State-dependent non-linear search and matching

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Overview and Findings

What if we solve the DMP model (Mortensen and Pissarides, 1994) with endogenous job destruction exactly?

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  These model properties are missed with approximation methods and help bring the model closer to the data.

Plan

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Literature (not comprehensive)

Lots of puzzles e.g.: (Shimer, 2005) and (Shimer et al., 2010)

Lots of puzzle resolution suggestions e.g. : (Hall, 2005) (Hagedorn and Manovskii, 2008)

Theoretical interpretations : (Ljungqvist and Sargent, 2017)

Computations : (Petrosky-Nadeau and Zhang, 2017) and (Petrosky-Nadeau, Zhang, and Kuehn, 2018)

Lots of other observations e.g. job destruction driving dynamics (Coles and Moghaddasi Kelishomi, 2018)

Baseline DMP Model

Model equations

Description	Equation
Aggregate productivity process	$\log(A_t) = \rho_a \log(A_{t-1}) + \epsilon_t$
Law of motion of employment	$n' = (1 - \Phi(\frac{\tilde{Z}' + \sigma_z^2/2}{\sigma_z}))[(1-\lambda)n + vq(\theta)]$
Job creation condition	$\frac{\kappa}{q(\theta)} = \beta E_t[(1 - \Phi(\frac{\tilde{Z}' + \sigma_z^2/2}{\sigma_z}))\frac{\mu'}{\mu}\left(A'\bar{Z}' - \bar{w}' + (1-\lambda)\frac{\kappa}{q(\theta')}\right)]$
Job destruction condition	$A\tilde{Z} = b + \frac{\eta}{1-\eta}\kappa\theta - \frac{1}{1-\eta}\frac{\kappa}{q(\theta)}$
Conditionally exp. of the idiosyncratic match productivity	$\bar{Z} = \frac{\Phi((\tilde{z} - \sigma_z^2/2)/\sigma_z)}{1 - \Phi((\sigma_z^2/2 + \tilde{z})/\sigma_z)}$
Bargained average wage	$\bar{w} = (1-\eta)b + \eta[A\bar{Z} + \kappa\theta]$
Probability of finding a job	$p(\theta) = \theta(1+\theta^l)^{-\frac{1}{l}}$ or alternative C-D: $m\theta^{1-\xi}$
Probability of filling a vacancy	$q(\theta) = (1+\theta^l)^{-\frac{1}{l}}$ or alternative C-D: $m\theta^{-\xi}$
Resource constraint	$c = A\bar{Z}n - \kappa v$
Tightness	$\theta = \frac{v}{u}$
Unemployed	$u = 1 - n$

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Model equations

Job creation condition determines vacancy posting policy tightness $\theta = \frac{v}{1-n}$:

$$\frac{\kappa}{q(\theta)} = \beta E_t[(1 - \Phi(\frac{\tilde{Z}' + \sigma_z^2/2}{\sigma_z}))\frac{\mu'}{\mu}\left(A'\bar{Z}' - \bar{w}' + (1-\lambda)\frac{\kappa}{q(\theta')}\right)]$$

Job destruction condition determines $\tilde{Z}$:

$$A\tilde{Z} = b + \frac{\eta}{1-\eta}\kappa\theta - \frac{1}{1-\eta}\frac{\kappa}{q(\theta)}$$

Law of motion of the employment rate:

$$n' = (1 - \underbrace{\Phi(\frac{\tilde{Z}' + \sigma_z^2/2}{\sigma_z}))}_{\substack{\text{Endogenous job destruction} \\ \text{driven by } \tilde{Z}}} [(1-\lambda)n + vq(\theta)]$$

Unemployment is $1 - n$.

Ergodicity - Based Solution Method

Usual ways of solving the model

Policy rule: $\theta = JC(A, n)$ with the law of motion implies path for n' and an expected path for A'. We know that the given the policy rule and the law of motion of n and A the policy rule should be such that the expectation is correct. Thus $LHS = \frac{\kappa}{q(\theta)}$.

$$RHS_t^{(i)} = \beta \mathbb{E}_t \left[(1 - \Phi_{t+1}^{(rv)}) \left(\frac{c_{t+1}^{(rv)}}{c_t^{(i)}} \right)^{-\sigma} \left(A_{t+1}^{(rv)} \overline{Z}_{t+1}^{(rv)} - w_{t+1}^{(rv)} + (1 - \lambda) \frac{\kappa}{q_{t+1}^{(rv)}} \right) \right] \quad (1)$$

$$R(A, n, JC(A, n)) = 0$$

Difficulty in computing for a stochastic problem JC.
Perturbation: Approximate the function with a Taylor series around the steady state.
Projection Approximate $JC(A, n)$ with a polynomial $\hat{JC}(A, n, \gamma)$

Ergodicity- based method Algorithm 1

1. Simulate a T-period path $\{A_t\}_{t=0}^{T}$. For an ith iteration, guess the time series of the following allocations:
$$\{n_t^{(i)}, \theta_t^{(i)}, v_t^{(i)}, u_t^{(i)}, c_t^{(i)}, w_t^{(i)}, \widetilde{Z}_t^{(i)}, \overline{Z}_t^{(i)}, \Phi_t^{(i)}\}_{t=1}^{T}.$$
2. Solve the problem backward from the terminal period T:
$$RHS_t^{(i)} = \beta \mathbb{E}_t \left[(1-\Phi_{t+1}^{(rv)}) \left(\frac{c_{t+1}^{(rv)}}{c_t^{(i)}} \right)^{-\sigma} \left(A_{t+1}^{(rv)} \overline{Z}_{t+1}^{(rv)} - w_{t+1}^{(rv)} + (1-\lambda) \frac{\kappa}{q_{t+1}^{(rv)}} \right) \right]$$
Then, using RHS, obtain $q(\theta_t^*)$ as follows:
$$q(\theta_t^*) = \frac{\kappa}{RHS_t^{(i)}} \text{ providing } \{q_t^*, v_t^*, w_t^*, c_t^*, \widetilde{Z}_t^*, \overline{Z}_t^*, \Phi_t^*\}_{t=1}^{T}$$
3. Using the conditionally optimal allocations with superscript $*$, we simulate the state forward:
$$n_t^* = (1-\Phi_t^*)[(1-\lambda)n_{t-1}^* + v_{t-1}^* q_{t-1}^*] \text{ providing } \{n_t^*, u_t^*\}_{t=1}^{T} \tag{2}$$
4. Check if the following inequality is satisfied:[1]
$$tol > \sup_{y \in \Omega} \sup_t ||y_t^* - y_t^{(i)}|| \qquad where \quad \Omega = \{n, q, v, u, c, w, \widetilde{Z}, \overline{Z}, \Phi\}$$
5. If the inequality is satisfied, ith guess is the solution. Otherwise, we update the guess using the following convex combination and go back to step 2.

Ergodicity- based method Algorithm 2

Part 1 identified state contingent allocations for the following variables:

$$\{\Phi^{rv}_{t+1}, c^{rv}_{t+1}, \overline{Z}^{rv}_{t+1}, w^{rv}_{t+1}, q^{rv}(\theta_{t+1})\}$$

Due to the ergodicity of the simulated path, we can find a period where the endogenous aggregate state is the same as period $t+1$ while the exogenous shocks are realized differently on the equilibrium path.

1. Form a partition $\mathcal{P}^{\widetilde{A}}$ of the simulated path based on the aggregate productivity realizations.

$$\mathcal{P}^{\widetilde{A}} = \{\tau | A_\tau = \widetilde{A}\} \text{ for } \forall \widetilde{A} \in \mathcal{A}.$$

where $\mathcal{A}$ is the support (grids) of the productivity.

3-2. For each $\widetilde{A}$, find period $\tau^{\widetilde{A}} \in \mathcal{P}^{\widetilde{A}}$ such that the number of employed workers $n^{(i)}_\tau$ is identical (closest) to period $t+1$.

$$\tau^{\widetilde{A}} = \arg\min_{\tau \in \mathcal{P}^{\widetilde{A}}} ||n^{(i)}_\tau - n^{(i)}_{t+1}||$$

3-3. Compute the rationally expected future variables using the allocations in period $\tau^{\widetilde{A}}$ for each $\widetilde{A}$. For example, the computation of Equation (2) is as follows:

$$RHS^{(i)}_t = \beta \sum_{\widetilde{A} \in \mathcal{A}} \Gamma_{A,\widetilde{A}} \left[(1 - \Phi^{(i)}_{\tau^{\widetilde{A}}}) \left(\frac{c^{(i)}_{\tau^{\widetilde{A}}}}{c^{(i)}_t} \right)^{-\sigma} \left(\widetilde{A}\overline{Z}^{(i)}_{\tau^{\widetilde{A}}} - w^{(i)}_{\tau^{\widetilde{A}}} + (1-\lambda) \frac{\kappa}{q^{(i)}_{\tau^{\widetilde{A}}}} \right) \right].$$

where $\Gamma_{A,\widetilde{A}}$ is the transition probability of the aggregate productivity from a level A to a level $\widetilde{A}$.

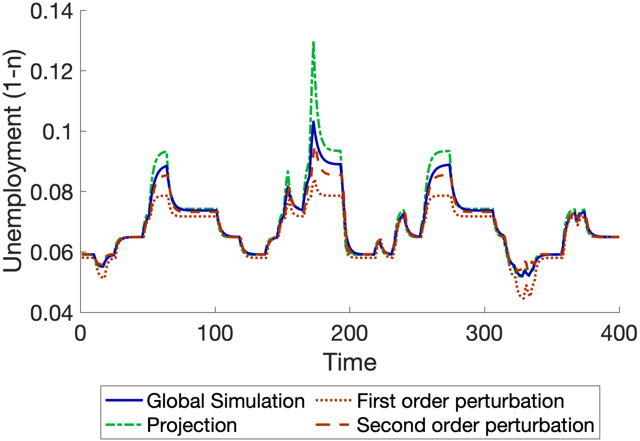
Ergodicity- based method Algorithm 2 simple terms

- Take each A' level for each t
- For each we have a set of corresponding tightness θ and outcomes $\{n', A'\}$
- $\{n', A'\}$ and θ uniquely define n
- We have an optimal tightness θ exactly fulfilling the policy function for each of state outcome $\{n', A'\}$ given any starting point $\{n, A\}$. With this we can simulate the model essentially without error.

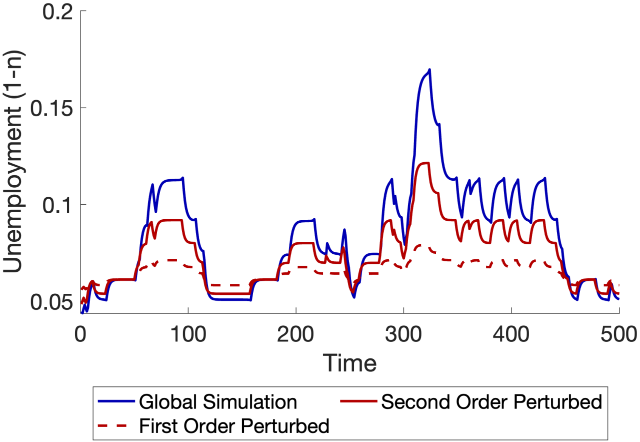
Results and Comparison

For the model with exogenous job destruction fixing $\Phi()$

Allows us to compare our method to (Petrosky-Nadeau and Zhang, 2017).



Unemployment in much more sensitive with endogenous job destruction



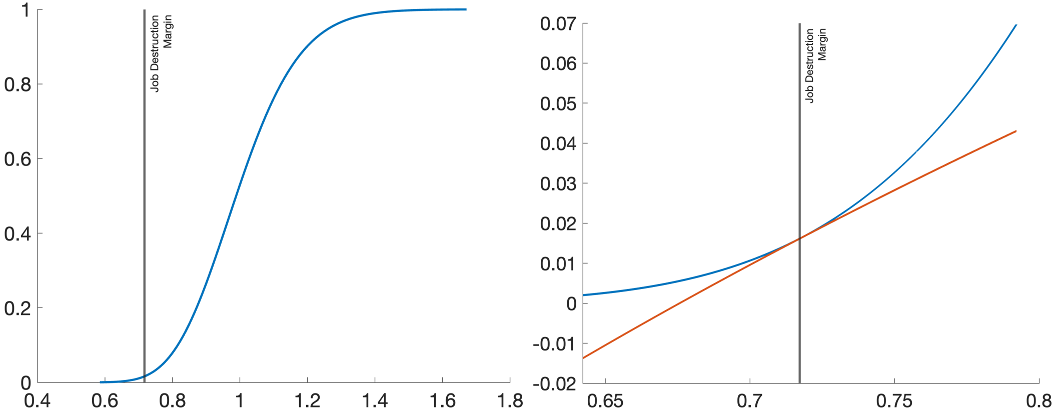
Calibration Steady State

	Global nonlinear	Linear	2nd-order
Mean (%)	6.641	6.132	6.15
Volatility	2.357	0.259	0.263
Skewness	0.95	-0.037	0.3
Kurtosis	3.937	2.67	2.817
Volatility: Quarterly + HPfilter (%)	12.383	2.783	2.784

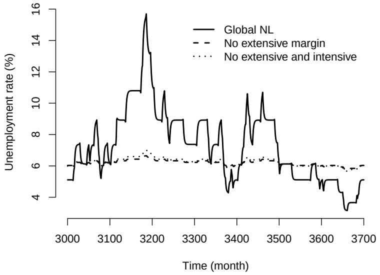
Calibration Steady State

What is going on?

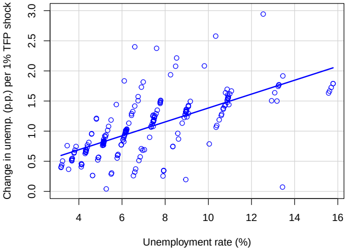
Difficult to approximate a distribution. Given normal times times a reasonable calibration requires being in a flat part. Response driven by the endogenous destruction margin (Coles and Moghaddasi Kelishomi, 2018).



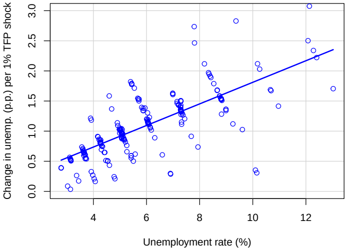
Most of unemployment driven by the destruction margin



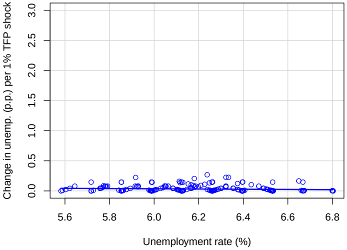
	Extensive	Intensive	Others	Total
Unemployment volatility (p.q.)	9.81	-0.22	2.57	12.38
Portion (%)	79.24	-1.8	20.76	100



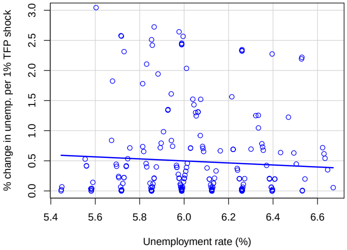
(a) Global nonlinear: $\Delta TFP > 0$



(b) Global nonlinear: $\Delta TFP < 0$



(c) Linear: $\Delta TFP > 0$

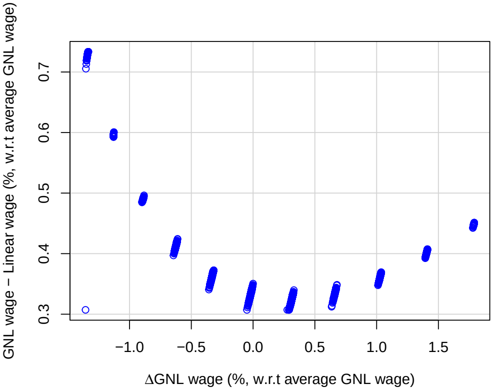


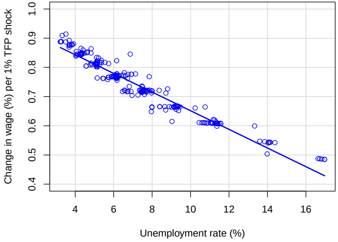
(d) Linear: $\Delta TFP < 0$

Wages are actually asymmetric, non-linear and state dependent in the model

Wage: $\bar{w} = (1-\eta)b + \eta[A\bar{Z} + \kappa\theta]$

In line with (Hall, 2005), (Hagedorn and Manovskii, 2008) and (Moscarini, 2005) observations.





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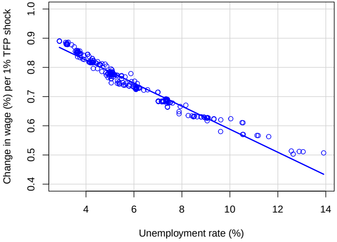

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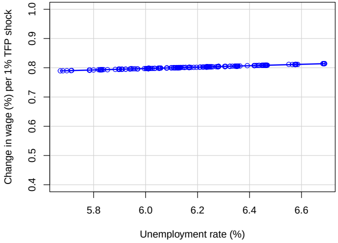

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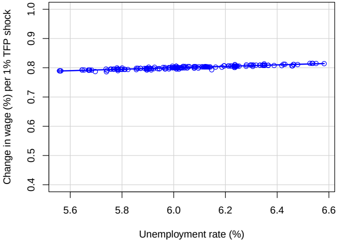

(d) Linear: $\Delta TFP < 0$

Conclusion

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- Solving old model with new accurate solution.
- Accurate solution provides novel insights.
- Unemployment respond non-linear and state dependent insights.
- Wages are non-linear asymmetric and state dependent.
- These results help bring the model closer to the data.
- More to come, e.g. stochastic discount factor, percent of puzzle explained by accurate solution...

Thank you for your attention!

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Appendix

Calibration

Parameter	**Value**	**Calibration**
b	0.672	Unemployment benefit from (Pissarides, 2009)
κ	0.067	Vacancy posting cost (Shimer, 2005)
m	0.291	To match the US job finding rate
λ	0.0272	Exogenous separations
ξ_1	0.635	Matching function parameter CD
ξ_2	0.525	Matching function parameter CES
β	0.9967	Discount factor
ρ_A	0.983	Persistence of the aggregate process (Leduc and Liu, 2016)
σ_ϵ	0.0295	Exogenous shocks
σ_z	0.2	Idiosyncratic productivity

Back

Steady State values

Steady State	Value	Description
q	0.1620	Probability of finding a Worker
p	0.4073	Probability of finding a job
n	0.9350	Employment rate
$u = 1 - n$	0.0649	Unemployment rate
θ	2.5147	Tightness
η	0.6353	Bargaining power of the worker
c	0.8915	Consumption

Back