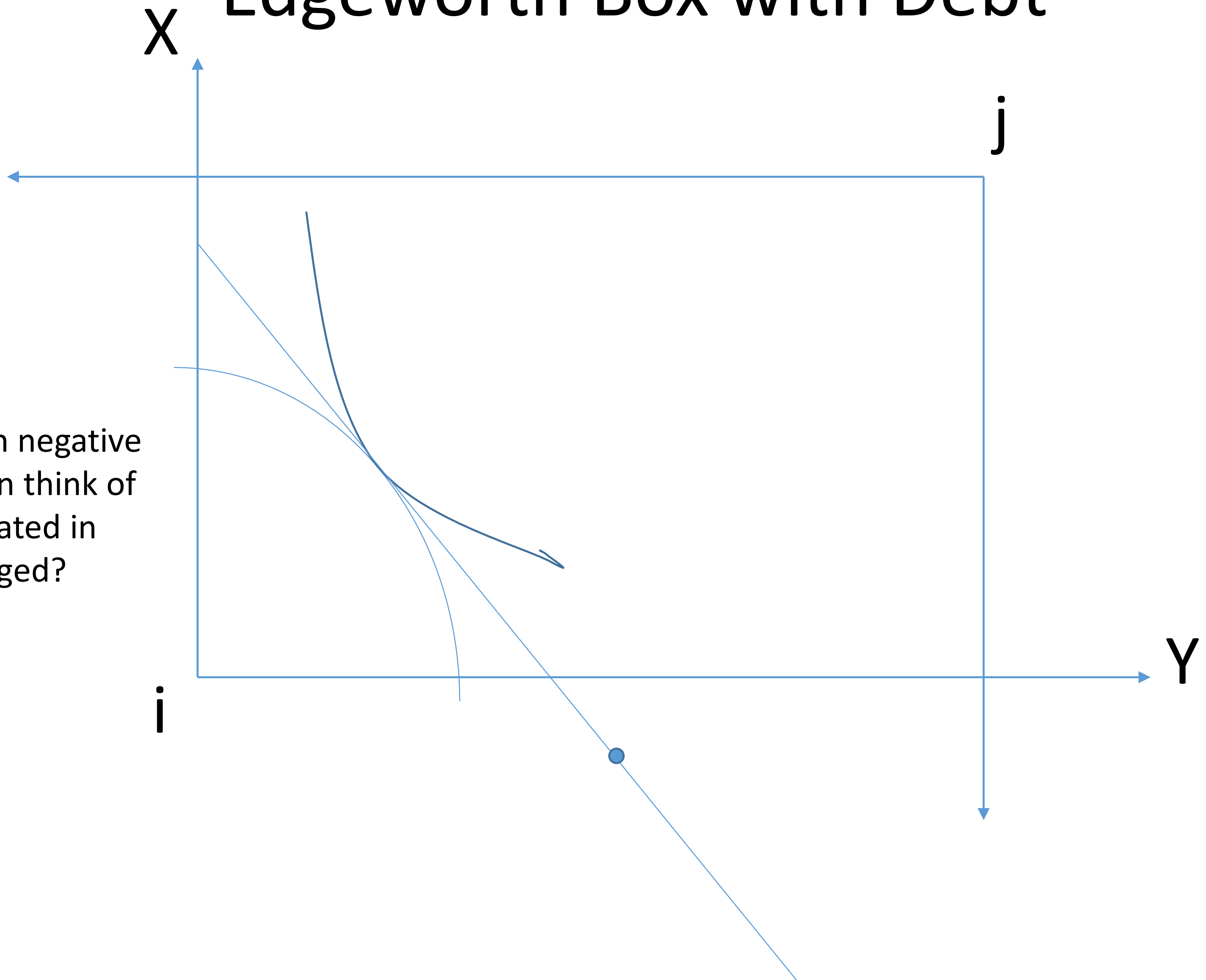
Aggregate Slutsky Equation at equilibrium:

$dy/dp_y = dy/dp_y|_{subs} + (MPC^i_y - MPC^j_y)(e^i_y - y^i)$

Probably these income effects offset, and get substitution effect dominating. Likely high MPC agent is buying. Substitution effect must be dominating for endowments near equilibrium.

In simplest cases like Cobb-Douglas economies, still get unique eq. If substitution effect always dominates, then must get unique equilibrium.

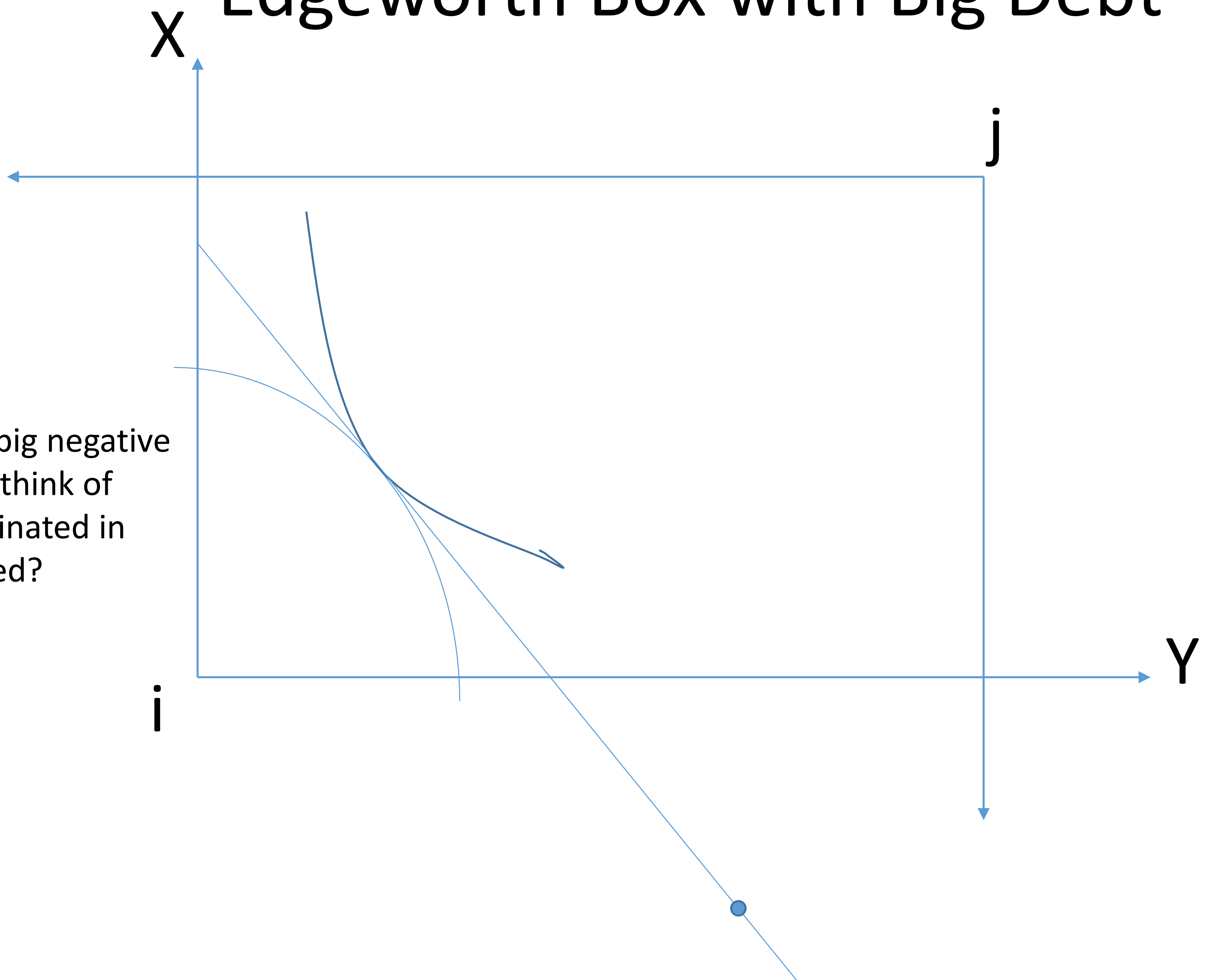
Edgeworth Box with Debt



Same equilibrium with negative Endowments. Now can think of this as debt, denominated in Units of X. What changed?

Now agent i with negative endowment of X must be the buyer of X.
Hence i is a forced seller of Y.
No presumption that buyer of Y has higher MPC at final consumption.
In fact, debt was probably written in X so that agent could get more Y.
At least half the time i has higher MPC for Y and lower MPC for X.

Edgeworth Box with Big Debt



Same equilibrium with big negative Endowments. Now can think of this as big debt, denominated in Units of X. What changed?

Slutsky Equation for agent h:

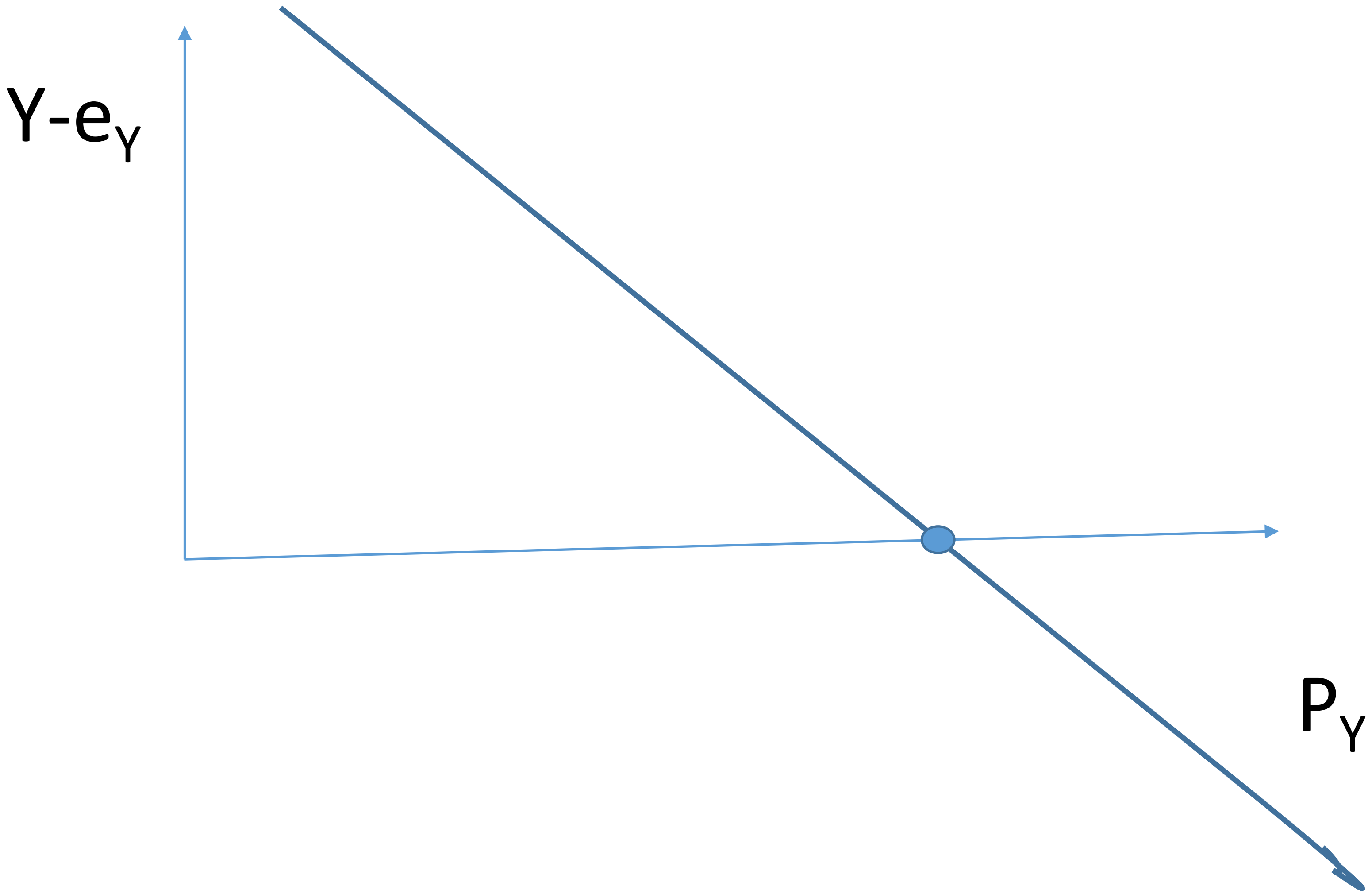
$$dy^h/dp_y = dy^h/dp_y|_{subs} + MPC^h_y(e^h_y - y^h)$$

As endowment moves along budget line, substitution effect is unchanged, because it is calculated at consumption, which is unchanged. But income effect gets bigger.

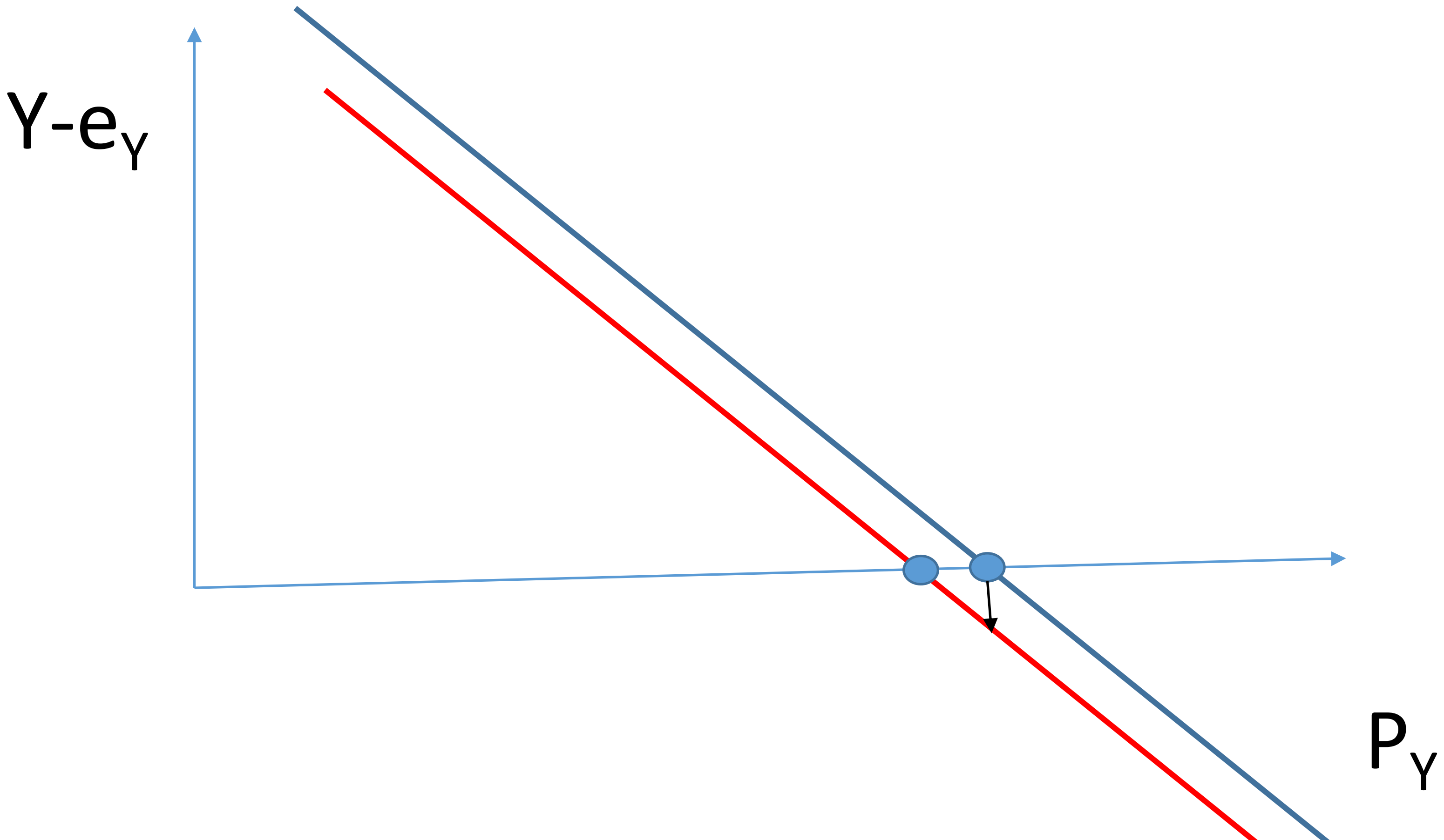
Aggregate Slutsky Equation at equilibrium:

$$dy/dp_y = dy/dp_y|_{subs} + (MPC^i_y - MPC^j_y)(e^i_y - y^i)$$

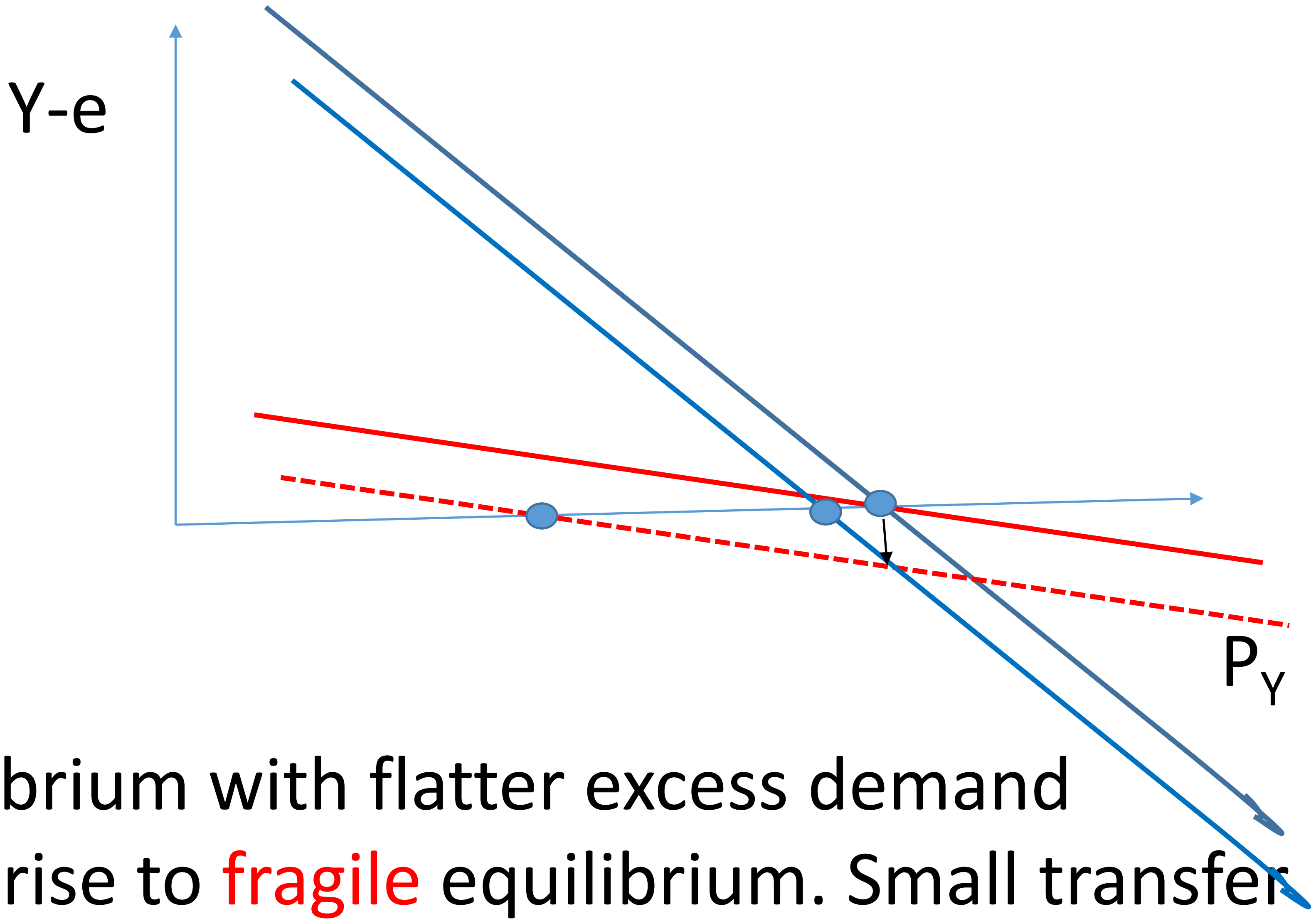
As endowment moves along budget line, substitution effect is unchanged. But income effect gets bigger. So if agent i who sells Y has higher marginal propensity to consume Y, the income effect can partly reverse the substitution effect, flattening the excess demand curve.



Equilibrium with downward sloping excess demand gives rise to unique, stable equilibrium.



Equilibrium with downward sloping excess demand gives rise to unique, stable equilibrium. Small transfer from i to j or change in endowments gives small change in price.

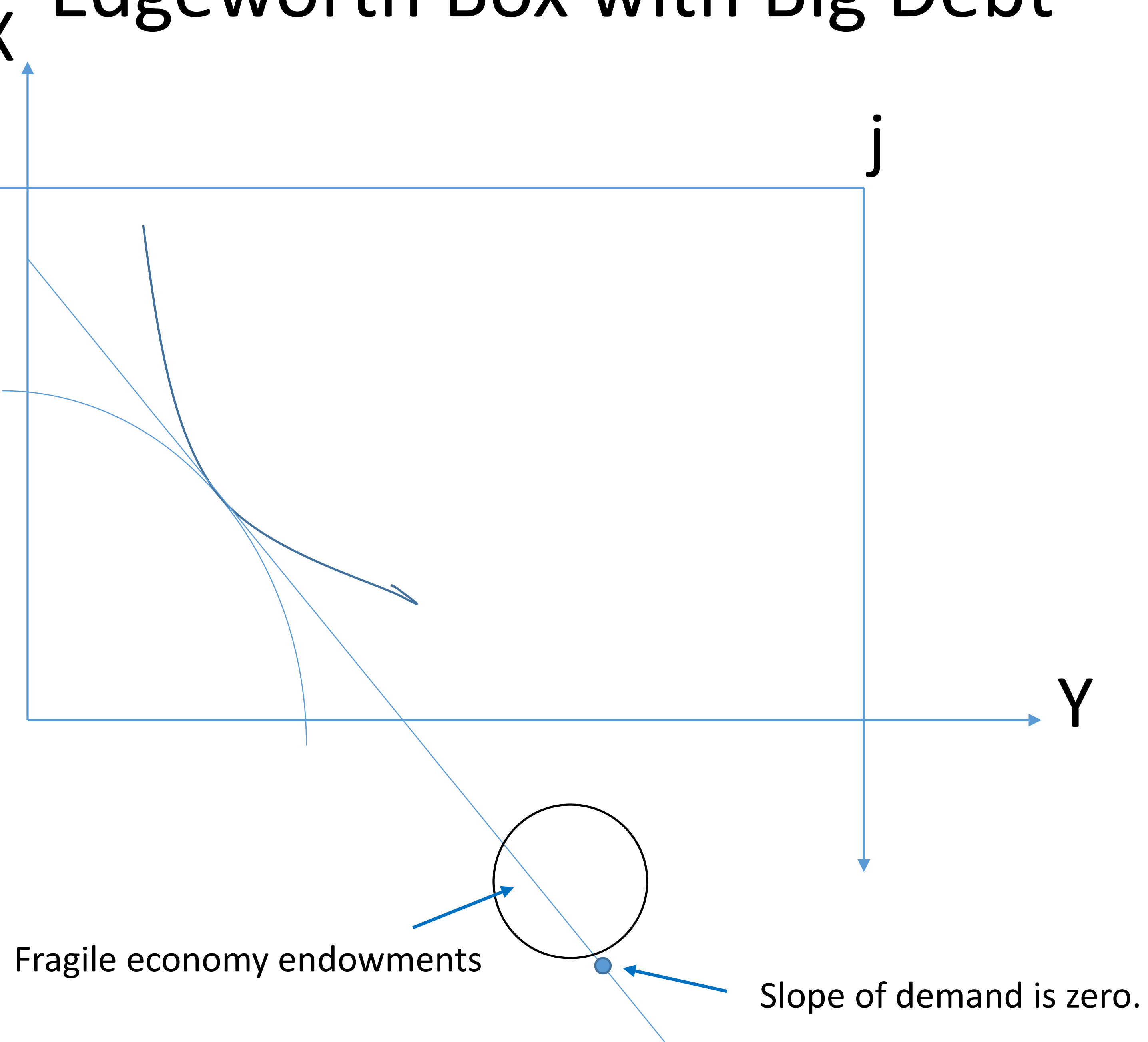


Equilibrium with flatter excess demand gives rise to fragile equilibrium. Small transfer from i to j or change in utility of Y gives large drop in price.

Fragility Theorem for Edgeworth Box economy

- Fix any 2-person, 2-good (Edgeworth Box) Walrasian economy with smooth preferences and positive endowments.
- Suppose that at some allocation on the Pareto frontier, the marginal propensity of i to consume Y out of extra income is greater (at the unique supporting prices) than the marginal propensity of j to consume Y.
- Fix any sensitivity $A > 0$.
- Then there is an open set G of endowments with debts (outside the Edgeworth Box) such that for any endowments e in G, there is an equilibrium price $p(e) = (p_Y(e), p_X(e)) = (p_Y(e), 1)$ at which there is no default and such that $|\partial p_Y(e)/\partial e_n| > A$ for some coordinate n.

Edgeworth Box with Big Debt



Same equilibrium with big negative Endowments. Now can think of this as big debt, denominated in Units of X. What changed?

Micro Story of Fire Sales = Fragility

- If agent i has higher marginal propensity to spend out of income on Y, and
- If agent i is selling enough Y in equilibrium
- Then equilibrium will be fragile: small shock to demand or supply of Y, e.g. via value of Y or via i's wealth, causes big drop in p_Y.

- All this can be accomplished by supposing that agent i with higher marginal propensity to consume Y owes a big enough debt in X. Big debt forces i to sell lots of Y in equilibrium to get positive consumption of X.
- Doesn't necessarily happen in standard Walrasian economies because sales of Y not big enough. Need to move outside the box.

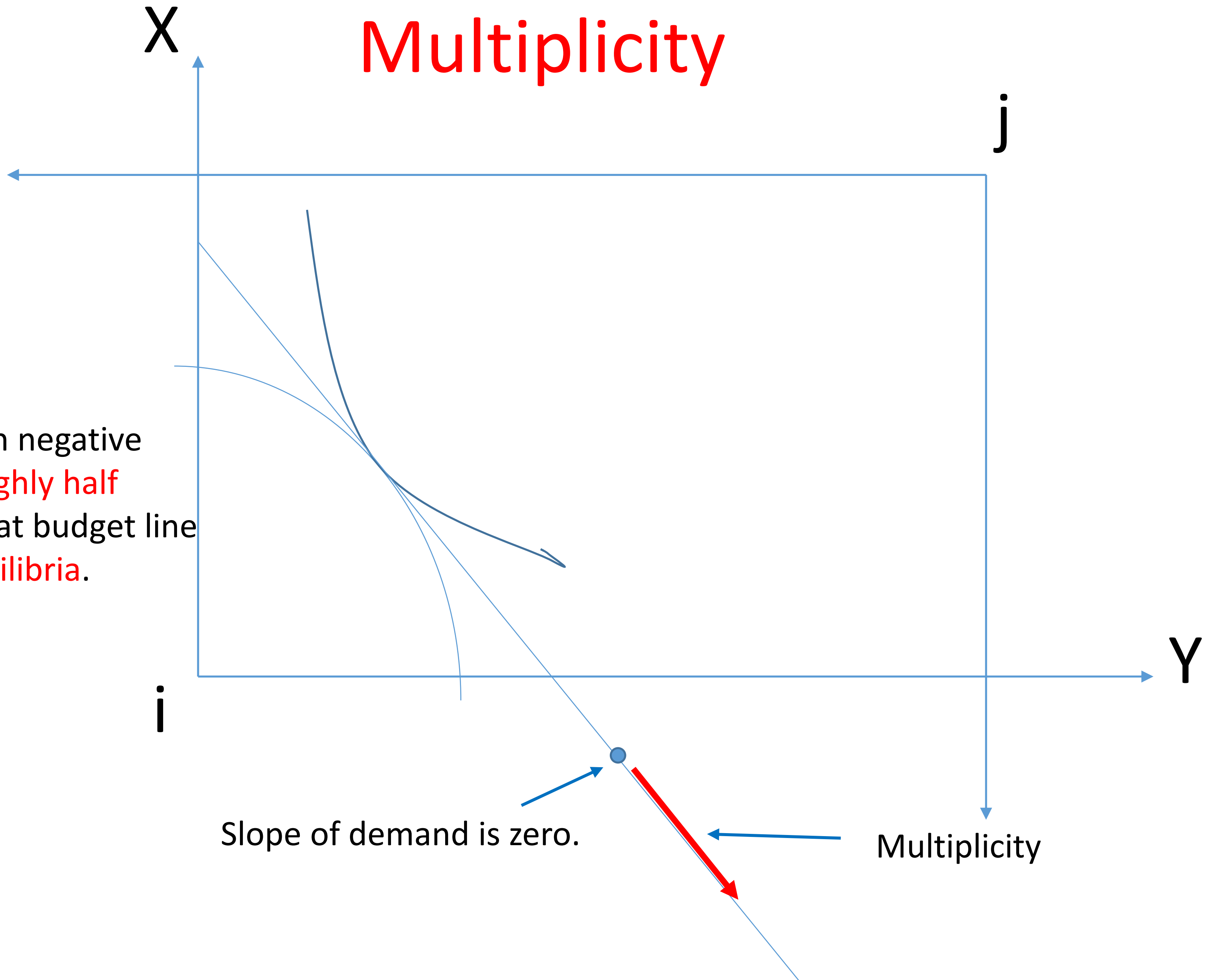
Standard Macro/Finance Story: Fire Sales

- If agent i is high valuation (high marginal utility) buyer of houses, and
- If agent i has a large debt and is holding lots of houses
- Then a small shock to i's wealth (endowment of food) can force i to sell houses, causing a fire sale of houses (sales reduce price so more sales to pay debt so more sales) and a huge drop in the price of housing, because the housing must be held by low valuation j.

- Macro story makes debt central, and different valuations (corner solution) central.

- Micro story of fragility eliminates differences in valuation (marginal utility). Replaces it with differences in MPC.
- Micro story reveals that big debt is not necessary, it is a sufficient instrument for creating the micro premises for fragility, i.e. for causing the agent with the higher marginal propensity to consume houses to be a big seller of houses. Debt not a qualitatively new factor, but a element that magnifies traditional factors.

Multiplicity

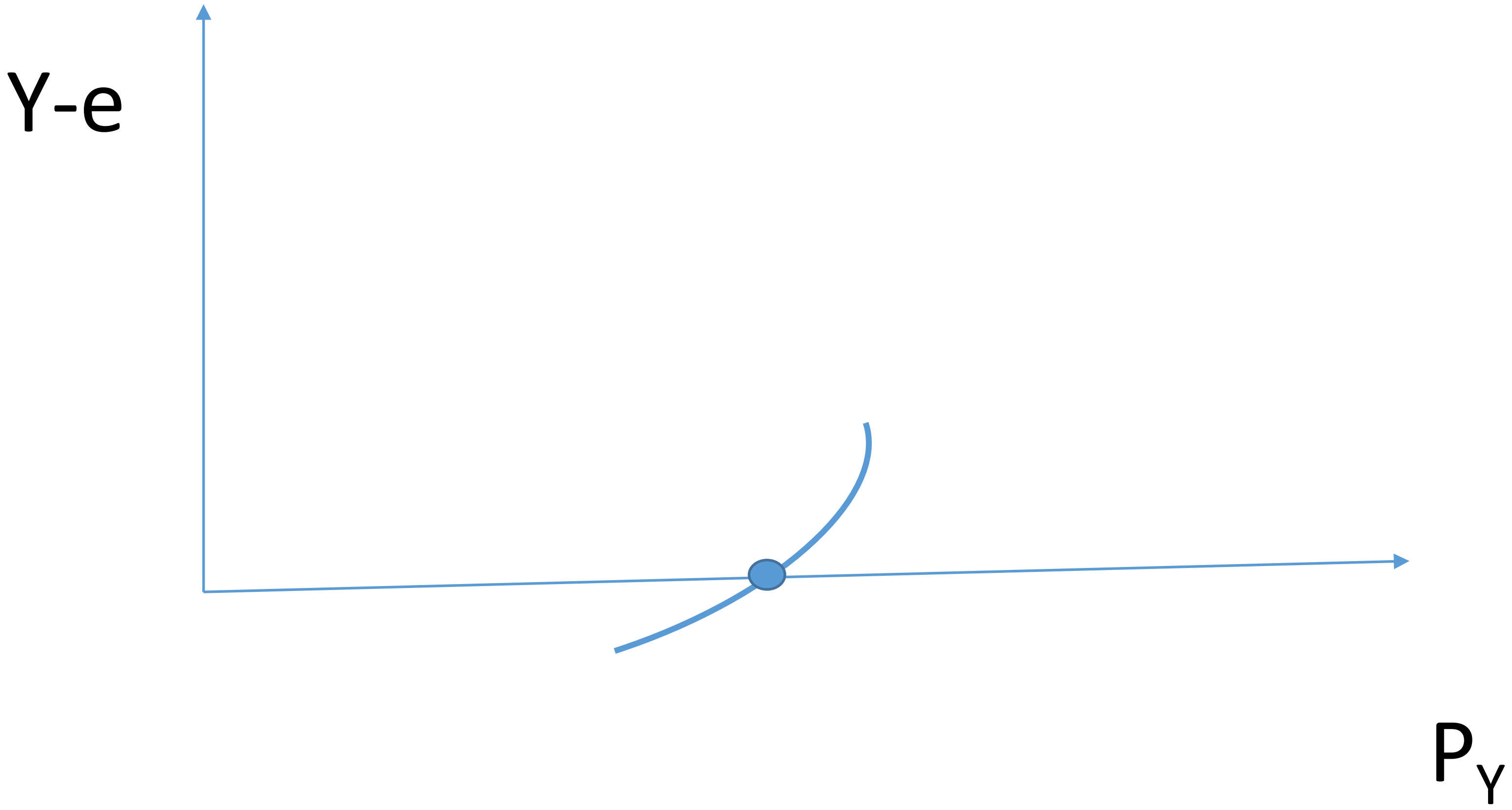


Same equilibrium with negative Endowments. For roughly half of endowments on that budget line get multiplicity of equilibria.

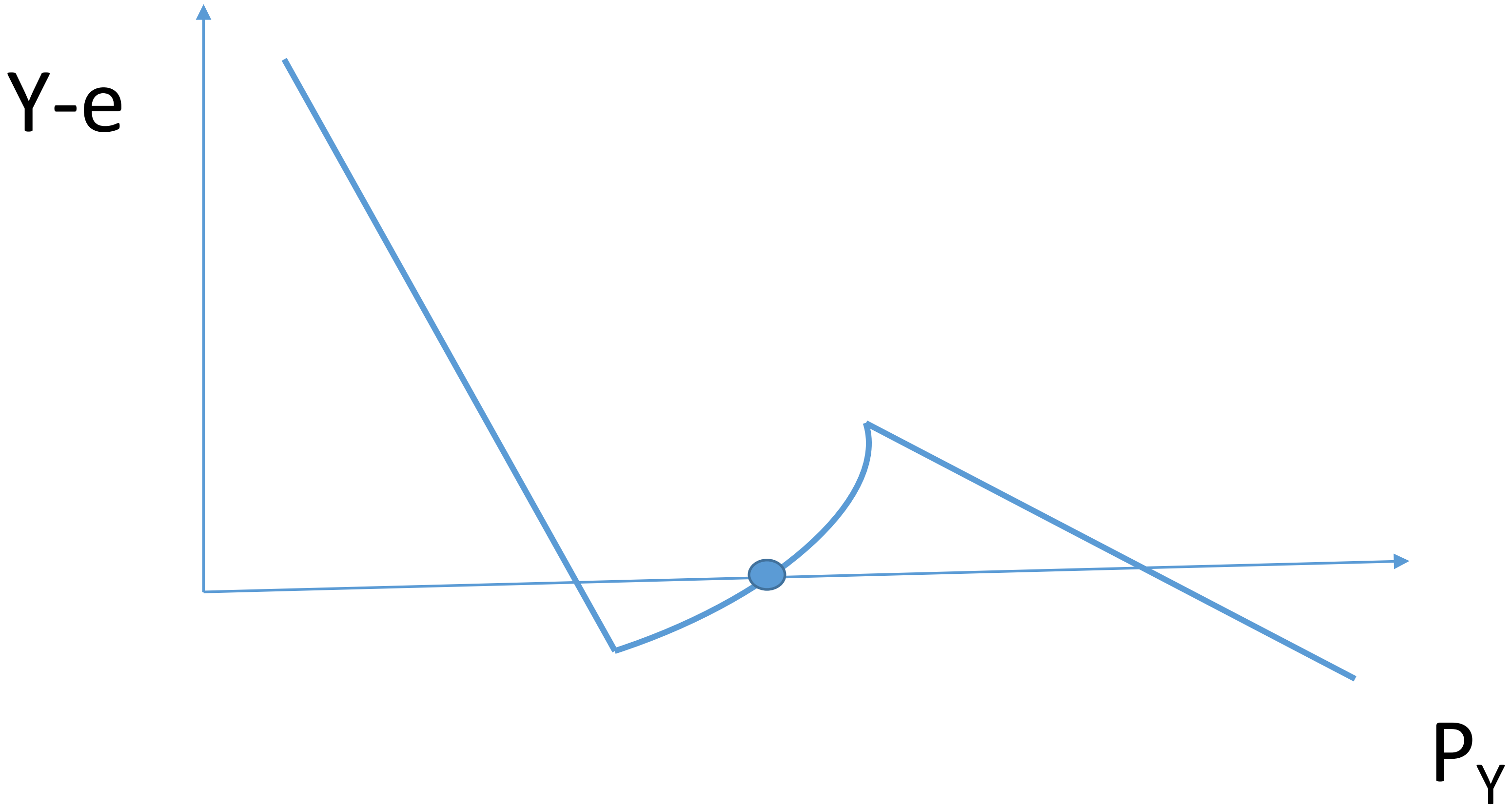
Aggregate Slutsky Equation at equilibrium:

$$dy/dp_y = dy/dp_y|_{subs} + (MPC^i_y - MPC^j_y)(e^i_y - y^i)$$

As endowment moves further along budget line, substitution effect is unchanged. But income effect gets bigger. So if agent i who sells Y has higher marginal propensity to consume y, the income effect will eventually reverse the substitution effect, creating a rising excess demand curve.

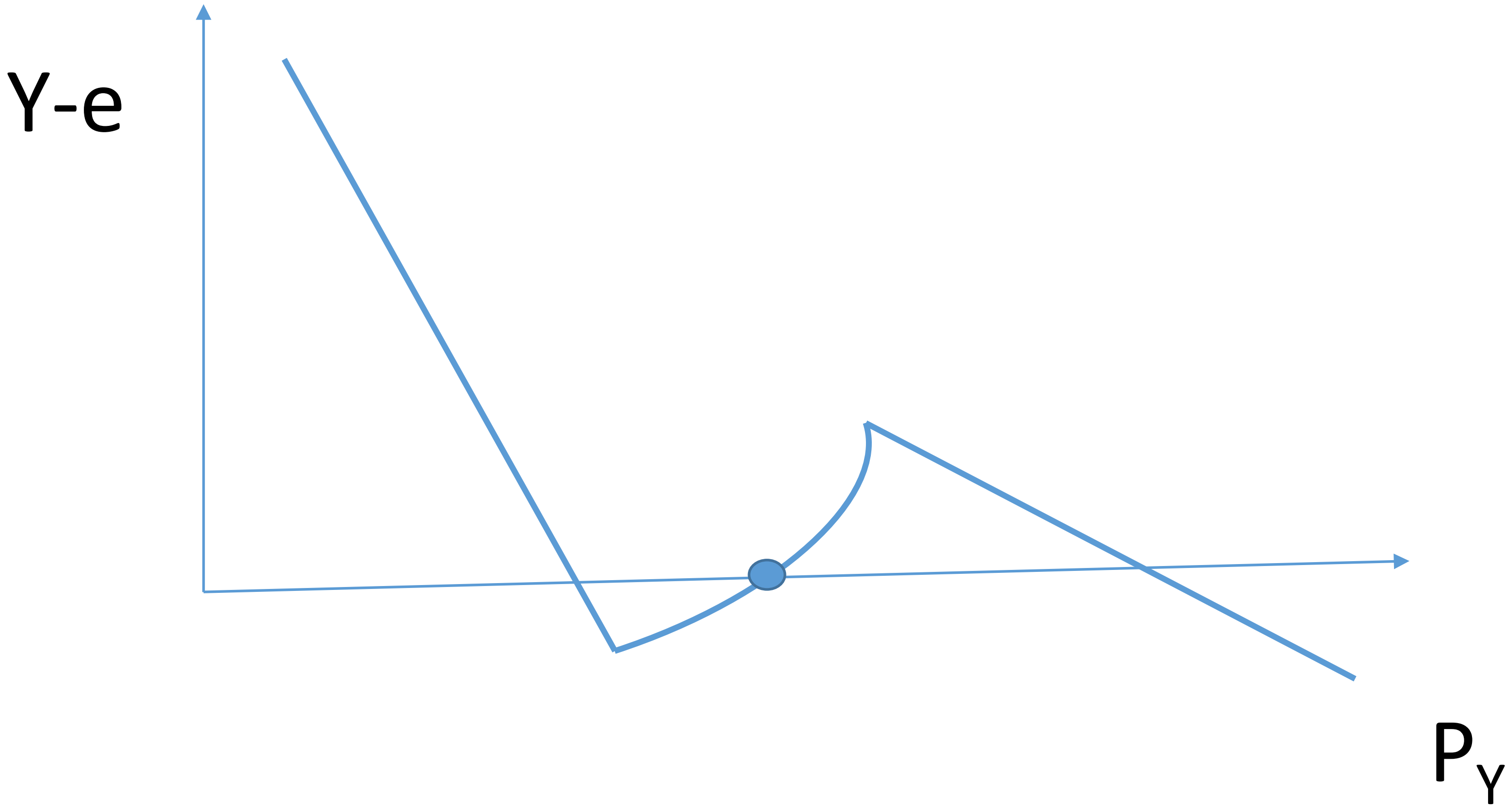


Equilibrium with upward sloping excess demand



Equilibrium with upward sloping excess demand gives rise to multiple equilibria, provided that excess demand goes to infinity as price goes to zero

In fact only way to get multiplicity (generically) is for one equilibrium to have positive slope.
So multiplicity if and only if some equilibrium has upward sloping excess demand.



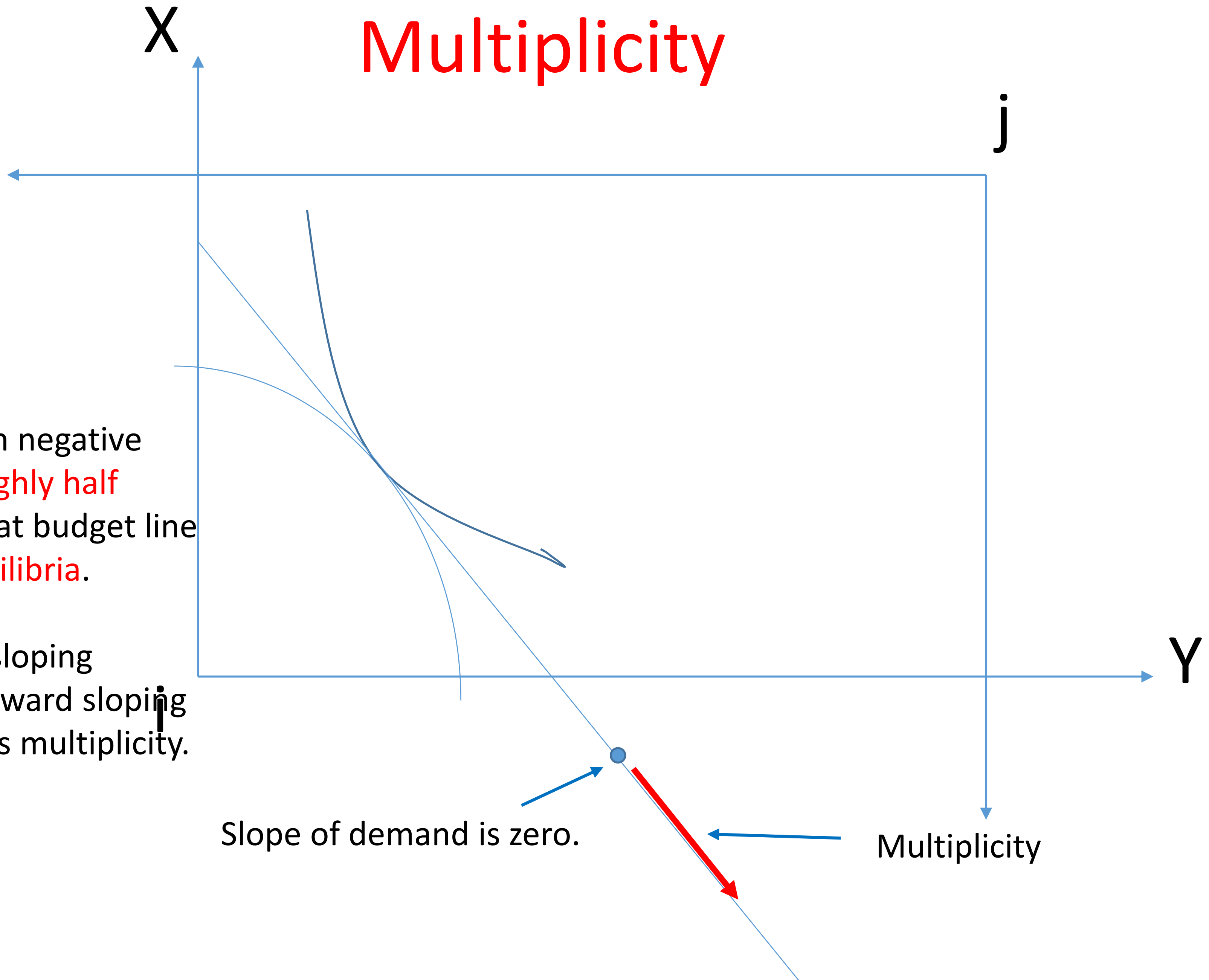
Equilibrium with upward sloping excess demand gives rise to multiple equilibria, provided that excess demand goes to infinity as price goes to zero

GE Fragility-Multiplicity Mechanism

- Sellers have higher MPC
- Big trade
- Normal Good

- Debt explains last two, especially big trade.

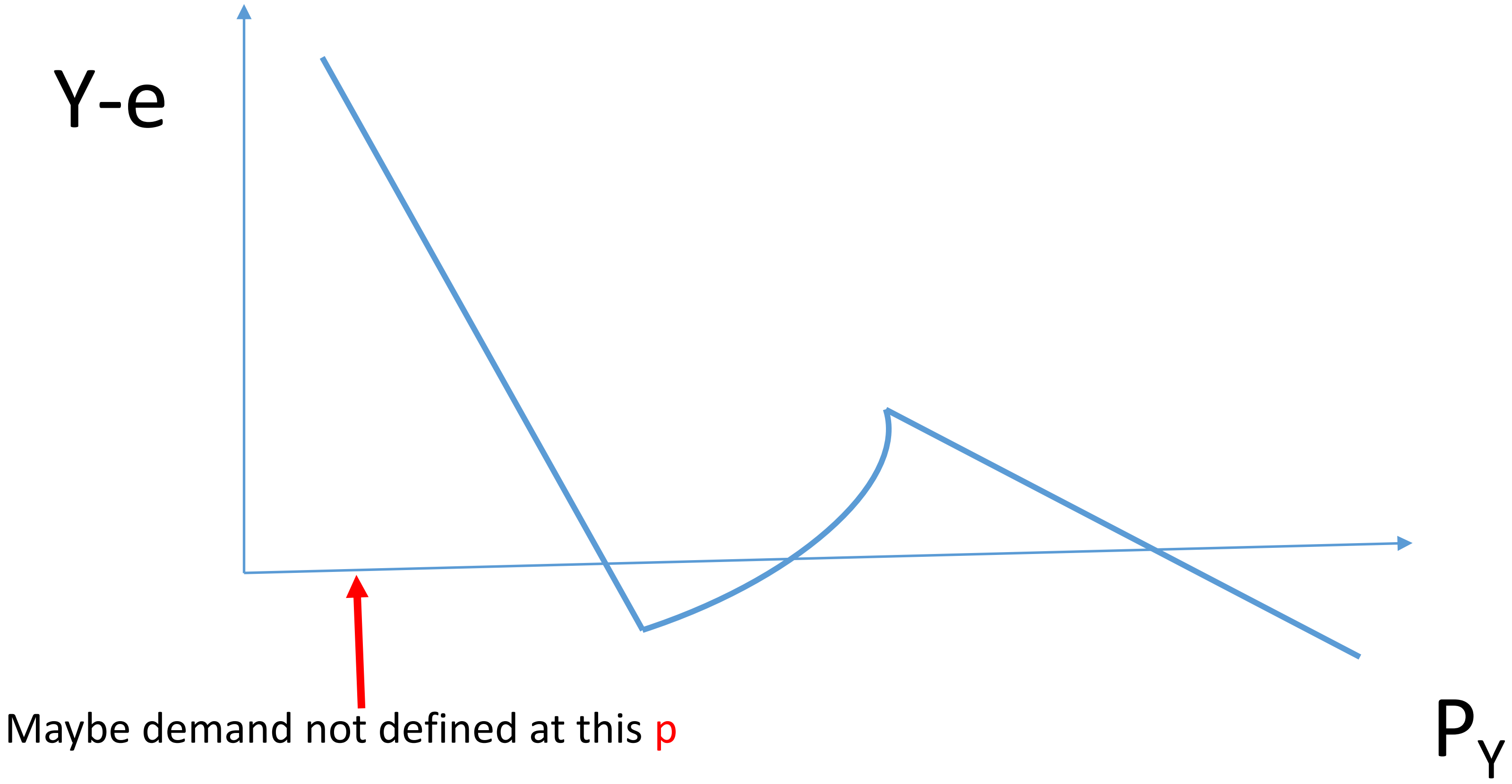
Multiplicity



Same equilibrium with negative Endowments. For roughly half of endowments on that budget line get multiplicity of equilibria.

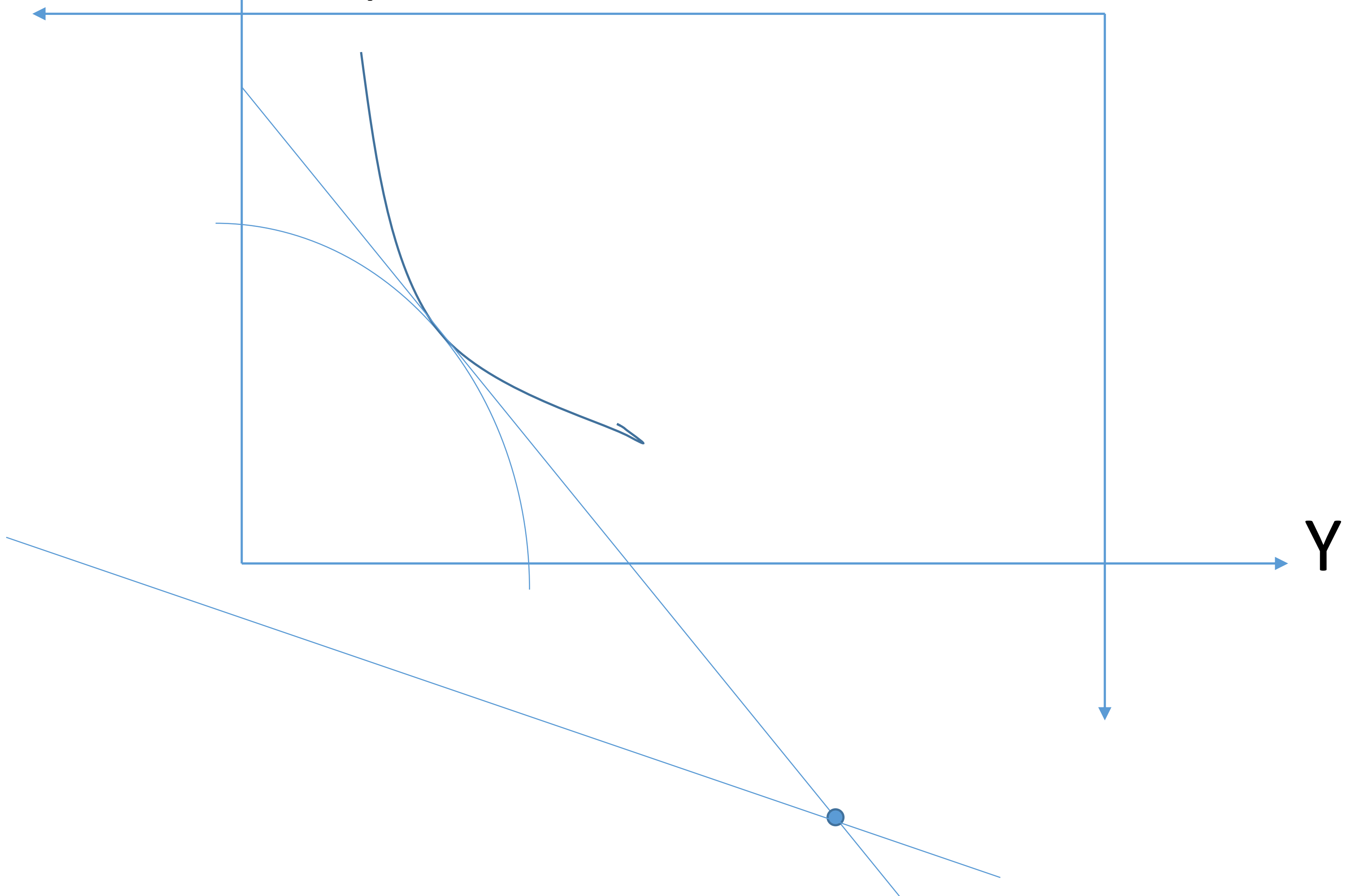
Half have downward sloping demand. Half have upward sloping Demand which implies multiplicity.

Not so fast: demand not defined at many p



Equilibrium with upward sloping excess demand gives rise to multiple equilibria, provided that excess demand goes to infinity as price goes to zero

If endowment is outside box, then at some prices Default is inevitable. So must define demand for arbitrary Endowments and prices.



Not so fast

- Need to take into account default.

- Need a little bit of mathematics
- If story so easy, it would have been told already.

- What was done?

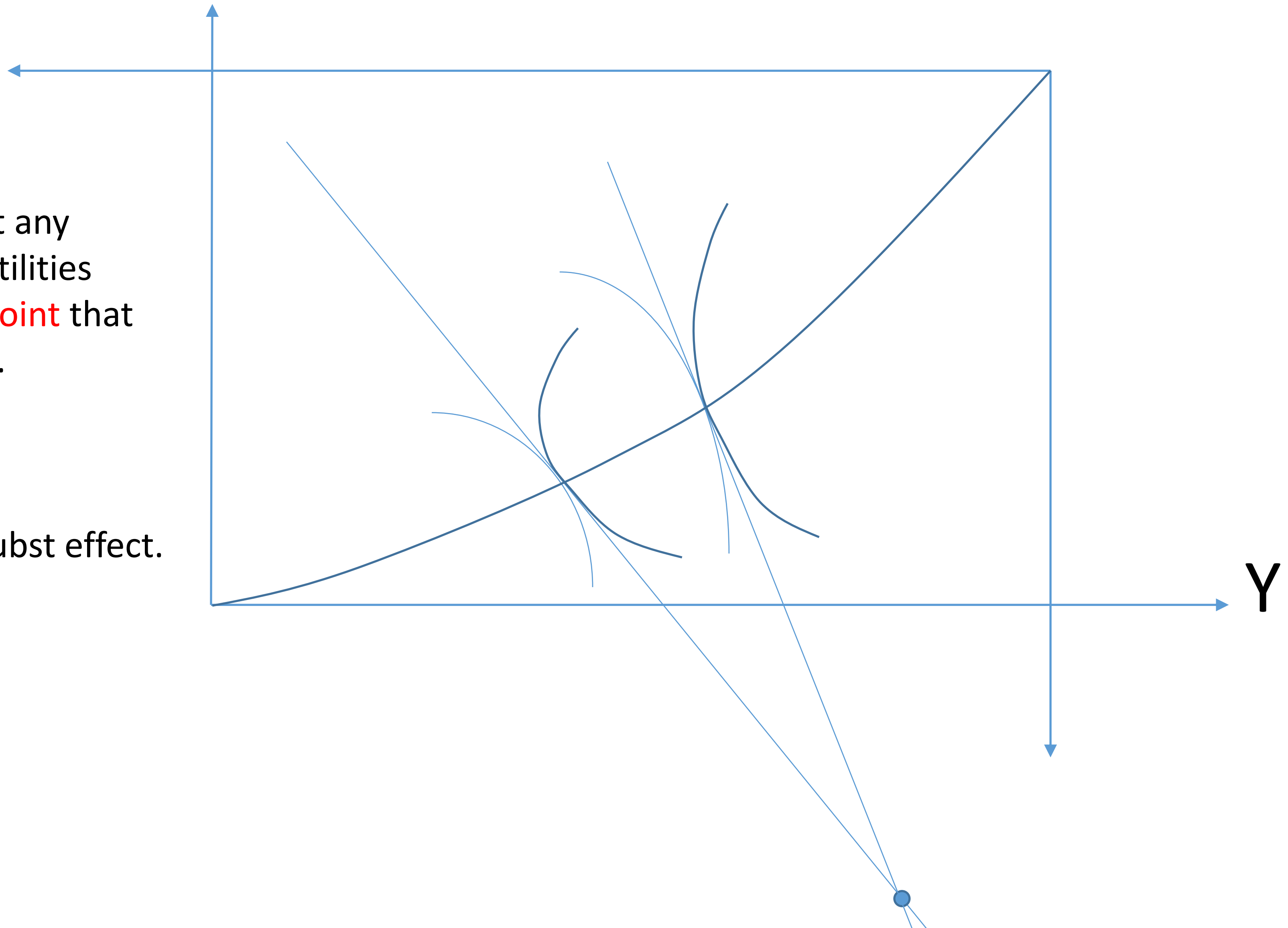
Gottardi and Kajii (1999)

Unless Budget Lines are parallel, they will eventually cross at an endowment (outside the Edgeworth Box) that generates two equilibria.

Proved that for almost any
Edgeworth box with utilities
There is at least one point that
Generates multiplicity.

No default.

No income effect vs subst effect.



Cobb-Douglas Example

Multiple equilibria when Pareto Budget lines intersect. In Cobb-Douglas all budget lines Intersect at one endowment. Get a continuum of equilibria At one endowment point.

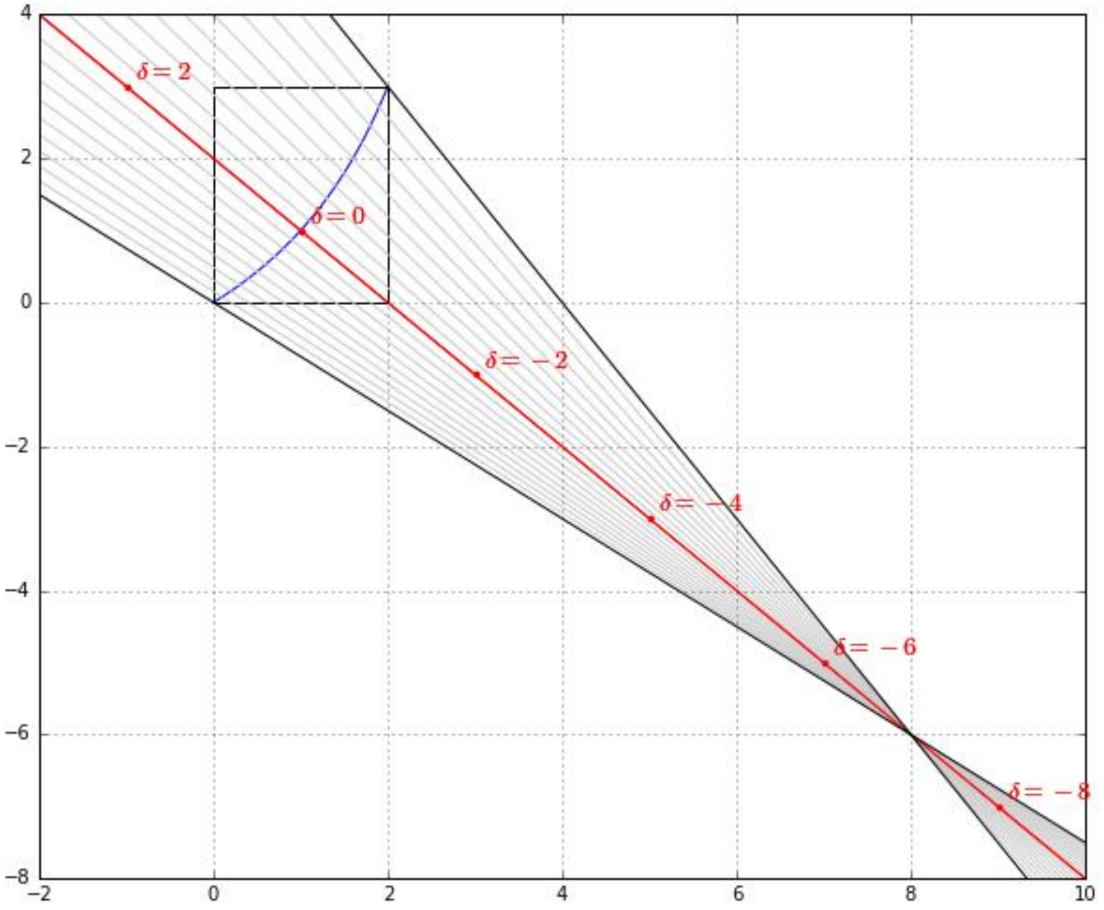
But what about other Endowments outside Box?

Does equilibrium exist?
Is there multiplicity?

No use of
income and substitution effects.

No mention of default.

$$u_i = x_{1i}^{\alpha_{1i}} x_{2i}^{\alpha_{2i}}, \; e_1 + e_2 = (2, 3)$$
$$\alpha_{11} = (1 - \alpha_{21}) = 1/2, \; \alpha_{12} = (1 - \alpha_{22}) = 1/3, \; s_1 = 1, \; s_2 = 1$$



Cass – Pavlova Example, used by Hens

Generalization to H Agents and L goods Look at all endowments outside box

- Must make it rigorous
- Cannot ignore Default
- Default Equilibria might be the most interesting equilibria

- Is it true that half of endowments give rise to multiple equilibria and fragility?
- Or is it true that any crowded trade gives rise to multiple equilibria and fragility?

Extending General Equilibrium to Debt and Default

Always existence.

Open Set of Fragile economies.

Half of endowments that give rise to interior no default equilibria give multiplicity.

Because Debt Makes Income Effects Become More Important.

Walrasian Economy with Debts

Agents have utilities

$u^h: R^L_{++} \to \mathrm{R}$

endowments of each agent $h \in H$ are

$e^h \in R^L_{++}$

but also by additional vectors

$d^{hi} \in R^L_{+}$;

where d^{hi}_k denotes the amount of good k that agent h owes agent i.

Model without Default

If there is no worry about default, we can simply take the gross endowments as the sum of the conventional endowments + debts

$$\bar{e}^h = e^h + \sum_i d^{ih} - \sum_i d^{hi}$$

And leave definition of Walrasian Equilibrium unchanged, except that now gross endowments might be negative, i.e. outside the Edgeworth Box $EB(\bar{e})$.

Gross endowments can be broken up into conventional endowments + debts in many ways

Where do Debts Come From?

- Debt is a fact of life.
- No need to go back and find out how they came about. Gross endowments are primitive just like conventional endowments.
- But many economists feel that debts should come with how creditors expected to get paid.

- So concentrate on case where gross endowments $(\bar{e}^h)_{h \varepsilon H}$
- Give rise to a conventional Walrasian equilibrium $(p, (\bar{e}^h)_{h \varepsilon H})$ with no default.

- Fragility means if discover endowments slightly different, get radically different new equilibrium.
- Multiplicity means that if agents' beliefs change, can get radically different equilibrium, perhaps with default.

Rationalizable Debts for $(u^h, e^h, d^h)_{h \varepsilon H})$

- Imagine two period economy with additively separable common utilities, common discounts, strictly positive individual endowments, identical aggregate endowments.

- $u^h(x_1) + \delta u^h(x_2)$

- $\sum_h e^h{}_1 = \sum_h e^h{}_2 = \sum_h e^h$

- Two period Walrasian equilibrium without default in which debts carried into second period are the debts of one period economy.
- Equilibrium of one period economy with debt is intended second period of Walrasian economy.
- Then say debts are rationalized.

Rationalizable Debt Proposition

- Any one period economy with arbitrary debts that has a Walrasian equilibrium without default can be rationalized as the second period of a two period economy.

Bankruptcy Law and Deliveries D^{hi}_k Given Prices and Given Debt Receipts

- Fixing prices and i's receipts, D^{hi}_k denotes the amount of money that agent h pays agent i on the debt owed in good k.
- We must make the deliveries endogenous.
- Deliveries should depend on debts and resources and prices and bankruptcy laws.
- Many ways of doing that.
- So we posit a delivery function that includes many of these possibilities, with one simple monotonic deliveries axiom.

Endogenous Deliveries

- Endogenous deliveries and endogenous consumption expenditures
- $D^{hi}_{k} = \underline{\mathbf{D}}^{hi}_{k}(p, u^h, e^h, d^h, D^{-h})$; $I^h = \underline{\mathbf{I}}^h(p, u^h, e^h, d^h, D^{-h})$
- Assumption
- Monotone Deliveries Assumption:
- Each delivery and consumption expenditure is continuous, and weakly increasing in deliveries received: $D^{-h}\uparrow$ implies $[\underline{D}^{hi}_{k}, \underline{I}^h]\uparrow$
- All our examples of Delivery Rules satisfy the monotone deliveries assumption.

Equal Priority Delivery Rule
Green (1974) and Eisenberg and Noe (2001)

- Assume each agent h gradually increases paying common percentage r_h of each debt until her resources run out or all debts paid.
- Money left over after debts paid goes into consumption.
- With same debts and more resources, all deliveries and total consumption expenditures weakly rise.
- Monotonic deliveries assumption

Seniority Delivery Rule

- Assume all debts (i,k) of each agent h are strictly ordered.
- Assume pay off first debt in full, then if any money remaining, pay off second debt, etc until resources run out, or until all debts paid, after which all remaining resources go to consumption.

- Clear satisfies monotonic deliveries assumption.

Default Penalties Delivery Rule

- Suppose that for each debt (i,k), agent h faces a utility loss that is an increasing strictly convex function of his real default on (i,k), (measured in units of good k).
- Suppose the total utility penalty is the sum of the penalties across all debts (i,k).

- Clearly this also satisfies the monotonic deliveries axiom.

More Variations on Delivery Rules

- Don’t have to deliver any of endowment in some goods g (like labor endowment) in first two rules.
- Or only have to deliver ε of some goods.
- Or default penalties very small on some goods.

Model with Default and no seizure costs

- Dollar deliveries D^{hi}_k
- $$0 \leq D^{hi}_k \leq p_k d^{hi}_k$$
- Budget Constraint
- $$p \cdot x^h + \sum_i \sum_k D^{hi}_k = p \cdot e^h + \sum_i \sum_k D^{ih}_k$$

- Lemma: Whatever the deliveries,
- $$\sum_h [p \cdot e^h + \sum_i \sum_k D^{ih}_k - \sum_i \sum_k D^{hi}_k] = \sum_h p \cdot e^h$$

Proof: D^{hi}_k appears twice, once subtracted from h and once added to i and so cancels out.

Endogenous Deliveries

- Endogenous deliveries and endogenous consumption expenditures
- $D^{hi}_k = \underline{\mathbf{D}}^{hi}_k(p, u^h, e^h, d^h, D^{-h})$; $I^h = \underline{\mathbf{I}}^h(p, u^h, e^h, d^h, D^{-h})$
- Assumption
- Monotone Deliveries Assumption:
- Each delivery and consumption expenditure is continuous, and weakly increasing in deliveries received: $D^{-h}\uparrow$ implies $[\underline{\mathbf{D}}^{hi}_k, \underline{\mathbf{I}}^h]\uparrow$

- All our examples of Delivery Rules satisfy the monotone deliveries assumption.

Consistent Deliveries

- Fixing $(\, p,\ (u^h, e^h, d^h)_{h \varepsilon H})$ and the delivery functions
-
- $\underline{\mathbf{D}}^{hi}{}_k(\, p,\ u^h,\ e^h, d^h, D^{-h})$

- We say that the deliveries $(D^h)_{h \varepsilon H}$ are consistent iff

- $D^h = \underline{\mathbf{D}}^h(\, p,\ u^h,\ e^h, d^h, D^{-h})$ for all h

Consistent Deliveries?

- Since h deliveries depend (increase) on i deliveries, might think no consistent deliveries.
- Or might think could have consistent low deliveries and consistent high deliveries.
- Get existence and uniqueness!
- So deliveries multiplicity has nothing to do with Walrasian multiplicity.

- If dropped no seizure losses, could get multiple consistent deliveries.

Unique Consistent Deliveries and Continuous Consumption Expenditures Theorem

- For any $(\, p,\ (u^h, e^h, d^h)_{h \varepsilon H})$
- And any delivery rules $\underline{\mathbf{D}}^{hi}{}_k(\, p,\ u^h, e^h, d^h, D^{-h})$
  - satisfying the monotonic delivery assumption
- There is a vector of consistent deliveries D,
- And all consistent deliveries give the same consumption expenditures,
- Which are continuous functions $E^h(\, p,\ (u^h, e^h, d^h)_{h \varepsilon H})$.

- Green (1974) and Eisenberg and Noe (2001) proved this for the case of equal priority delivery rules.

Proof of Unique Consistent Deliveries Theorem

- Fix $(p, (u^h, e^h, d^h)_{h \varepsilon H})$
- The dollar deliveries $(D^h)_{h \varepsilon H})$ lie in the product $\mathbf{D}$ of finite intervals
- $$0 \leq D^{hi}_k \leq p_k d^{hi}_k$$
- The functions $\underline{\mathbf{D}}^{hi}_k(p, u^h, e^h, d^h, D^{-h})$ define a continuous function g from $\mathbf{D}$ to itself that is weakly increasing.
- It has a fixed point.
  - The function g is weakly increasing in every coordinate.
  - g(0) ≥ 0
  - Iterating, g(g(0)) etc has LUB which must be a fixed point.

Proof continued

- If $D > D'$ are fixed points of g, then
- $\underline{\mathbf{I}}^h(p, u^h, e^h, d^h, D^{-h}) = \underline{\mathbf{I}}^h(p, u^h, e^h, d^h, D'^{-h})$
  - Because the spending function $\underline{I}^h$ is weakly increasing in every coordinate,
  - Yet total spending is equal to the value of the aggregate endowment.
- If s,r distinct fixed points, then there is third solution $t > s \vee r$, showing incomes are the same at all three.
  - Proof: $g(s \vee r) \geq g(s) = s$
  - $\underline{g}(s \vee r) \geq g(r) = r$
  - $\underline{g}(s \vee r) \geq s \vee r$
  - Iterating, get fixed point at least as big as $s \vee r$, hence bigger than s or r.
  - QED

Definition of Walrasian Equilibrium with Debts and Default for L goods and H agents

- Given a Walrasian economy with debt and delivery rules
- $(u^h, e^h, d^h, \underline{\mathbf{D}}^h, \underline{\mathbf{I}}^h)_{h\varepsilon H}$
- Walrasian equilibrium is defined by $(p, (x^h, D^h)_{h\varepsilon H})$
- $\sum x^h = \sum e^h$ and for all h
- $D^h = \underline{\mathbf{D}}^h(p, u^h, e^h, d^h, D^{-h})$
- $x^h \ \varepsilon\ B^h(p, u^h, e^h, d^h, D^{-h}) = \{x : p \cdot x \leq \underline{I}^h(p, u^h, e^h, d^h, D^{-h}) = E^h(p)\}$
- $x \ \varepsilon\ B^h(p, u^h, e^h, d^h, D^{-h}) \rightarrow u^h(x^h) \geq u^h(x)$

Existence Theorem with Debt and Default for L goods and H agents

- Given a Walrasian economy with debt and delivery rules
- $(u^h, e^h, d^h, \underline{\mathbf{D}}^h, \underline{\mathbf{I}}^h)_{h\varepsilon H}$
- if u^h is strictly concave, strictly monotonic, continuous and
- If e^h strictly positive and
- If $(\underline{\mathbf{D}}^h, \underline{\mathbf{I}}^h)_{h\varepsilon H}$ satisfy the monotonic deliveries assumption
- Then, no matter what the debts d^{hi}
- Walrasian equilibrium with debt exists
- $(p, (x^h, D^h)_{h\varepsilon H})$

Demand Continuity Boundary Lemma:

- Given a Walrasian economy with debt and delivery rules
- $(u^h, e^h, d^h, \underline{\mathbf{D}}^h, \underline{\mathbf{I}}^h)_{h\varepsilon H}$
- If u^h is strictly concave, strictly monotonic, continuous, and
- if e^h is strictly positive,
- And $(\underline{\mathbf{D}}^h, \underline{\mathbf{I}}^h)_{h\varepsilon H}$ satisfy the monotonic deliveries assumption,
- Then no matter what the debts d^{hi}

- Demand $D^h(p) = \text{Argmax}\,\{u^h(x): x \,\varepsilon\, B^h(p) = \{x: p \cdot x \leq E^h(p)\}$ is well-defined and continuous at all $p >> 0$,

- Moreover, demand goes to infinity in some good if any price goes to zero.

Proof of Continuity Lemma

- Continuity: Budget constraint defined by continuous income function $E^h(p)$, by the unique deliveries theorem. Hence budget set is a continuous correspondence for $p >> 0$.
- Hence by maximum principle, and strict concavity, demand is continuous.
- Boundary Behavior: Adding up over people gives summation of incomes equal to value of endowment for all p and r
- $$\sum_h E^h(p) = \sum_h p \cdot e^h$$
- So somebody always has positive wealth, bounded away from zero. Hence demand goes to infinity as any price goes to zero.

Proof of Existence for L goods and H agents

- Because of boundary behavior of demand, the usual fixed point argument can be applied on the trimmed simplex where each price is at least $\varepsilon > 0$.
- Hence need only deal with positive prices at which budget set is continuous, and demand is USC.
- QED.

Multiplicity and Fragility

- Depend on some power of the bankruptcy law. If people defaulted on all their debts, then debts would be irrelevant and we would be back in the conventional Walrasian economy inside the Edgeworth Box.
- It is surprising how weak the bankruptcy laws can be to guarantee multiplicity and fragility.

Strong Bankruptcy Law

- The delivery rule $\underline{\mathbf{D}}^{hi}_{k}(p, u^h, e^h, d^h, D^{-h})$ displays strong bankruptcy law if h pays all his debts whenever he has the resources to do so.

- Both the equal priority delivery rule and the seniority delivery rule display strong bankruptcy law.
- So does the default penalty delivery rule if the penalties are harsh enough.
  - More precisely, if consumption utilities are continuously differentiable and bounded, marginal utilities must be bounded, and setting marginal penalties higher than the upper bound of marginal utilities does the trick.

ε - Strong Bankruptcy Law

- The delivery rule $\underline{\mathbf{D}}^{hi}{}_{k}(p, u^{h}, e^{h}, d^{h}, D^{-h})$ displays ε - strong bankruptcy law if h pays all his debts whenever he can afford to out of delivery receipts from others plus the fraction ε of any one of his endowment goods.

- This is a great weakening of the strong bankruptcy law. It allows an agent to default completely whenever he would need more than the fraction ε of his endowments to pay.
- ε - Strong Bankruptcy Law holds even if only one good in endowment can be seized.

Multiple Equilibrium Theorem for 2 goods, 2 agents:
Consider any economy with two agents and two goods,
Satisfying assumptions of the existence theorem.

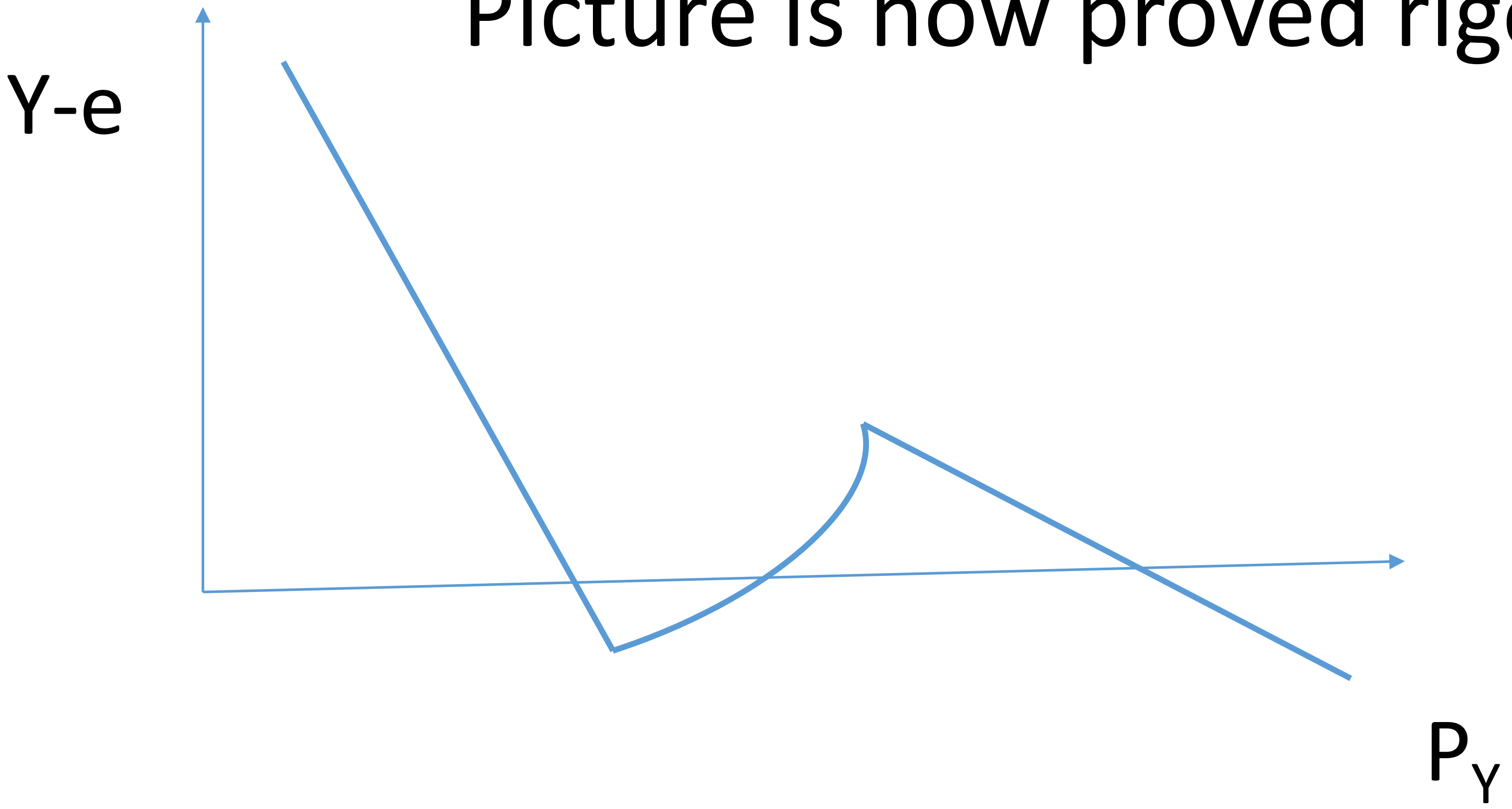
Suppose deliveries also display strong bankruptcy law.

Suppose at some Pareto efficient point, the marginal propensities to consume of the two agents are different.

Then half of gross endowments on budget line generate at least three equilibria.

Holds with ε - strong bankruptcy law, with right debts.

Picture is now proved rigorous



Equilibrium with upward sloping excess demand gives rise to multiple equilibria, provided that excess demand goes to infinity at price goes to zero

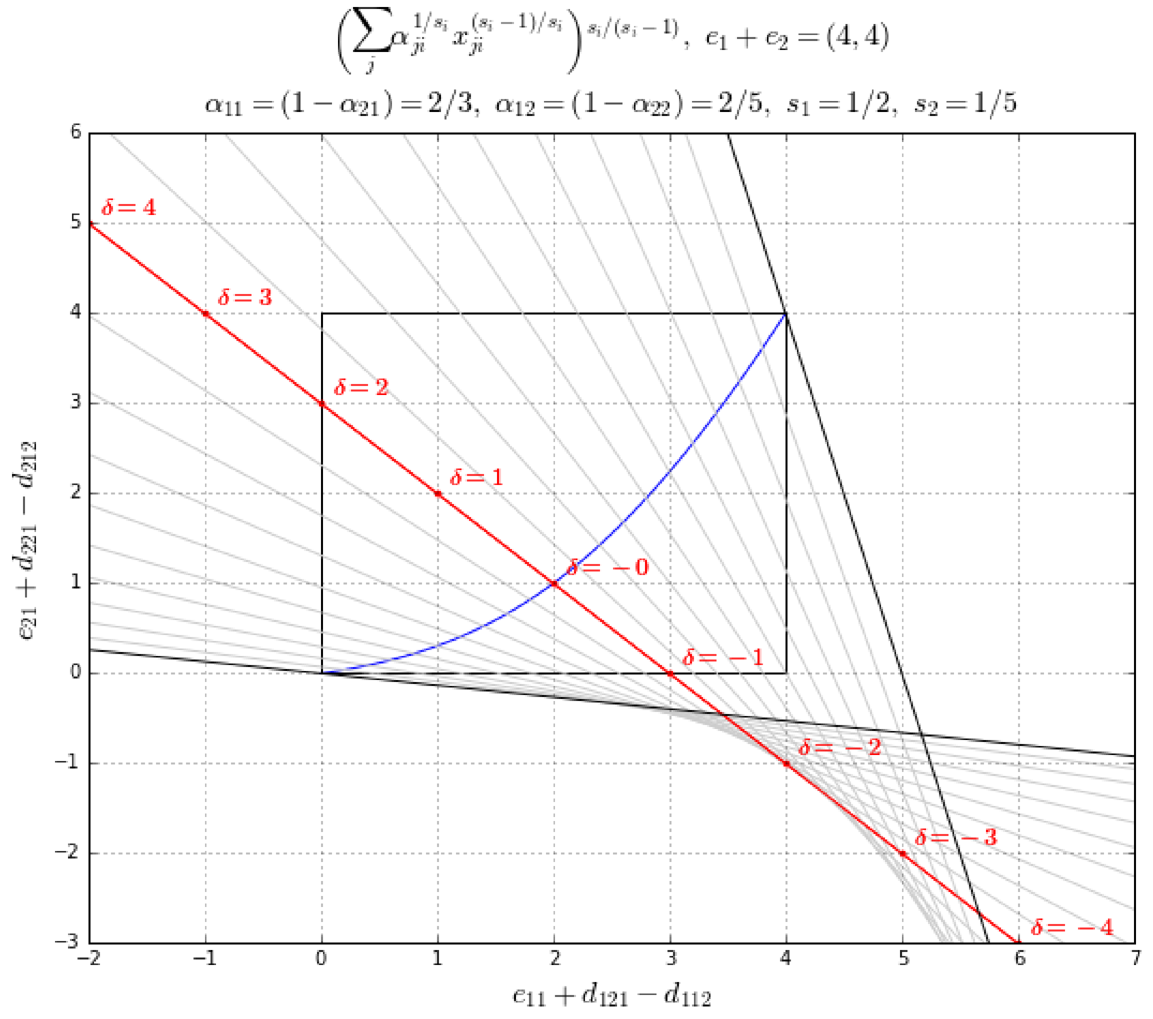
Multiple Equilibria driven by

- Differences in marginal propensities to consume (MPC)
- Enough Debt

Proof of multiple equilibria for 2 goods, 2 agents

- Start at Pareto efficient point.
- Move endowment along budget line in direction so that agent i is selling the good k (call it housing) he has higher MPC.
- Since on budget line, everybody can pay debts at old prices if everybody else pays
- Set debts so value of debt every agent h owes is equal to the value of the debt he is owed.
- From ε strong bankruptcy law, by delivery uniqueness, everybody does pay.
- So old equilibrium maintained.
- Will all pay even with very nearby prices.
- As endowments go out, eventually get upward sloping demand locally.
- So get two more equilibria because of boundary behavior, as in picture.

Example with CES utilities and equal priority delivery rule



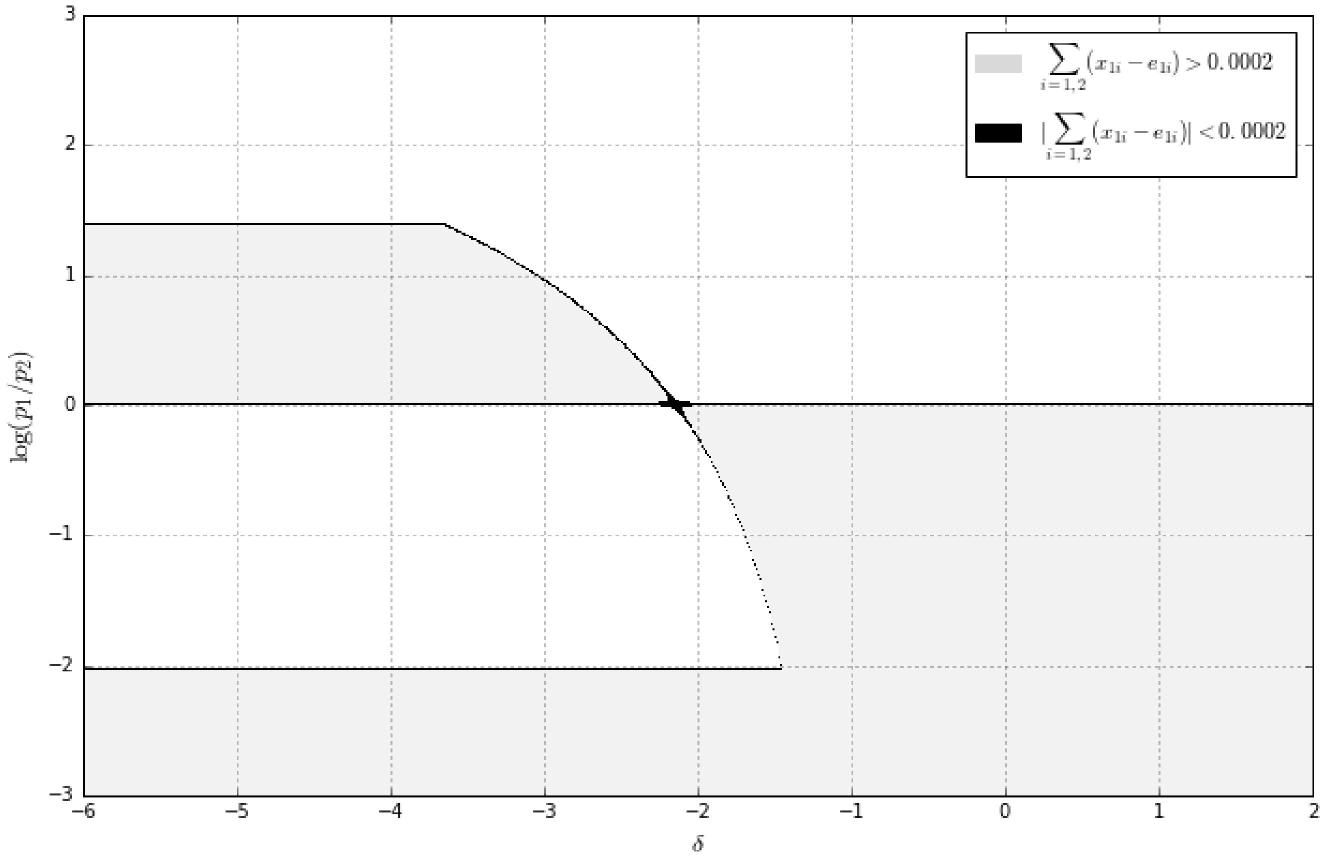
Edgeworth Box with equilibrium budget line through each Pareto point on blue Pareto frontier.
When two budget lines intersect at endowment (outside the Box) get multiple equilibria.
Moving to the southeast on red line gives agent i at bottom bigger debt in vertical good X.

$$u_i = \left(\sum_j \alpha_{ji}^{1/s_i} x_{ji}^{(s_i-1)/s_i}\right)^{s_i/(s_i-1)},\ e_1=(2,1),\ e_2=(2,3)$$

$$d_{12}=(\max(0,\delta),\max(0,-\delta)) \quad d_{21}=(\max(0,-\delta),\max(0,\delta))$$

$$\alpha_{11}=(1-\alpha_{21})=2/3,\ \alpha_{12}=(1-\alpha_{22})=2/5,\ s_1=1/2,\ s_2=1/5$$

Equilibria on top or bottom
Straight lines are default equilibria

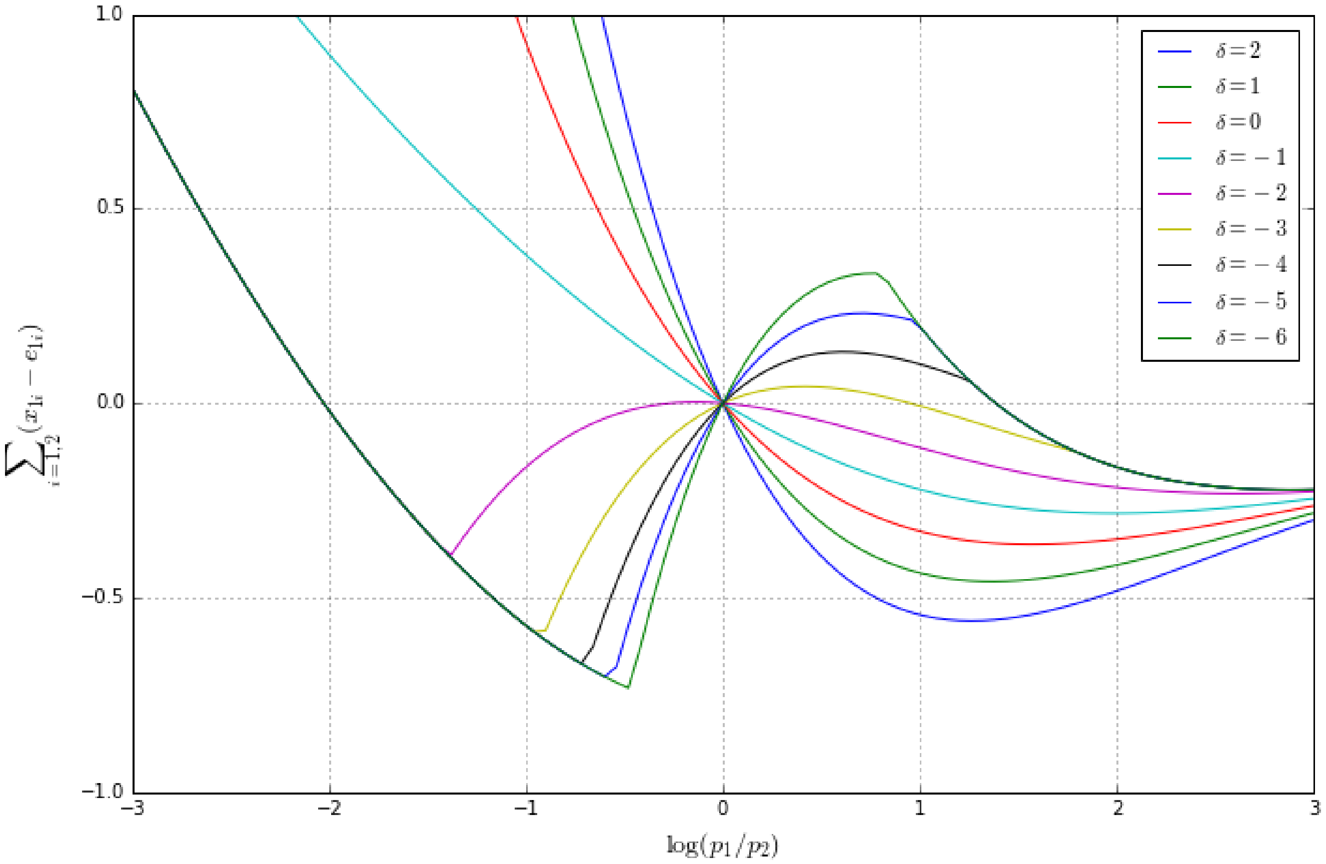


Equilibrium prices for different δ. Notice that unique equilibrium inside Edgeworth Box δ > -1
Eventually get 3 equilibria. For -3.7 < δ < -1.6, 2 equilibria involve no default.

$$u_i = \left(\sum_j \alpha_{ji}^{1/s_i} x_{ji}^{(s_i-1)/s_i}\right)^{s_i/(s_i-1)}, \;\; e_1 = (2,1), \;\; e_2 = (2,3)$$

$$d_{12} = (\max(0,\delta), \max(0,-\delta)) \quad d_{21} = (\max(0,-\delta), \max(0,\delta))$$

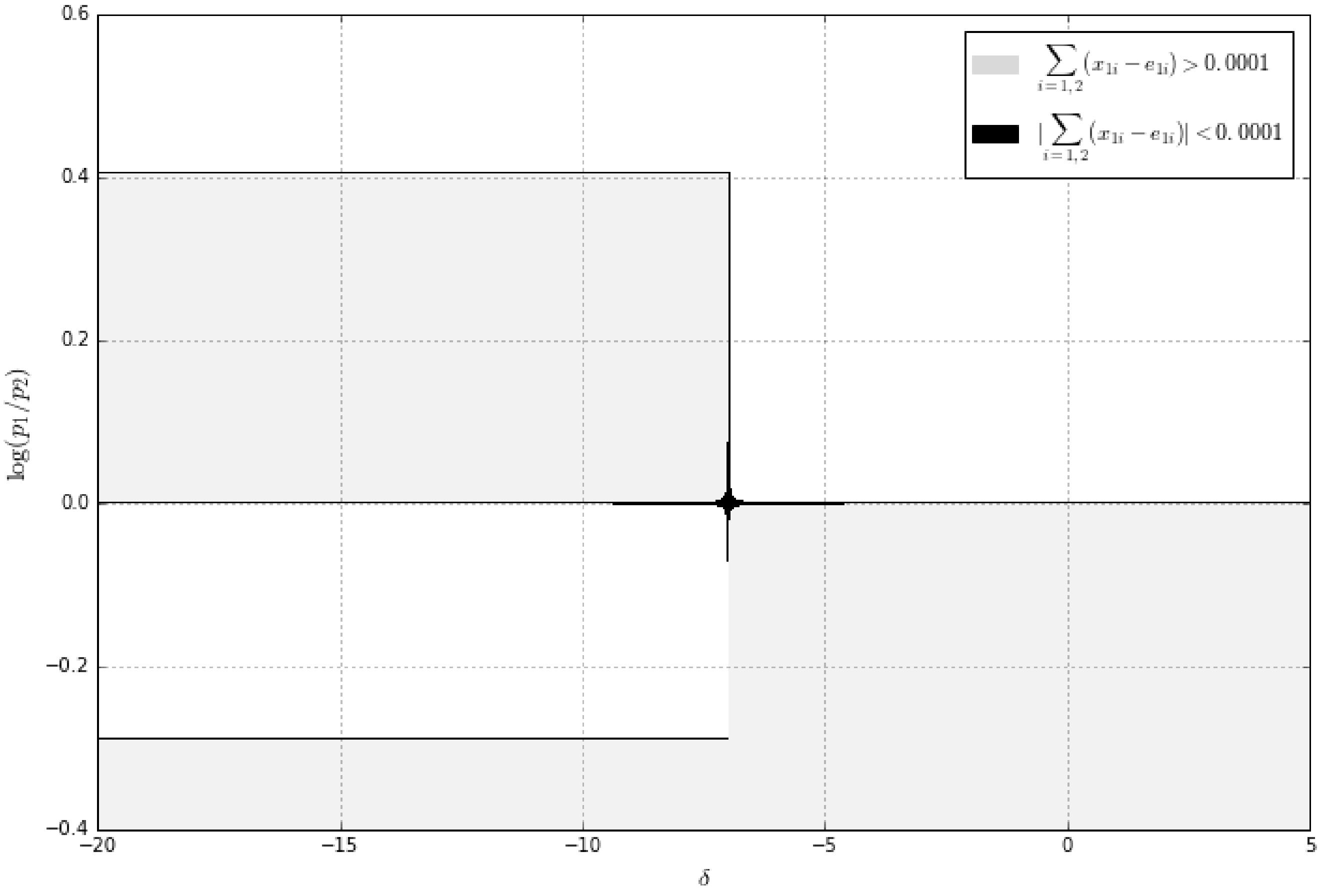
$$\alpha_{11} = (1-\alpha_{21}) = 2/3, \;\; \alpha_{12} = (1-\alpha_{22}) = 2/5, \;\; s_1 = 1/2, \;\; s_2 = 1/5$$



Excess Demand Curves move from downward sloping to upward sloping as debt rises (δ goes more negative).
At edge of Edgeworth Box (δ = -1) excess demand is still downward sloping.

Cobb-Douglas Example revisited

$$u_i = x_{1i}^{\alpha_{1i}} x_{2i}^{\alpha_{2i}},\ \ e_1 = (1,1),\ \ e_2 = (1,2)$$
$$d_{12} = (\max(0,\delta), \max(0,-\delta)) \quad d_{21} = (\max(0,-\delta), \max(0,\delta))$$
$$\alpha_{11} = (1-\alpha_{21}) = 1/2,\ \ \alpha_{12} = (1-\alpha_{22}) = 1/3,\ \ s_1 = 1,\ \ s_2 = 1$$



Cobb-Douglas example goes from one equilibrium to continuum, to three, two of which involve default.

Definition of Utility Diversity at Pareto Efficient Allocation

- Every Pareto efficient allocation in an economy with smooth utilities gives rise to marginal propensity to consume out of income vectors
- $MPC^h = v^h$
- Weak diversity assumption means those vectors not all identical.
  - Holds generically in utilities and endowments
- Diversity Assumption: Dropping their last coordinates, the vectors
- $v^1 - v^H, \ldots\ldots, v^h - v^H, \ldots\ldots, v^{H-1} - v^H$
- Span all of R^{L-1}.
  - Holds generically if $H \geq L$.

Multiplicity and Fragility Theorem with L goods and H ≥ L agents

- Consider any economy with $L > 1$ goods and $H \geq L > 1$ agents
- Satisfying smooth utilities and positive endowments.
- Suppose some Pareto efficient point satisfies the Diversity Assumption.
- Suppose the delivery rules satisfy the monotonic delivery assumptions, and display strong bankruptcy law.
- Then half the gross endowments that produce the Pareto efficient point as an equilibrium also generate other equilibria (i.e. multiplicity), no matter how they are divided into conventional endowments and debts. Ie half of all rationalizable debt economies have multiple equilibria.
- On the ray connecting the Pareto efficient point to a gross endowment generating multiplicity, there is an open interval of gross endowments that render the original Pareto efficient point a fragile equilibrium.

Proof of fragility and multiplicity for $H \geq L > 1$

- 1) Start with point on Pareto frontier with diversity assumption.
- 2) Show that for any gross endowment assignment, there is another mirror image endowment that preserves the equilibrium but flips the determinant of the derivative of aggregate excess demand at the equilibrium.
- 3) Apply Poincare-Hopf Index Theorem to conclude that there must be multiple equilibria from one of the two endowments.

Slutsky

- $Dx^h/dp = K^h + v^h(e^h – x^h)'$

- Where K^h is negative semi-definite and $p'K = 0$ and v^h is column of MPC for each good and $(e^h – x^h)'$ is row vector of net supply at equilibrium prices.

- Adding up over agents

- $Dx/dp = K + \sum v^h(e^h – x^h)'$

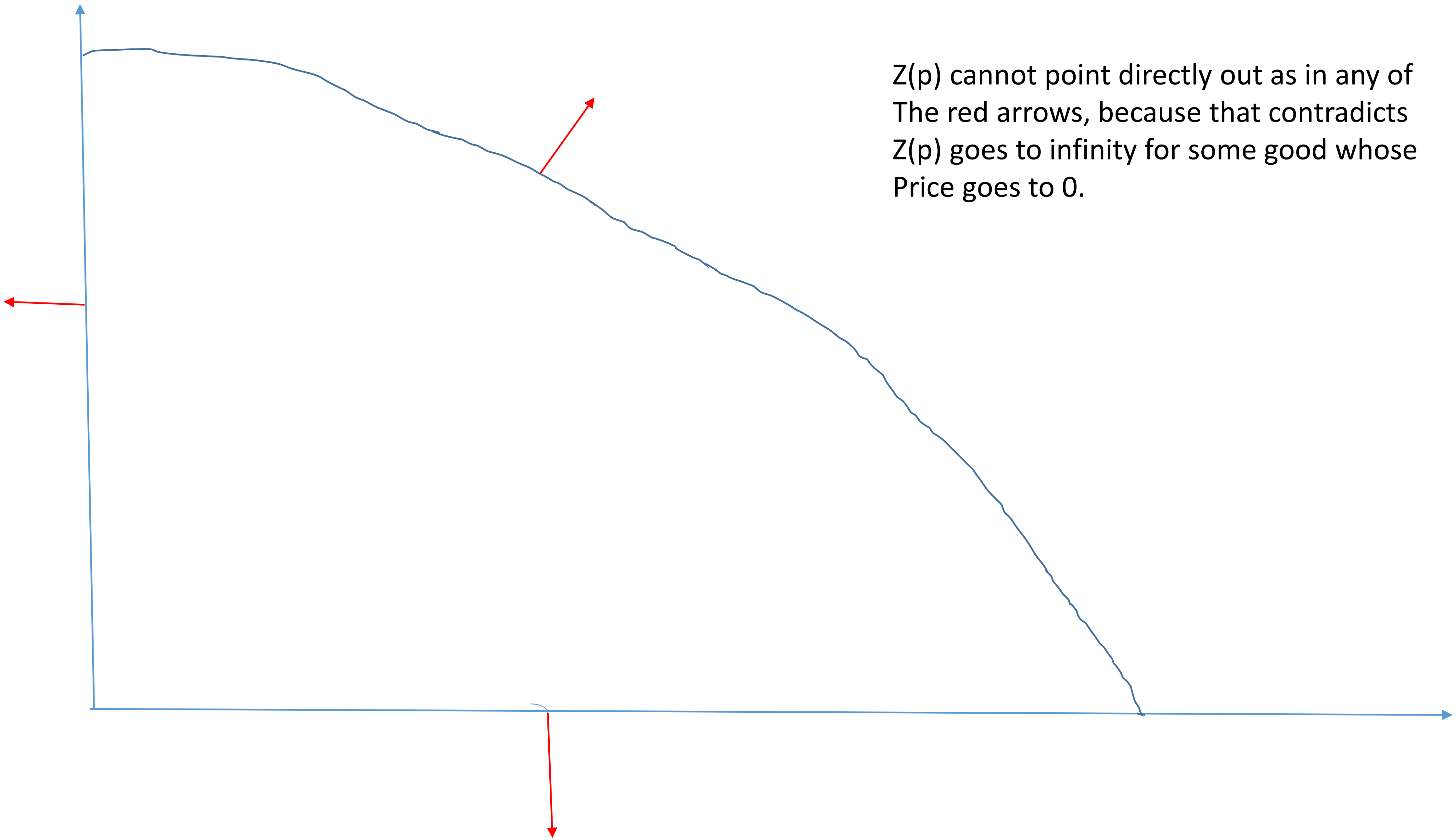
- Only need to look at L-1 rows and columns by Walras Law.

2) Flipping the determinant by changing endowments

- For every gross endowment, there is another gross endowment where the first coordinate $e^h_1 - x^h_1$ is reversed for every h and moved over by a constant.
- If the MPC vectors v^h span R^{L-1}, and in particular, first column of K, then the constant can be chosen so that this reverses the sign of the first column of the derivative matrix without changing any other column.
- That reverses the sign of the determinant.
- Coordinate e^h_L is adjusted so income of every agent h stays the same at equilibrium prices p.

3) Poincare – Hopf Index Theorem

- Suppose in shape P below in L-1 dimensions, every point p in P is mapped into L-1 dimensional vector z(p), where z is smooth and transverse to 0.
- Suppose z(p) never points directly out at a boundary point p of P.
- Then the sum of the signs of the determinants of Dz(p) over all zeroes of z is equal to $(-1)^{L-1}$
- $\sum_{\{p:z(p)=0\}} \mathrm{sign}(\det(Dz(p)) = (-1)^{L-1}$



Z(p) cannot point directly out as in any of
The red arrows, because that contradicts
Z(p) goes to infinity for some good whose
Price goes to 0.

3b) apply Poincare- Hopf

- Z(p) is continuous and demand goes to infinity when price goes to 0.
- Every continuous function can be approximated on a compact set by a smooth function transverse to 0.
- This smooth function has same determinant sign near original equilibrium.
- Hence approximation has at least two more equilibria.
- Take limits of approximations, showing z(p) has multiple equilibria.

- QED

Extension to Arbitrary Economies

$(u^h, e^h, d^h)_{h \varepsilon H})$

- Must have an equilibrium $(p, (x^h, D^h)_{h\varepsilon H})$
- Could involve default.
  - For example, if some agent has huge debts in all goods, she will default.
  - At least one agent does not default.
- What about fragility and multiplicity?

- Suppose $H' \geq L > 1$ agents do not default.
- Suppose other $H - H'$ agents can't afford to pay all debts at equilibrium p.
- Then same theorem holds, fixing endowments in $H - H'$.

Extension to Economies with $L > H > 1$

- Now can’t show for half of all endowments
- But always will be one direction to go in where everybody is taking a side of the same “crowded” trade.

Definition of Crowded and Consequential Trade

- Begin with a Walrasian equilibrium $(p, (x^h)_{h\varepsilon H})$ with no default coming from gross endowments $(\bar{e}^h)_{h\varepsilon H}$. Let aggregate Slutsky matrix = K.
- Augment the endowments by a common trade $(\delta, \lambda, (s^h)_{h\varepsilon H})$
- $\bar{e}^h \rightarrow \bar{e}^h + s^h\delta\lambda$
- $\delta \varepsilon R^L, p \cdot \delta = 0, \lambda > 0,$
- $\sum_h s^h = 0$
- Now common trade δ is made in same equilibrium. Crowded when λ large.
- Now drop last coordinate of δ and last row and column of K
- Crowded Common trade is consequential, i.e. it has aggregate income effect.
- $-\delta \cdot K^{-1}\sum_h v^h s^h > 0$

Crowded Trade Fragility and Multiplicity Theorem with L goods and H agents

- Consider any economy with $H > 1$ agents and $L > 1$ goods,
- Satisfying smooth utilities and positive endowments
- Suppose the delivery rules satisfy the monotonic delivery assumptions, and displays strong bankruptcy law.
- Consider any gross endowments that generate a regular equilibrium at the Pareto efficient point.
- Suppose the original gross endowments do not already generate multiple equilibria.
- Then adding any crowded and consequential trade $(\delta, \lambda, (s^h)_{h \varepsilon H})$ creates a new economy with multiple equilibria for all big enough λ.
- For an open interval of λ, the original equilibrium is fragile.

Proof of fragility and multiplicity for any $H > 1, L > 1$

- 1) Start with point on Pareto frontier with diversity assumption.
- 2) Show that for any gross endowment assignment, adding crowded trade flips the determinant of the derivative of aggregate excess demand at equilibrium.
- 3) Apply Poincare-Hopf Index Theorem to conclude that there must be multiple equilibria.

Key idea of proof

- Now adding a new matrix with every row given by crowded trade vector.
- But that is one dimensional. So can use Sylvester's determinant theorem to show that determinant is linear in λ, with non-zero slope, given income effects.
- Hence for big enough λ must have a change in sign of determinant.
- Finish again with Poincare Hopf theorem.

Weakening the bankruptcy law

- As said earlier, without some bankruptcy power, debts become irrelevant and going outside the Edgeworth Box is meaningless. However can weaken the laws substantially.
- Both theorems have analogues with delivery rules that display ε – strong bankruptcy law.

Pareto Inefficiency

- Some of the multiple equilibria may involve default.
- Default equilibria are the most interesting.
- They may be Pareto inferior.
- 1) If there are default penalties, that is a deadweight loss.
- 2) Could put production into economy. Then if some goods not seizable, agents who are far into debt will not use unseizable goods to produce seizable good, even if it makes a profit.
- For example, unused labor cannot be seized. A refurbished house can be. An agent in enough debt will take the effort to refurbish his house even if the increase in market value of the house is greater than the vaue of the required labor.

Policy Solutions at Point of Fragility or Multiplicity ?

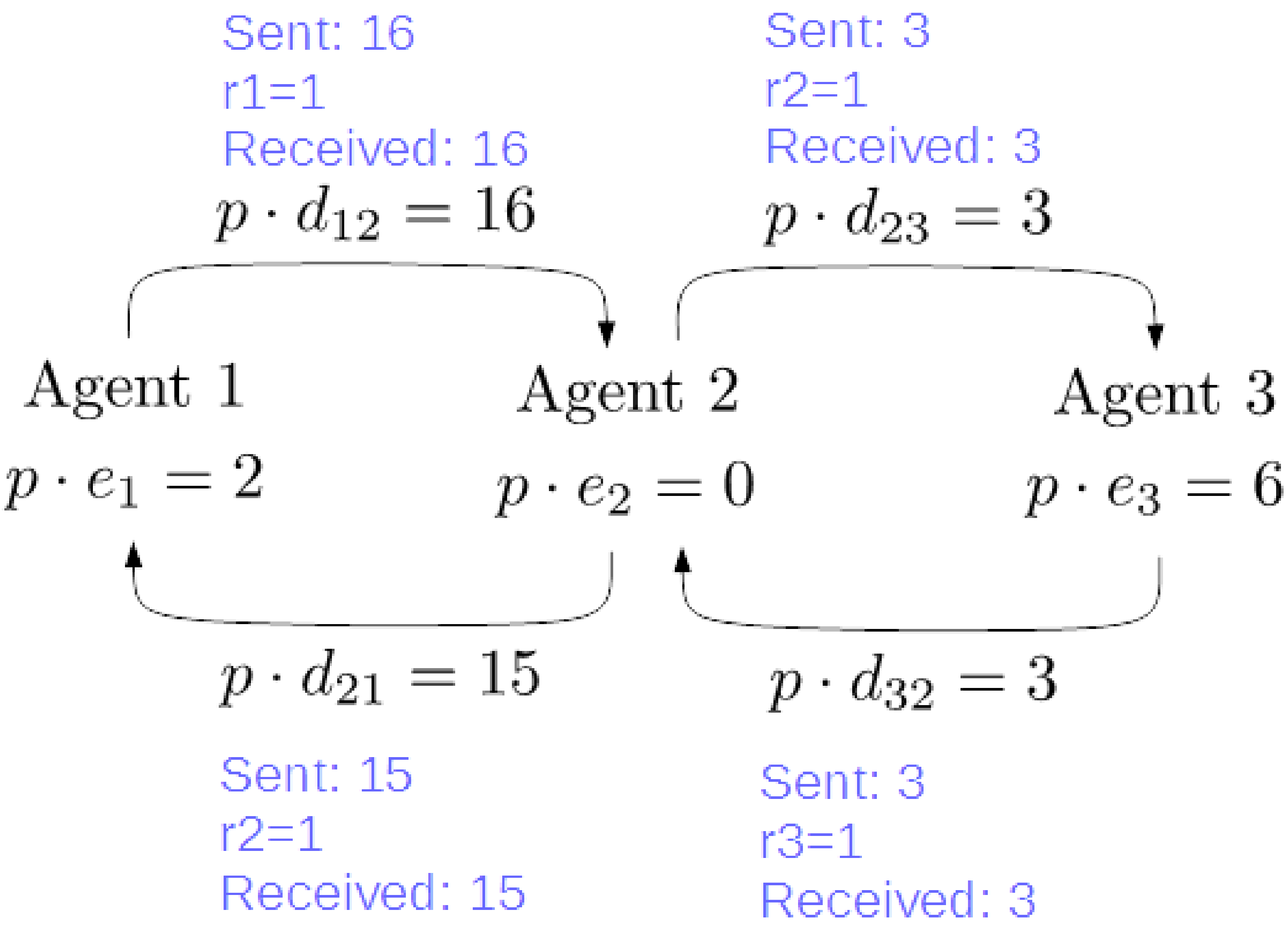
- Net debts.
- Partially Forgive debts can Pareto improve, eliminating the bad equilibria.
- Creating the right new debt?

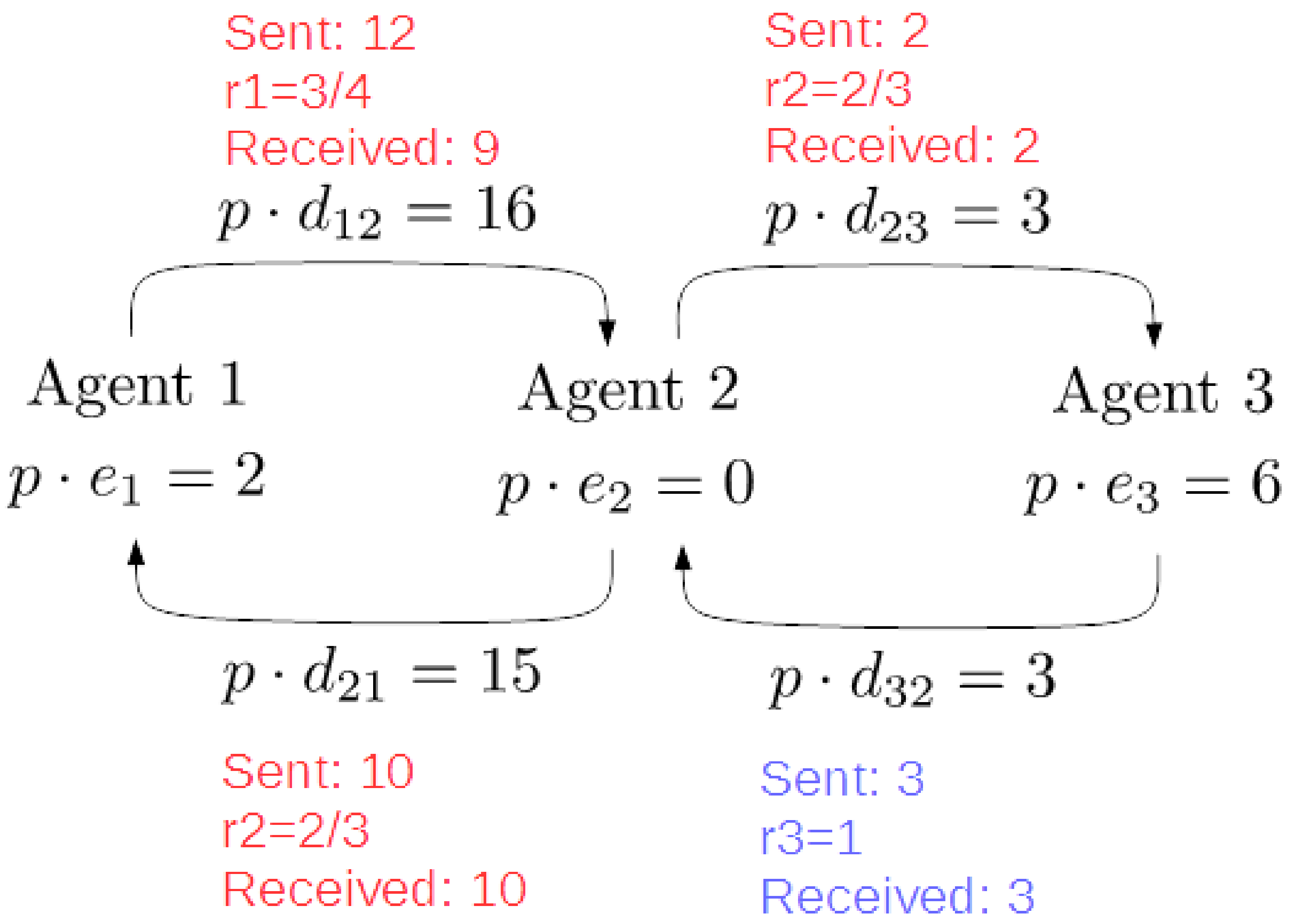
Appendix

Default Multiplicity

- When seizing collateral destroys part of the collateral
- Equilibrium with default definitely might be Pareto inferior.
- But also
- No longer get unique $r^*(p)$.
- So can get more multiplicity.
- More Pareto ranking of default equilibria.

$$
\begin{aligned}
r_1 &= \min\left(1, \frac{p\cdot e_1 + r_2 p\cdot d_{21} + r_3 p\cdot d_{31}}{p\cdot(d_{12}+d_{13})}\right) \\
r_2 &= \min\left(1, \frac{p\cdot e_2 + r_1^2 p\cdot d_{12} + r_3 p\cdot d_{32}}{p\cdot(d_{21}+d_{23})}\right) \\
r_3 &= \min\left(1, \frac{p\cdot e_3 + r_1^2 p\cdot d_{13} + r_2 p\cdot d_{23}}{p\cdot(d_{31}+d_{32})}\right)
\end{aligned}
$$


Sent: 16
r1=1
Received: 16
$p \cdot d_{12} = 16$
Sent: 3
r2=1
Received: 3
$p \cdot d_{23} = 3$
Agent 1
$p \cdot e_1 = 2$
Agent 2
$p \cdot e_2 = 0$
Agent 3
$p \cdot e_3 = 6$
$p \cdot d_{21} = 15$
$p \cdot d_{32} = 3$
Sent: 15
r2=1
Received: 15
Sent: 3
r3=1
Received: 3


Sent: 12
r1=3/4
Received: 9
$p \cdot d_{12} = 16$
Sent: 2
r2=2/3
Received: 2
$p \cdot d_{23} = 3$
Agent 1
$p \cdot e_1 = 2$
Agent 2
$p \cdot e_2 = 0$
Agent 3
$p \cdot e_3 = 6$
$p \cdot d_{21} = 15$
$p \cdot d_{32} = 3$
Sent: 10
r2=2/3
Received: 10
Sent: 3
r3=1
Received: 3