The Econometrics of Financial Duration Modeling

Asymptotics and Inference

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A key issue

If $x_1, x_2, ..., x_n$ nice (iid, or stationary, ergodic etc.) then

LLN:	$n^{-1} \sum_{i=1}^{n} x_i \rightarrow_p \mathbb{E}[x_i]$ as $n \rightarrow \infty$

CLT:	$n^{-1/2} \sum_{i=1}^{n} (x_i - \mathbb{E}[x_i]) \rightarrow_d N(0, \sigma_x^2)$ as $n \rightarrow \infty$

What if n is random and depends on x_i?

What the paper is about

We consider the Autoregressive Conditional Duration [ACD] model introduced by Engle and Russell (1998, ECMA) and Engle (2000, ECMA)

These models are strictly connected to ARCH (stationarity and geometric ergodicity; likelihood function, QMLE, etc), with x_i (duration) replacing r_i^2 (squared returns)

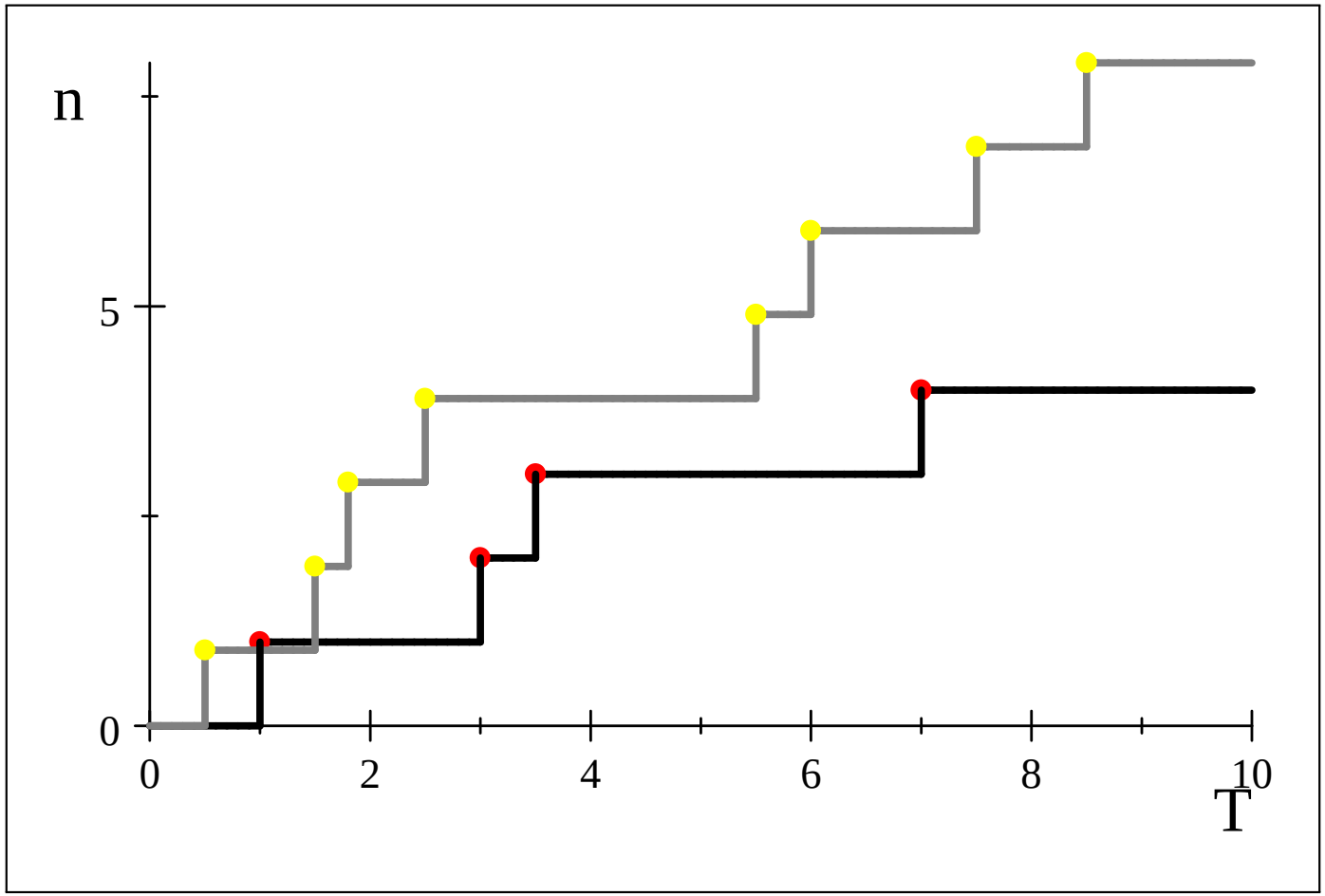
Common wisdom: the QMLE of ACD has the same properties of the QMLE for ARCH (same conditions for, e.g., consistency and asymptotic normality)

We show that this is not the case: the QMLE for ACD has very different asymptotic properties.

Key feature: the marginal (unconditional) distribution of the duration matters (while ARCH asymptotics is independent of the marginal distribution of the returns).

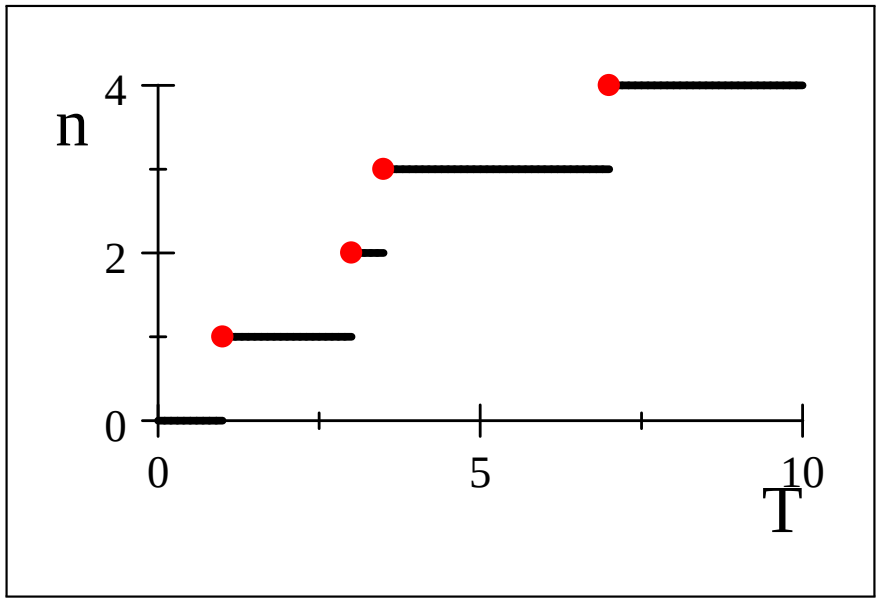
ACD Model – setup

Engle and Russell (1998, ECMA):
Fixed interval $[0, T]$ with events $0 = t_0 < t_1 < t_2 < ... < t_{n_T} \leq T$
Observe n_T durations $\{x_i\}_{i=1}^{n_T}$ (waiting times between events, $x_i = t_i - t_{i-1}$).



ACD Model – setup

$$n_T = 4 = \max_k \{x_1 + x_2 + ... + x_k < T\}$$



ACD Model – setup

With the **random** n_T defined by,

$$n_T = \max_k \{x_1 + x_2 + ... + x_k < T\}$$

the simplest ACD model for x_i is given by

$$\begin{aligned} x_i &= \psi_i \varepsilon_i, \quad i = 1, 2, ..., n_T \\ \psi_i &= \omega + \alpha x_{i-1} \end{aligned}$$

and ε_i i.i.d., exponentially distributed.

ACD Model – properties

Consider the ACD with ε_i iid exp distributed:

$$x_i = \psi_i \varepsilon_i,\ \psi_i = \omega + \alpha x_{i-1}$$

The coefficient α determines the properties of $\{x_i\}$.

α	Geo. Ergodic	Mean	Variance	Tail index
$0 \leq \alpha < 0.7$	$\surd$	$E[x_i] < \infty$	$E[x_i^2] < \infty$	$\kappa > 2$
$0.7 \leq \alpha < 1$	$\surd$	$E[x_i] < \infty$	$E[x_i^2] = \infty$	$1 < \kappa \leq 2$
$1 \leq \alpha < 1.8$	$\surd$	$E[x_i] = \infty$	$E[x_i^2] = \infty$	$\kappa \leq 1$
$\alpha > 1.8$	$\times$	$E[x_i] = \infty$	$E[x_i^2] = \infty$	?

Notes:

1. Tail index κ: $P(x_i > x) \sim x^{-\kappa}$ as $x \to \infty$.
2. With γ denoting the Euler's constant, $\exp(\gamma) = 1.8...$
3. We have the following surprisingly simple result (Lemma 2.2)

$$\alpha = [\Gamma(\kappa + 1)]^{-1/\kappa}$$

ACD Model – QMLE

The log-likelihood function is given by

$$\begin{aligned} L_T(\alpha) &= -\sum_{i=1}^{n_T} (\log \psi_i + x_i/\psi_i) \\ \psi_i &= \omega + \alpha x_{i-1} \end{aligned}$$

and the QMLE is defined as

$$\hat{\alpha} = \arg\max L_T(\alpha)$$

- Note: $L_T(\alpha)$ identical to ARCH(1) likelihood function with x_i squared returns

$$L_T(\alpha) = -\sum_{i=1}^{n_T} \left(\log \psi_i + r_i^2/\psi_i\right), \ \psi_i = \omega + \alpha r_{i-1}^2$$

- ARCH(1) theory: Asymptotic normality if $\{x_i\}$ geo ergodic (and even if not).

ACD Model – QMLE

Engle and Russell (1998, Econometrica):

COROLLARY TO LEE AND HANSEN (1994): *If*:

(A) $E_{i-1}(x_i) \equiv \psi_{0,i} = \omega_0 + \alpha_0 x_{i-1} + \beta_0 \psi_{0,i-1}$;

(B) $\varepsilon_i \equiv x_i / \psi_{0,i}$ *is* (i) *strictly stationary and ergodic*, (ii) *nondegenerate*, (iii) *has bounded conditional second moments*, (iv) $\sup_i E[\ln(\beta_0 + \alpha_0 \varepsilon_i) \mid F_{i-1}] < 0$ *a.s.*;

(C) $\theta_0 \equiv (\omega_0, \alpha_0, \beta_0)$ *is in the interior of* Θ;

(D) $$L(\theta) = -\sum_{i=1}^{N(T)} \left\{ \log(\psi_i) + \frac{x_i}{\psi_i} \right\} \quad \textit{where}$$

$$\psi_i = \omega + \alpha x_i + \beta \psi_{i-1} \quad \textit{for } i > 1,$$

$$\psi_i = \omega / (1 - \beta) \quad \textit{for } i = 1;$$

then:

(a) *the maximizer of L will be consistent and asymptotically normal with a covariance matrix given by the familiar robust standard errors from Lee-Hansen; and*

(b) *the model can be estimated with ARCH software by taking* $\sqrt{x_i}$ *as the dependent variable and setting the mean to zero.*

ACD Model – Main result

Consider behavior of $\hat{\alpha}$ as $T \to \infty$ (and hence $n_T \to \infty$):

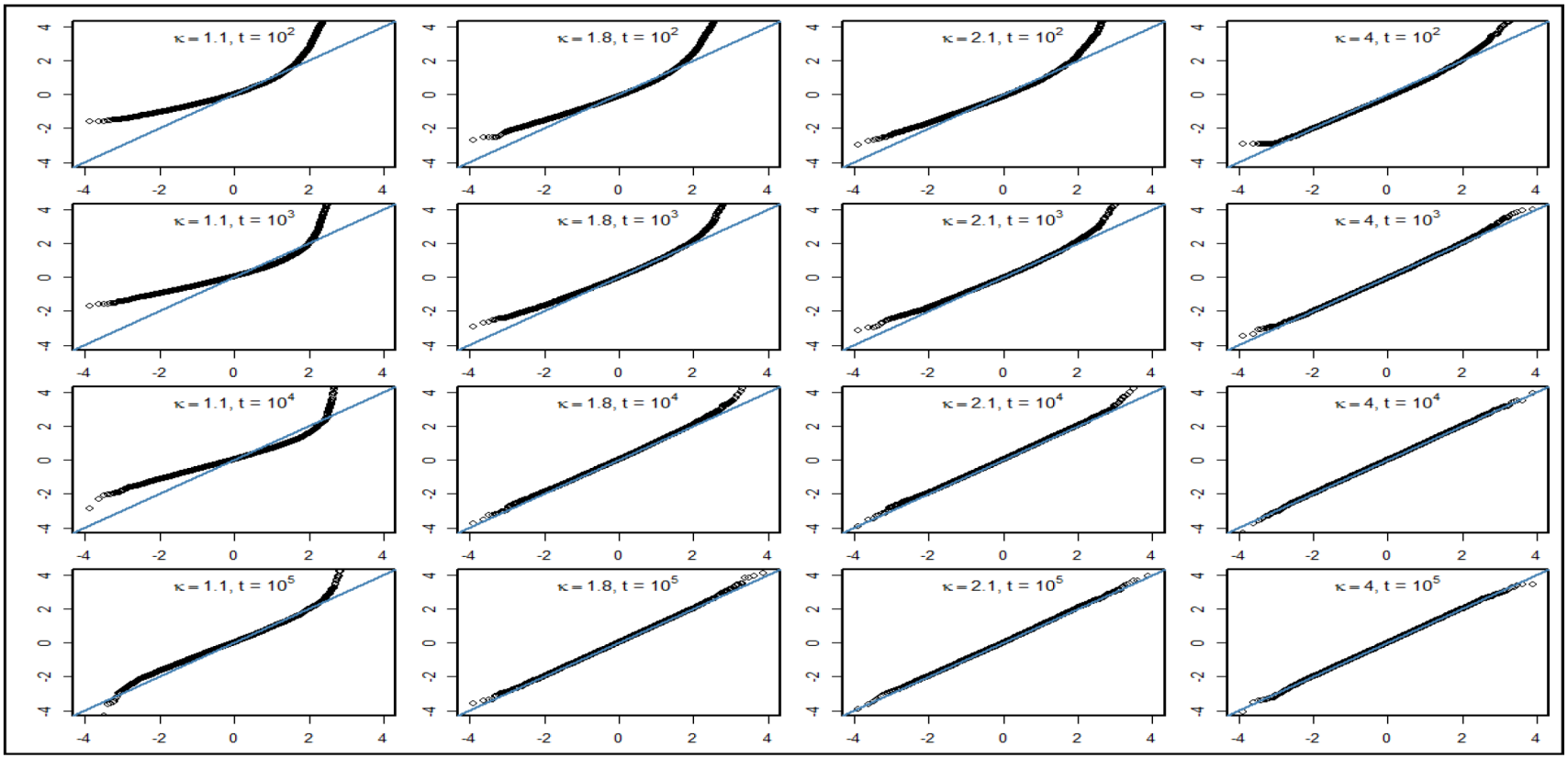
$\alpha_0 \in (0,1)$	$T^{1/2}(\hat{\alpha} - \alpha_0) \xrightarrow{d} \mathrm{N}\left(0, \mu\sigma_\alpha^2\right)$	$\mu = E[x_i]$
$\alpha_0 \in (1, \exp(\gamma))$	$T^{\kappa/2}(\hat{\alpha} - \alpha_0) \xrightarrow{d} \mathrm{MN}\left(0, \eta_\kappa\sigma_\alpha^2\right)$	η_κ r.v.

- New asymptotic result: New asymptotic limit and rate of convergence depends on tail index κ.
- Mixed Gaussian if $\kappa < 1$ and rate is $T^{\kappa/2}$ (and hence, t ratios asymptotically N).
- Similar to cointegration analysis (Johansen, Phillips): $T(\hat{\beta} - \beta_0) \xrightarrow{d} \mathrm{MN}$.
- Different from ARCH: $T^{1/2}(\hat{\alpha} - \alpha_0) \xrightarrow{d} \mathrm{N}$ for any α_0 (and tail index).
- Case $\alpha_0 = 1 \mid \kappa = 1$: asymptotic distribution unknown.

Simulations

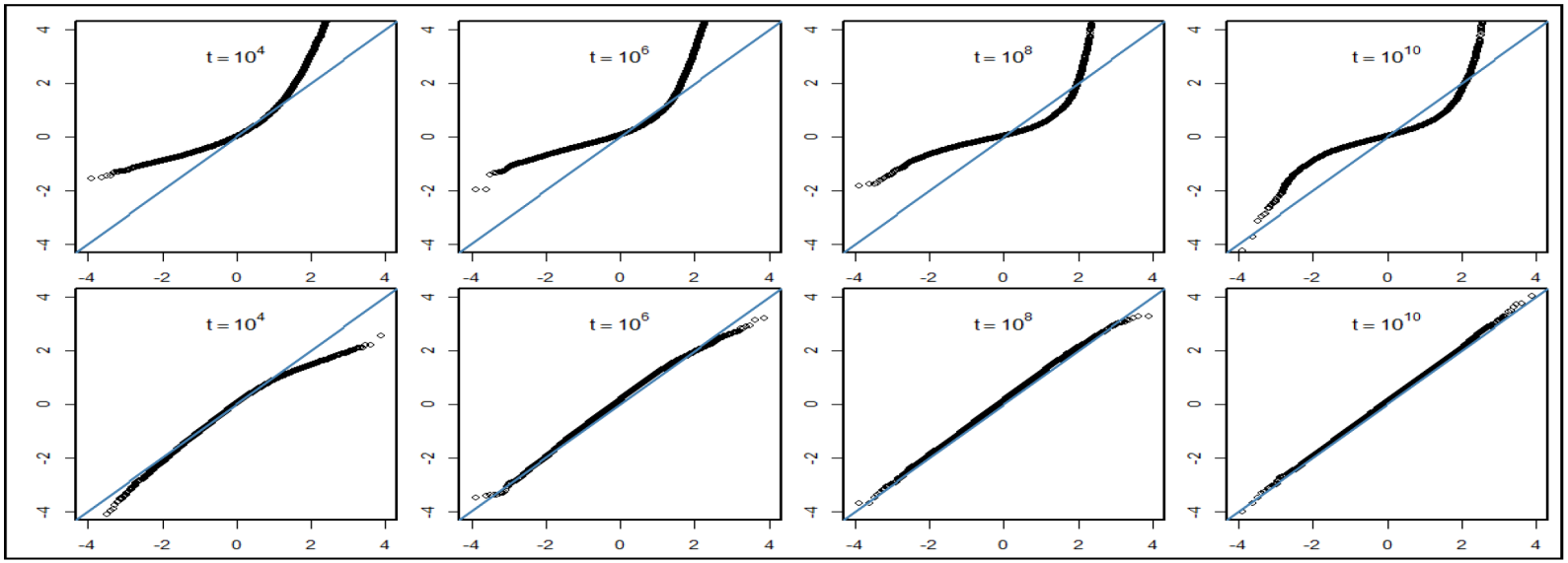
Simulate $\{x_i\}_{i=1}^{n_T}$ for different $\kappa > 1$ ($\alpha < 1$) - QQ plots of $\hat{\alpha}$ against Gaussian distribution for

$$\left\{\left(\kappa, \alpha = \left[\Gamma\left(\kappa + 1\right)\right]^{-1/\kappa}\right)\right\} = \left\{\left(1.1, 0.96\right); \left(1.8, 0.75\right); \left(2.1, 0.68\right); \left(4, 0.45\right)\right\}$$



Simulations

Simulations of $\{x_i\}_{i=1}^{n_T}$ for $\kappa = 0.5 < 1$ ($\alpha = 1.3 > 1$).
Top: QQ plots of $\hat{\alpha}$ against Gaussian
Bottom: QQ plots of t ratios.



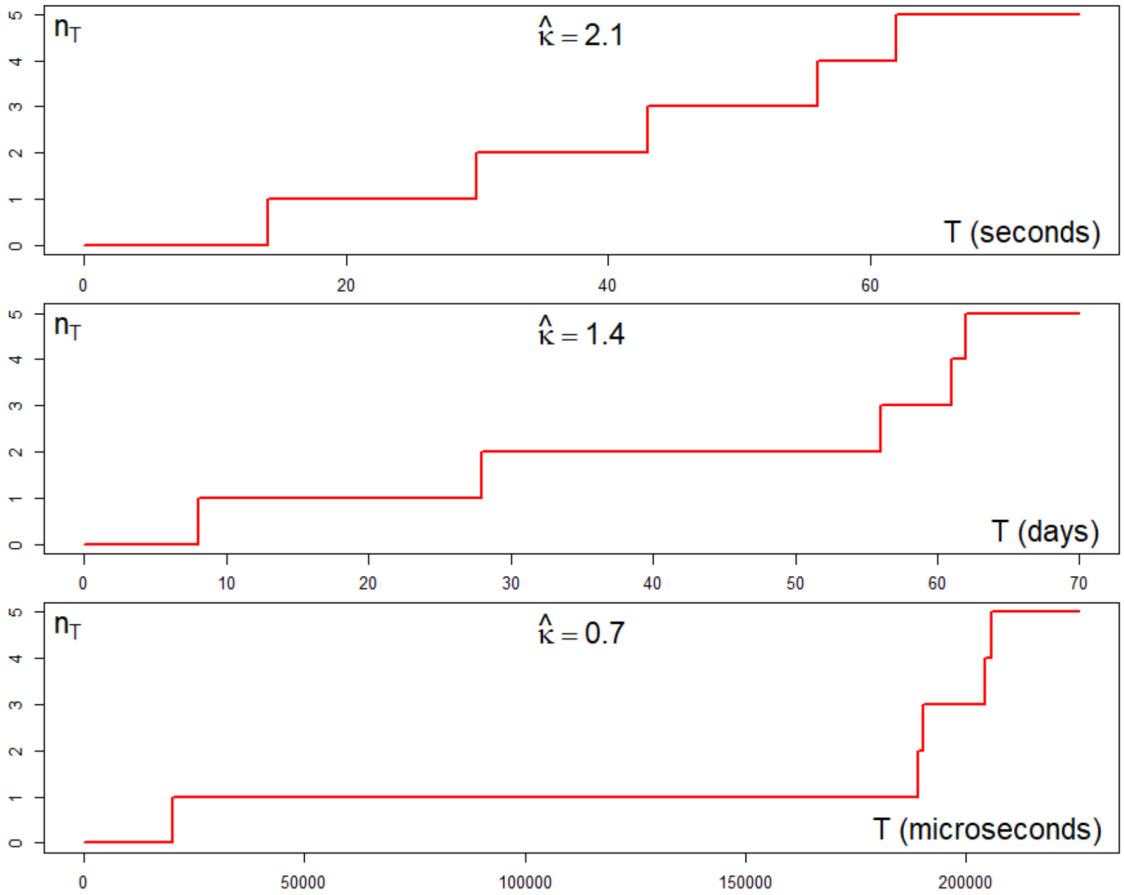
Empirical Relevance and Implications

- Used several places: $\psi_i = \omega + \alpha x_{i-1} + \beta \psi_{i-1}$ (GARCH).

- Tail index empirically:
  $\hat{\kappa} = 2.1 > 2$ in Engle and Russell (1998)
  $\hat{\kappa} = 1.4 > 1$ in DJ data Embrechts, Liniger and Lin (2011)
  $\hat{\kappa} = 0.7 < 1$ SPY transaction data (single day: 2019:07:31)

- Gaussian approximation: N
  For $\kappa > 2$ Gaussian approximation appears fine.
  For $1 < \kappa < 2$ Gaussian approximation not always good in finite samples.
  For $\kappa < 1$ not Gaussian (but MN so t-ratios ok – large samples).

Empirical



	Emp. Mean	Emp. Median	Emp. StdDev.
IBM	29	15	45
DJ	10	4	16
SPY	0.1	10^{-4}	3

Essential Insight

Recall

$$n_T = \max_k \{x_1 + x_2 + ... + x_k < T\}$$

- $n_T \to \infty$ not sufficient for CLT to apply, $n_T^{-1/2} \sum_{i=1}^{n_T} \xi_i$
- Existing (renewal) literature place conditions on $\{x_i\}$ (usually i.i.d. or m-dependent)
- Under these conditions, if $E(x_i) < \infty$ (finite mean):

$$\frac{n_T}{T} \xrightarrow{\text{a.s.}} 1/E[x_i]$$

and as $T \to \infty$, $n_T^{-1/2} \sum_{i=1}^{n_T} \xi_i$ behaves similarly to $n^{-1/2} \sum_{i=1}^{n} \xi_i$ with $n \to \infty$.

Essential Insight

Extension to ACD:

- If x_i geometrically ergodic with finite mean, i.e. $\kappa > 1$:

$$\frac{n_T}{T} \xrightarrow{\text{a.s.}} 1/E\left[x_i\right]$$

- If x_i geometrically ergodic with infinite mean, i.e. $\kappa < 1$:

$$\frac{n_T}{T^\kappa} \xrightarrow{d} 1/\eta_\kappa$$

  where η_κ is a non-standard random variable.

- The latter result does not imply $n_T^{-1/2} \sum_{i=1}^{n_T} \xi_i$ behaves like $n^{-1/2} \sum_{i=1}^{n} \xi_i$.
- Need conditions on the joint convergence of

$$\left(\frac{n_T}{T^\kappa}, \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor n \cdot \rfloor} \xi_i \right) \in \mathbb{R}^+ \cup \mathbb{D}[0, \infty).$$

Essential Insight (score and information)

With $S_T = \partial L_T(\alpha_0)/\partial\alpha$ and $I_T = -\partial^2 L_T(\alpha_0)/\partial\alpha^2$.

Theorem

For $\kappa > 1$, then $\frac{n_T}{T} \xrightarrow{a.s.}$ constant and

$$\left(T^{-1/2} S_T, T^{-1} I_T\right) \xrightarrow{d} (\mathrm{N}, \textit{constant})$$

Theorem

For $\kappa < 1$, then $\frac{n_T}{T^\kappa} \xrightarrow{d}$ r.v. and

$$\left(T^{-\kappa/2} S_T, T^{-\kappa} I_T\right) \xrightarrow{d} (\mathrm{MN}, \textit{r.v.})$$

Comparison to ARCH/MEM

Here

$$x_i = \psi_i^{1/2} z_i, \quad \psi_i = \omega + \alpha x_{i-1}^2 \quad i = 1, 2, 3, ..., n$$

and x_i geom. ergodic for $\alpha < 2\exp(\gamma) = 3.6$. The log-likelihood has the same form as the ACD likelihood:

$$L_T(\alpha) = -\sum_{i=1}^{n} \left(\log \psi_i + x_i^2 / \psi_i \right).$$

However, for any $\alpha < 2\exp(\gamma)$,

$$\left(n^{-1/2} S_T, n^{-1} I_T \right) \rightarrow (\mathrm{N}, \text{constant}) \quad \text{as } n \rightarrow \infty$$

- and in fact also for $\alpha > 2\exp(\gamma)$.

Comparison to ARCH/MEM

- There is a crucial difference between the setting of ACD models and the classic ARCH framework.
- Given a time span $[0, T]$:
  - in the ACD setting, the number of observations n_T over $[0, T]$ is random;
  - for ARCH models, the number of observations $n_T = n$ is a deterministic function of T (e.g., T/δ, δ being the sampling frequency)
- As a consequence, the classic asymptotic results for ARCH/MEM can equivalently be given in form $\sqrt{T}(\theta_T - \theta_0)$ or $\sqrt{n}(\theta_T - \theta_0)$.
- This is not the case for ACD, where $\sqrt{T}(\theta_T - \theta_0)$ and $\sqrt{n_T}(\theta_T - \theta_0)$ may have distinct distributional limits.

Conclusion

- New asymptotic theory for ACD models.
  Here exemplified by the simple ACD of order one.
- Tail index κ is important for both asymptotic and finite sample properties.
- MN/t-ratios: Small sample issues.
  Bootstrap: Cavaliere, Lu, Rahbek and Østergaard, 2022, JoE.