Games under Network Uncertainty

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Motivation

- Network participants rarely have complete information about the full architecture of the network.
- Common knowledge of network architecture is a typical assumption in the network games literature.
- How does the agent behavior change in the absence of complete information about the network architecture?
- How will agents employ local information towards equilibrium play?

Related Literature

- **Games on Networks**
  - Ballester, Calvo and Zenou (2006).
- **Unknown Network Architecture**
  - Breza, Chandrasekhar and Salehi (2018).
- **Incomplete Information**
  - Galeotti, Goyal, Jackson, Vega-Redondo and Yariv. (2010).

Notations

- Adjacency representation of a graph:

$$\mathbf{g} = [g_{ij}] \in \{0,1\}^{n \times n}, \mathbf{g} = \mathbf{g}^T \text{and } g_{ii} = 0$$

- $N_i(\mathbf{g}) = \{j : g_{ij} = 1\}$ denotes the *neighborhood* of player i.
- $d_i(\mathbf{g}) = |N_i(\mathbf{g})|$ denotes the *degree* of player i.

Benchmark: Complete Information

- Agents are members of a network.
- Simultaneously exert $a_i \in \mathbb{R}$ to maximize linear quadratic payoffs:

$$u_i(a_i, \mathbf{a}_{-i}; \mathbf{g}) = a_i - \frac{1}{2}a_i^2 + \lambda \sum_{j=1}^{n} g_{ij} a_i a_j$$

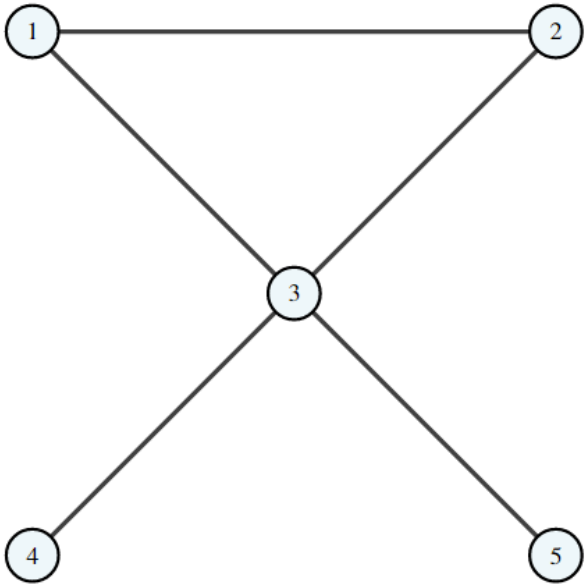
Benchmark: Complete Information

- Best response of each agent i:

$$a_i = 1 + \lambda \sum_{j=1}^{n} g_{ij} a_j \implies \mathbf{a} = \mathbb{1} + \lambda \mathbf{g}\mathbf{a}$$

- Nash Equilibrium efforts (K-B Centrality):

$$\mathbf{a} = (I - \lambda \mathbf{g})^{-1} . \mathbb{1} \equiv b(\mathbf{g}, \lambda) \quad [\text{for, } \lambda < \frac{1}{n-1}]$$



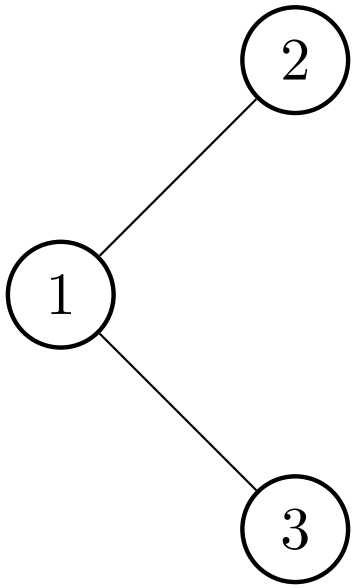
- $b(\mathbf{g}, \lambda) = I + \lambda \mathbf{g}.\mathbb{1} + \lambda^2 \mathbf{g}^2.\mathbb{1} + \ldots$
- $\mathbf{g}^k.\mathbb{1}$ is the number of walks of length k.

Timing of Events (Incomplete Information)

- Nature moves first and picks an undirected and unweighted network on n vertices from an ex-ante probability distribution over all possible graphs on n vertices.
- Each player realizes their connections (neighbors) but fails to see beyond that (cannot see the neighbors of their neighbors and so on).
- They update their belief on what the whole graph might look like with respect to the connections they realize.
- Agents exert efforts simultaneously to maximize the interim linear quadratic payoffs.

Graph Realization

Suppose the following graph involving 3 players is realized:



$$\Longrightarrow \mathbf{g} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Realization of each player

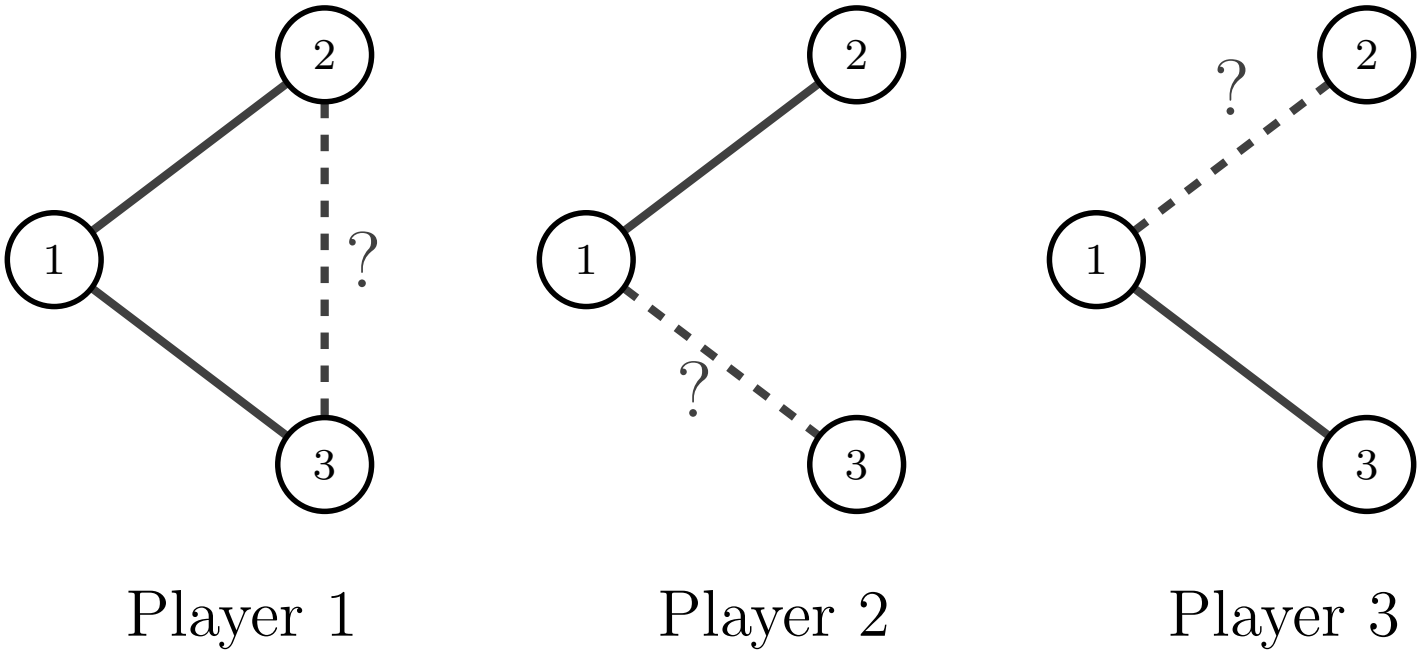
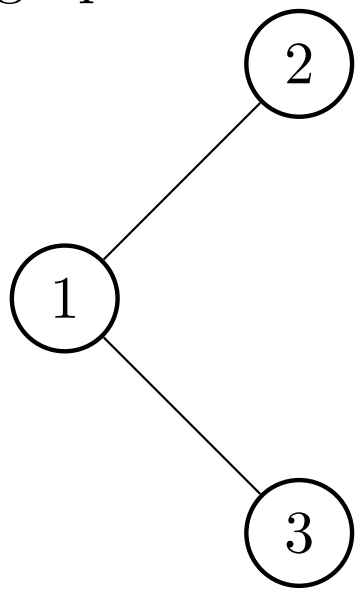


Figure 1: Realizations of the 3-players

Graph Realizations and Types

The realized graph:



$$\Longrightarrow \mathbf{g} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The rows of the adjacency matrix gives the types of each players.

Types for players

Definition 1 (Types)

G_i is the set of types for each player $i \in N$, where

$$G_i = \{\mathbf{g}_i = (g_{i1}, g_{i2}, ..., g_{in})_i \in \{0,1\}^n : g_{ii} = 0\}$$

- $\mathbf{g}_i^{t_i} \in G_i$, i.e. an element of G_i is essentially the identity of the neighbors of player i.
- $G = \times_{i \in N} G_i$ is the *type space*.
- But there might be $\mathbf{g} \in G$

$$\mathbf{g}_1^{t_1} = (0,1,1)_1, \quad \mathbf{g}_2^{t_2} = (0,0,1)_2 \quad \text{and} \quad \mathbf{g}_3^{t_3} = (1,0,1)_3$$

Ex-ante Belief

- p is the *admissible probability distribution* over the *type space*:

$$p(\mathbf{g}) = 0, \quad \forall \mathbf{g} \in G, \textit{such that } \mathbf{g} \neq \mathbf{g}^T$$

- $\Delta_A(G)$ is the set of all admissible probability distributions.
- **Assumption**: $p \in \Delta_A(G)$ and this is common knowledge.

Belief Updating

- The ex-post updated beliefs can be written as:

$$p(\mathbf{g}_j|\mathbf{g}_i) = \frac{p(\mathbf{g}_i, \mathbf{g}_j)}{p(\mathbf{g}_i)} = \frac{\sum_q p(G^q) \cdot \mathbb{I}\{\mathbf{g}_i, \mathbf{g}_j \in G^q\}}{\sum_k p(G^k) \cdot \mathbb{I}\{\mathbf{g}_i \in G^k\}}$$

  where $G^q, G^k \in G$.

- Since $p \in \Delta_A(G)$, the following *consistency* condition holds for any $\mathbf{g}_i \in G_i$:

$$p(\mathbf{g}_j|\mathbf{g}_i) = 0 \quad \forall \mathbf{g}_j \in G_j \ni g_{ij} \neq g_{ji}, \quad \forall i, j \in N$$

Interim utility

- $a_i(\mathbf{g}_i^{t_i}) \in A \equiv [0, \infty)$ is the type dependent effort exerted by each player.
- The interim payoff structure:

$$u_i(a_i(\mathbf{g}_i^{t_i}); \mathbf{a}_i(\mathbf{g}_i^{-t_i}), \mathbf{a}_{-i}) =$$
$$a_i(\mathbf{g}_i^{t_i}) - \frac{1}{2} a_i(\mathbf{g}_i^{t_i})^2 + \lambda a_i(\mathbf{g}_i^{t_i}) \sum_{j=1}^{n} g_{ij}^{t_i} \sum_{\mathbf{g}_j \in G_j} p(\mathbf{g}_j | \mathbf{g}_i^{t_i}) a_j(\mathbf{g}_j)$$

where $\lambda > 0$.

Best Responses and Equilibrium

- Given the payoff structure the best response for the i^{th} player of type $\mathbf{g}_i$ is given by:

$$a_i(\mathbf{g}_i^{t_i}) = 1 + \lambda \sum_{j=1}^{n} g_{ij}^{t_i} \sum_{\mathbf{g}_j \in G_j} p(\mathbf{g}_j | \mathbf{g}_i^{t_i}) a_j(\mathbf{g}_j)$$

- We use the Bayesian Nash Equilibrium notion.

▸ Utility

Best Responses (Vector Notation)

In vector notation the best responses are:

$$\mathbf{a} = \mathbf{1}_{n\gamma} + \lambda \mathbb{B}\mathbf{a}$$

where: $\mathbf{a} = [\mathbf{a}_i]_{i=1}^{n}$ and $\mathbf{a}_i = \left[a_i(\mathbf{g}_i^{t_i})\right]_{t_i=1}^{\gamma}$, with $\gamma = 2^{n-1}$ is the total number of types of each player.

$$[\mathbb{B}(n)] = \begin{pmatrix} \mathbf{0} & G_{1\sim 2} & \dots & G_{1\sim n} \\ G_{2\sim 1} & \mathbf{0} & \dots & G_{2\sim n} \\ \dots & \dots & \dots & \dots \\ G_{n\sim 1} & G_{n\sim 2} & \dots & \mathbf{0} \end{pmatrix}_{n\gamma \times n\gamma}$$

$$[G_{i\sim j}]_{t_i t_j} = g_{ij}^{t_i} p(\mathbf{g}_j^{t_j} | \mathbf{g}_i^{t_i}), \quad t_i, t_j = 1, 2, \dots \gamma \qquad \forall \mathbf{g}_j^{t_j} \in G_j, \mathbf{g}_i^{t_i} \in G_i$$

Equilibrium characterization

Theorem 1

For $0 \leq \lambda < \frac{1}{n-1}$, there exists an unique pure strategy BNE.

- For the incomplete information, the maximum complementarity strength is attained by decomposing it across the types of each neighbors than just links.

Walk Characterization

Theorem 2

With any admissible probability distribution, the equilibrium actions of the agents for a realized graph $\mathbf{g} \in G$ is given by:

$$a_i^*(\mathbf{g}_i^{t_i}) = 1 + \lambda\beta_{i,t_i}^{(1)} + \lambda^2\beta_{i,t_i}^{(2)} + \lambda^3\beta_{i,t_i}^{(3)} + \lambda^4\beta_{i,t_i}^{(4)} + \ldots\ldots \quad (1)$$

where $\mathbf{g}_i^{t_i}$ is the type of the i^{th} agent in that realized graph.

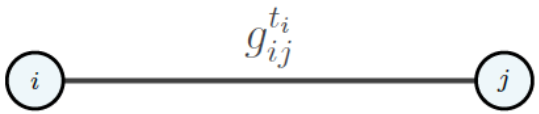
where;

$$\beta_{i,t_i}^{(1)} = \sum_{j=1}^{n} g_{ij}^{t_i}, \quad \beta_{i,t_i}^{(2)} = \sum_{j,k=1}^{n} \sum_{t_j=1}^{\gamma} g_{ij}^{t_i} g_{jk}^{t_j} p(\mathbf{g}_j^{t_j} | \mathbf{g}_i^{t_i})$$

$$\beta_{i,t_i}^{(3)} = \sum_{j,k,l=1}^{n} \sum_{t_j,t_k=1}^{\gamma} g_{ij}^{t_i} g_{jk}^{t_j} g_{kl}^{t_k} p(\mathbf{g}_k^{t_k} | \mathbf{g}_j^{t_j}) p(\mathbf{g}_j^{t_j} | \mathbf{g}_i^{t_i}) \quad \text{and so on.}$$

Expected Complementarity

$$\beta_{i,t_i}^{(3)} = \sum_{j,k=1}^{n} \sum_{l=1}^{n} g_{ij}^{t_i} \sum_{t_j=1}^{\gamma} g_{jk}^{t_j} p(\mathbf{g}_j^{t_j} | \mathbf{g}_i^{t_i}) \sum_{t_k=1}^{\gamma} g_{kl}^{t_k} p(\mathbf{g}_k^{t_k} | \mathbf{g}_j^{t_j})$$



- Certain about the links between i and j, i.e. $g_{ij}^{t_i}$.

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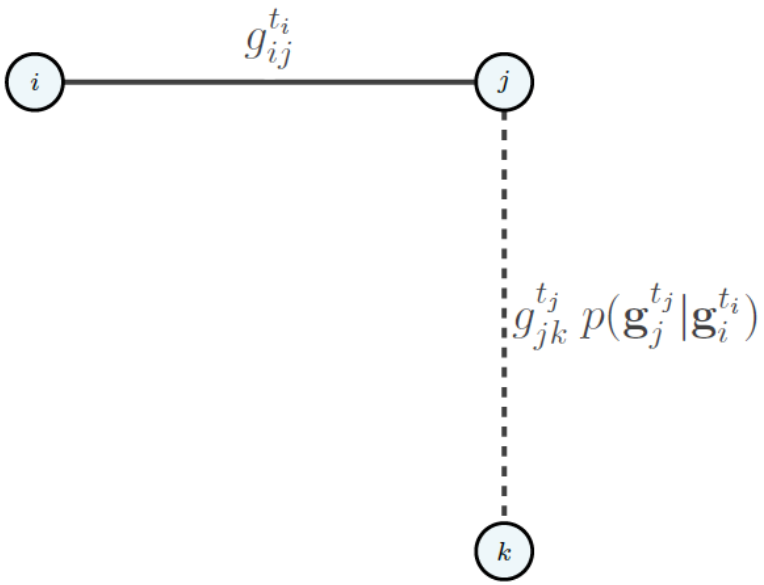

- Certain about the links between i and j, i.e. $g_{ij}^{t_i}$.
- Given the link with j, the possible paths from i to k through j is given by $g_{jk}^{t_j} p(\mathbf{g}_j^{t_j} | \mathbf{g}_i^{t_i})$.

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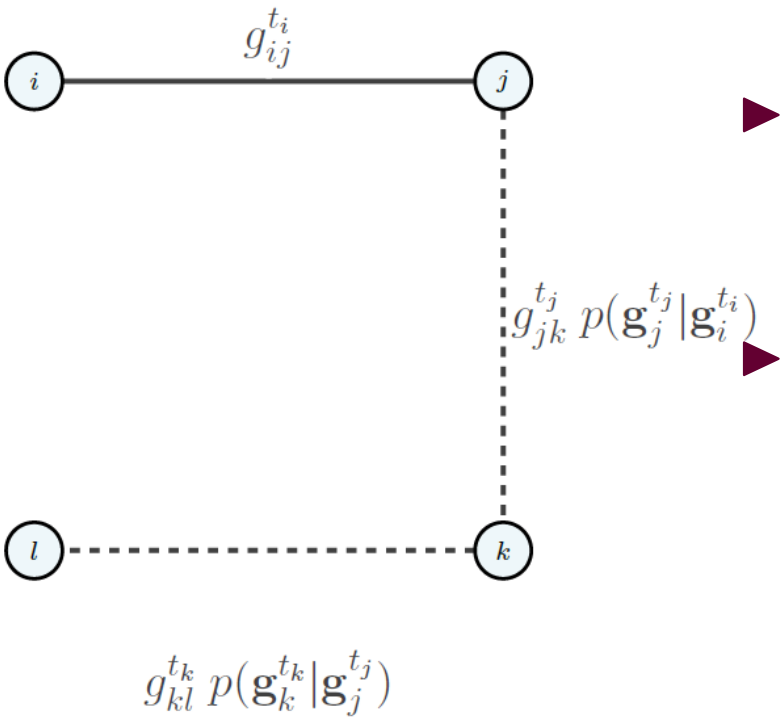

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- Given j is of type $\mathbf{g}_j^{t_j}$, i.e. given the link $g_{jk}^{t_j}$ the possible paths from j to l through k is given by, $g_{kl}^{t_k} p(\mathbf{g}_k^{t_k} | \mathbf{g}_j^{t_j})$.

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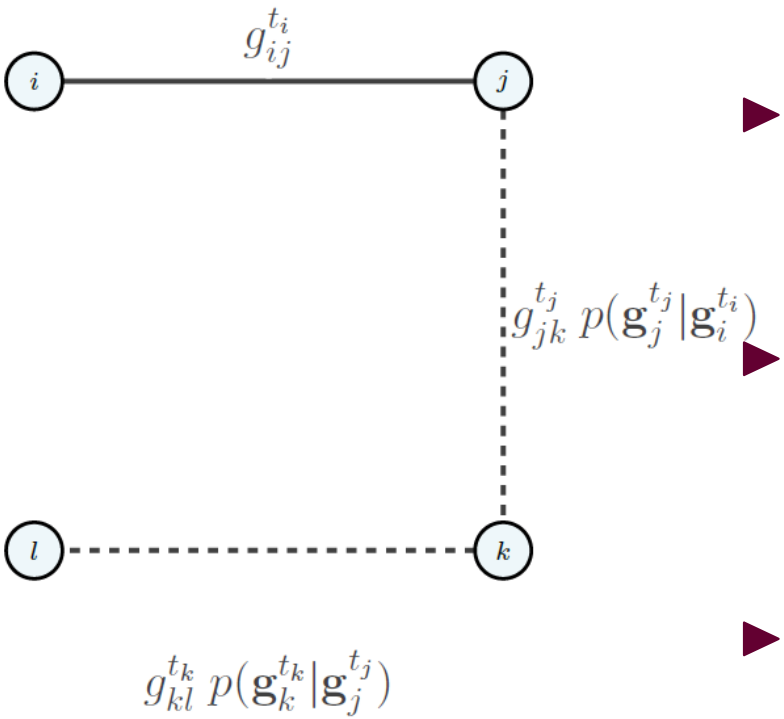

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- Given j is of type $\mathbf{g}_j^{t_j}$, i.e. given the link $g_{jk}^{t_j}$ the possible paths from j to l through k is given by, $g_{kl}^{t_k} p(\mathbf{g}_k^{t_k} | \mathbf{g}_j^{t_j})$.
- $g_{ij}^{t_i} g_{jk}^{t_j} p(\mathbf{g}_j^{t_j} | \mathbf{g}_i^{t_i}) g_{kl}^{t_k} p(\mathbf{g}_k^{t_k} | \mathbf{g}_j^{t_j})$ gives an expected complementarity due to walks of length 3.

Implications

$$a_i(\mathbf{g}_i^{t_i}) = 1 + \lambda\beta_{i,t_i}^{(1)} + \lambda^2\beta_{i,t_i}^{(2)} + \lambda^3\beta_{i,t_i}^{(3)} + \lambda^4\beta_{i,t_i}^{(4)} + \ldots\ldots$$

- Equilibrium is walk-based.
- Remark 1: $a_i(\mathbf{g}_i^{t_i}) \neq \mathbb{E}(b(\mathbf{g}, \lambda(n)))_i$.
- Remark 2: $a_i(\mathbf{g}_i^{t_i}) \neq \mathbb{E}(b(\mathbf{g}, \lambda(n))_i|\mathbf{g}_i^{t_i})$.

Uniform Case

- Monotonicity in degrees is not a general property of the equilibrium actions.
- But if the underlying distribution is uniform,

Proposition 2

When the underlying probability distribution is uniform, the equilibrium actions of the agents for a realized graph is given by:

$$a_i(d_i) = 1 + \frac{\lambda d_i}{1 - \frac{n\lambda}{2}}, \quad \forall d_i \in \{0, 1, 2, \ldots, (n-1)\} \qquad (2)$$

where d_i is the degree of i^{th} agent in that realized graph.

Uniform Case

$$a_i(d_i) = 1 + \frac{\lambda d_i}{1 - \frac{n\lambda}{2}}$$

- This is closely related to the result of Galeotti et al. (2010).
- They show that the equilibrium action of an agent is monotonically increasing in their degrees.
- Unlike them, here the agents internalize the local information about the identities of their neighbors and employ them in the equilibrium play.

Uniform Case

Lemma 1

For any player i denote by $\mathbb{E}_i(d_{j1})$,$\mathbb{E}_i(d_{j2})$,... the agent's ex-post expectations of any of their neighbor's degree, any of their possible neighbor's neighbor's degree and so on. When the underlying probability distribution is uniform, we have:

$$\mathbb{E}_i\left[d_{j1}\right] = \mathbb{E}_i\left[d_{j2}\right] = \ldots\ldots = \frac{n}{2}$$

- Anonymity is a consequence of the uniform assumption.
- A similar property as degree independence arises endogenously when the underlying distribution is uniform.

Also in the Paper

- Closed form characterizations of BNEs under:
  1. Uniform beliefs over all core-periphery networks
  2. Erdos-Renyi network generation
  3. Stochastic block generation and Homphilic Linkage

Conclusion

- Agents compute the expected complementarity strengths that they can extract from walks of different lengths.
- Equilibrium doesn't map to expectations over complete information outcomes.
- Monotonicity is not a general property of the equilibrium.
- Structure of BNE is robust to any ex-ante beliefs.